POTENTIAL OPERATORS FOR MARKOV PROCESSES

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1. Introduction

Yosida's definition of potential operators for semigroups [17] makes it possible to deal with transient Markov processes and a class of recurrent Markov processes in a unified operator theoretical way. In this paper, we prove some general properties of his potential operators, show which Markov processes admit the potential operators, and investigate the cases of processes with stationary independent increments as typical examples.

Let T_t be a strongly continuous semigroup of linear operators on a Banach space $\mathscr{B}$ satisfying

$$\sup_{t \geqq 0} \|T_t\| < \infty, \tag{1.1}$$

with infinitesimal generator A and resolvent

$$J_\lambda = (\lambda - A)^{-1}, \qquad \lambda > 0. \tag{1.2}$$

Following Yosida, we define potential operator V for the semigroup by

$$Vf = s \lim_{\lambda \to 0} J_\lambda f, \tag{1.3}$$

when and only when the limit exists for f in a dense subset of $\mathscr{B}$. The domain $\mathscr{D}(V)$ is the collection of f such that the limit exists. We will give conditions for the existence of the potential operator (Theorem 2.2) and prove some general properties (Theorem 2.3), summarizing Yosida's results [17], [19] with a few results added. The relation with other definitions of potential operators is shown in Theorem 2.4. In Section 3, we consider the case where $\mathscr{B}$ is the Banach space $C_0(S)$ of real valued continuous functions on S vanishing at infinity, S being a locally compact Hausdorff space with a countable base, and T_t is a semigroup induced by a Markov process transition probability. We will prove that the semigroup admits a potential operator if the Markov process is either transient or null recurrent, and that it does not admit a potential operator if the process is positive recurrent. Processes with stationary independent increments on Euclidean spaces are examined in Section 4. The fact that they admit potential operators (Theorem 4.1) is a generalization of Yosida's result [18] on Brownian motions. The domain and the representation of potential operators are investigated for Brownian motions, stable processes, and some other processes in Section 5. We return in Section 6 to a general situation and consider generalization of maximum principles for classical potential operators to operators in Banach lattices. New types of maximum principles are introduced for the adjoints of potential operators.

Several works have been done recently on potential operators of recurrent Markov processes ([7], [10], and others). Authors use different definitions of potential operators. It seems that an advantage for Yosida's potential operators lies in their direct connection with infinitesimal generators.

2. Potential operators for semigroups on Banach spaces

Let $\mathscr{B}$ be a Banach space, and $\mathscr{B}^*$ be its adjoint space. We use the notation $(\varphi, f) = \varphi(f)$ for $\varphi \in \mathscr{B}^*$ and $f \in \mathscr{B}$. The limit in the strong, weak, or weak* convergence is denoted by s lim, w-lim, or w^* lim, respectively. By *dense*, we mean strongly dense. We say that a subset $\mathscr{M}$ of $\mathscr{B}^*$ is w^* *dense* if for each $\varphi \in \mathscr{B}^*$ there is a sequence $\{\varphi_n\}$ in $\mathscr{M}$ such that φ_n weakly* converges to φ. Thus w^* denseness implies denseness in the sense of weak* topology. The symbols $\mathscr{D}$, $\mathscr{R}$, and $\mathscr{N}$ mean domain, range, and null space of an operator. In this section, $\{T_t; t \geqq 0\}$ is always a strongly continuous semigroup of linear operators on $\mathscr{B}$ satisfying (1.1), A is its infinitesimal generator, and J_λ is the resolvent operator (1.2). It is known that A has dense domain and determines the semigroup uniquely and that

$$J_\lambda f = \int_0^\infty e^{-\lambda t} T_t f \, dt, \tag{2.1}$$

(see [3] or [16]). Let T_t^*, J_λ^*, and A^* be the adjoint operators of T_t, J_λ, and A, respectively. The following theorem has a preliminary character, but is interesting in itself.

THEOREM 2.1.

(i) *The semigroup* $\{T_t^*; t \geqq 0\}$ *is a weakly* continuous semigroup on* $\mathscr{B}^*$.

(ii) *The operator* A^* *has* w^* *dense domain and*

$$A^*\psi = w^* \lim_{t \to 0} t^{-1}(T_t^*\psi - \psi), \qquad \psi \in \mathscr{D}(A^*). \tag{2.2}$$

Conversely, if the right side of (2.2) *exists, then* $\psi \in \mathscr{D}(A^*)$.

(iii) *The following relations hold*

$$J_\lambda^* = (\lambda - A^*)^{-1}, \tag{2.3}$$

$$J_\lambda^*\varphi = \int_0^\infty e^{-\lambda t} T_t^*\varphi \, dt, \qquad \varphi \in \mathscr{B}^*. \tag{2.4}$$

(iv) *The operator A^* determines T_t uniquely.*

The integral in (2.4) is defined to be the element $\psi \in \mathscr{B}^*$ which satisfies

$$(2.5) \qquad (\psi, f) = \int_0^\infty e^{-\lambda t} (T_t^* \varphi, f)\, dt, \qquad f \in \mathscr{B}.$$

PROOF. Part (i) is obvious. Equation (2.3) is a consequence of (1.2), (see [16], p. 224). The domain of A^* is w^* dense since $\lambda J_\lambda^* \varphi$ converges to φ weakly* as $\lambda \to \infty$ and $\lambda J_\lambda^* \varphi \in \mathscr{D}(A^*)$. If φ is the right side of (2.2), then $J_\lambda^*(\lambda\psi - \varphi) = \psi$, and hence $\psi \in \mathscr{D}(A^*)$ and $A^*\psi = \varphi$, because we have

$$(2.6) \qquad \begin{aligned}(J_\lambda^* \varphi, f) &= \lim_{t \to 0} t^{-1}(T_t^* \psi - \psi, J_\lambda f) = (\psi, A J_\lambda f) \\ &= (\psi, \lambda J_\lambda f - f) = (\lambda J_\lambda^* \psi - \psi, f)\end{aligned}$$

for any $f \in \mathscr{B}$. If ψ is the right side of (2.4), then

$$(2.7) \qquad \begin{aligned}(T_t^* \psi, f) &= \int_0^\infty e^{-\lambda s} (T_s^* \varphi, T_t f)\, ds = \int_0^\infty e^{-\lambda s} (T_{t+s}^* \varphi, f)\, ds \\ &= e^{\lambda t} \int_t^\infty e^{-\lambda s} (T_s^* \varphi, f)\, ds\end{aligned}$$

for any f, which implies $t^{-1}(T_t^* \psi - \psi, f) \to (\lambda\psi - \varphi, f)$, and hence $\psi \in \mathscr{D}(A^*)$ and $A^*\psi = \lambda\psi - \varphi$. This shows (2.4) by (2.3). In order to finish the proof of (2.2), note that any $\psi \in \mathscr{D}(A^*)$ is represented as $\psi = J_\lambda^* \varphi$ by (2.3), and hence $w^* \lim t^{-1}(T_t^* \psi - \psi)$ exists by (2.4) and the above argument. The operator A^* determines T_t uniquely since A^* determines J_λ^* by (2.3) and J_λ^* determines J_λ. The proof is complete.

The potential operator defined in Section 1 does not always exist. But we have simple criteria for its existence.

THEOREM 2.2. *The following conditions are equivalent:*

(a) *T_t admits a potential operator;*
(b) *$\mathscr{R}(A)$ is dense;*
(c) *$\lambda J_\lambda f \to 0$ $(\lambda \to 0)$ strongly for all f;*
(d) *$\lambda J_\lambda f \to 0$ $(\lambda \to 0)$ weakly for all f;*
(e) *$t^{-1} \int_0^t T_s f\, ds \to 0$ $(t \to \infty)$ strongly for all f;*
(f) *$t^{-1} \int_0^t T_s f\, ds \to 0$ $(t \to \infty)$ weakly for all f;*
(g) *A^* is one to one;*
(h) *$\lambda J_\lambda^* \varphi \to 0$ $(\lambda \to 0)$ weakly* for all φ;*
(i) *$t^{-1} \int_0^t T_s^* \varphi\, ds \to 0$ $(t \to \infty)$ weakly* for all φ.*

The equivalence of the first four conditions is proved by Yosida [17]. He introduces also the condition (h) in [19]. Conditions (e), (f), (i) are new. The following condition is also equivalent, though apparently weaker: for each f in a dense set in $\mathscr{B}$ there is a sequence $\{\lambda_n\}$ decreasing to 0 such that $\lambda_n J_{\lambda_n} f$ converges to 0 weakly (see [17]).

PROOF. That (d) is equivalent to (h) is evident. The equivalence of (f) and (i) is also evident, since we have

$$\left(t^{-1}\int_0^t T_s^*\varphi\,ds, f\right) = \left(\varphi, t^{-1}\int_0^t T_s f\,ds\right). \tag{2.8}$$

Equivalence of (b) and (g) is of general character ([16], p. 224). Using

$$\lambda\|J_\lambda\| \leqq M, \tag{2.9}$$

where M is the bound of $\|T_t\|$, we get the implication (a) $\Rightarrow$ (c); (c) implies (d); (d) implies (b) because the closure of $\mathscr{R}(A)$ is closed in weak topology by the Hahn-Banach theorem ([16], p. 125) and $AJ_\lambda f = \lambda J_\lambda f - f \to f$ weakly as $\lambda \to 0$; (b) implies (c), since if $f = Au$ then

$$\|\lambda J_\lambda f\| = \|\lambda(\lambda J_\lambda - 1)u\| \leqq \lambda(1 + M)\|u\| \to 0, \tag{2.10}$$

and since we can use (2.9) and (b) for general f. On the other hand, (b) and (c) together imply (a) because $J_\lambda Au = \lambda J_\lambda u - u \to -u$ strongly. Thus, (a), (b), (c), (d) are equivalent. If $f = Au$, then

$$t^{-1}\int_0^t T_s f\,ds = t^{-1}\int_0^t AT_s u\,ds = t^{-1}(T_t u - u) \to 0, \qquad t \to \infty. \tag{2.11}$$

Since we have

$$\left\|t^{-1}\int_0^t T_s f\,ds\right\| \leqq M\|f\|, \qquad f \in \mathscr{B}, \tag{2.12}$$

(b) implies (e); (e) $\Rightarrow$ (f) is evident. If (f) holds, then

$$t^{-1}\int_0^t (\varphi, T_s f)\,ds = \left(\varphi, t^{-1}\int_0^t T_s f\,ds\right) \to 0, \qquad t \to \infty, \tag{2.13}$$

for each φ and f, which implies

$$\lambda\int_0^\infty e^{-\lambda s}(\varphi, T_s f)\,ds \to 0, \qquad \lambda \to 0, \tag{2.14}$$

by the Abelian theorem for Laplace transforms ([15], p. 181), that is, the condition (d). The proof is complete.

THEOREM 2.3. *Suppose that* $\{T_t;\, t \geqq 0\}$ *admits a potential operator* V *and let* V^* *be the adjoint operator of* V. *Then,* A, A^*, V, V^* *are all one to one,* $V = -A^{-1}$, *and* $V^* = -(A^*)^{-1}$. *Subspaces* $\mathscr{D}(V) = \mathscr{R}(A)$ *and* $\mathscr{R}(V) = \mathscr{D}(A)$ *are both dense in* $\mathscr{B}$; *similarly,* $\mathscr{D}(V^*) = \mathscr{R}(A^*)$ *and* $\mathscr{R}(V^*) = \mathscr{D}(A^*)$ *are both* w^* *dense in* $\mathscr{B}^*$. *Furthermore,*

$$V^*\varphi = w^*\lim_{\lambda\to 0} J_\lambda^*\varphi, \qquad \varphi \in \mathscr{D}(V^*) \tag{2.15}$$

holds. The collection of φ *such that the limit in the right side of* (2.15) *exists coincides with* $\mathscr{D}(V^*)$.

The theorem is proved by Yosida [17], [19]. We give the proof for completeness.

PROOF. If $u \in \mathscr{D}(A)$, then $J_\lambda Au = \lambda J_\lambda u - u \to -u$ strongly by Theorem 2.2 (c), and hence $Au \in \mathscr{D}(V)$ and $-VAu = u$. If $f \in \mathscr{D}(V)$, then $AJ_\lambda f \to -f$ strongly likewise, and hence $Vf \in \mathscr{D}(A)$ and $-AVf = f$ by the closedness of A. Thus, $\mathscr{N}(A) = \mathscr{N}(V) = \{0\}$ and $V = -A^{-1}$. This implies $\mathscr{N}(A^*) = \mathscr{N}(V^*) = \{0\}$ and $V^* = -(A^*)^{-1}$ by [16], p. 224. If $J_\lambda^* \varphi$ has weak* limit ψ as $\lambda \to 0$, then it follows from

$$(2.16) \qquad (\varphi - \lambda J_\lambda^* \varphi, Vf) = (-A^* J_\lambda^* \varphi, Vf) = (J_\lambda^* \varphi, f),$$

that $(\varphi, Vf) = (\psi, f)$ for all $f \in \mathscr{D}(V)$, which means $\varphi \in \mathscr{D}(V^*)$ and $V^*\varphi = \psi$. Conversely, if $\varphi \in \mathscr{D}(V^*)$ and $V^*\varphi = \psi$, then $J_\lambda^*\varphi = -J_\lambda^* A^* \psi = \psi - \lambda J_\lambda^* \psi$ converges to ψ weakly*. The remaining assertions are trivial consequences of the previous theorems.

COROLLARY 2.1. *The operator V determines $\{T_t\}$ uniquely, and so does V^*.*

THEOREM 2.4. *Suppose that $\{T_t\}$ admits a potential operator V.*

(i) *The following five conditions are equivalent:*

(a) $f \in \mathscr{D}(V)$ *and* $Vf = u$;

(b) $J_\lambda f \to u(\lambda \to 0)$ *weakly;*

(c) $(1 - \lambda J_\lambda)u = J_\lambda f$ *for some* $\lambda > 0$;

(d) $(1 - \lambda J_\lambda)u = J_\lambda f$ *for all* $\lambda > 0$;

(e) $(1 - T_t)u = \int_0^t T_s f\, ds$ *for all* $t \geqq 0$.

(ii) *In order that $f \in \mathscr{D}(V)$, $Vf = u$, and $w \lim T_t u\ (t \to \infty)$ exists, it is necessary and sufficient that*

$$(2.17) \qquad u = w \lim_{t \to \infty} \int_0^t T_s f\, ds.$$

(iii) *Assertion* (ii) *remains valid with* w lim *replaced by* s lim.

Equivalence of (d) and (e) is observed by Kondō, and he studies solution of (d) and (e) in an extended sense for recurrent Markov processes [7].

PROOF. The implication (a) ⇒ (b) is trivial. If $u = w \lim J_\mu f$, then by the resolvent equation

$$(2.18) \qquad J_\lambda f - J_\mu f + (\lambda - \mu) J_\lambda J_\mu f = 0,$$

we have (d). Thus, (b) implies (d). The implication (d) ⇒ (c) is trivial. If (c) holds, then it follows from (2.18) that

$$(2.19) \qquad J_\mu f = J_\lambda f + (\lambda - \mu) J_\mu (1 - \lambda J_\lambda) u = J_\lambda f - (\lambda - \mu) J_\mu A J_\lambda u,$$

which strongly converges to $J_\lambda f + \lambda J_\lambda u = u$, and we have (*a*). Also, we see equivalence of (a) and (e), noting that $f = -Au$. For (ii) let u be defined by (2.17). Then u satisfies (e), and hence (a). According to (e) and (2.17), $T_t u$ weakly converges to 0 as $t \to \infty$. Conversely, let $Vf = u$ and $T_t u \to v$ weakly as $t \to \infty$. Then we have $T_s v = v$ for every s, and hence $v = 0$ by $\mathscr{N}(A) = \{0\}$. Thus, (2.17) follows from (e). Replacing w lim by s lim, we get the proof of (iii).

A set $\mathscr{M}$ is called a *core* of a closed operator T, if $\mathscr{M} \subset \mathscr{D}(T)$ and if the smallest closed extension of $T \mid \mathscr{M}$ coincides with T, where $T \mid \mathscr{M}$ is the restriction of T to $\mathscr{M}$ (see [6], p. 166). The notion of core is important, because if $\mathscr{M}$ is a core of the potential operator V, then $V \mid \mathscr{M}$ determines the semigroup. Note that V is a closed operator since A is closed. Although it is usually difficult to find explicit expression of V, it is sometimes possible to find the expression on some core. See Section 5 for examples. If $\mathscr{M}$ is a core of T, then $\mathscr{M}$ is dense in $\mathscr{D}(T)$ and $T(\mathscr{M})$ is dense in $\mathscr{R}(T)$. But the converse is not true in general. We can prove the following assertion: let T be a closed linear operator and let $\mathscr{M}$ be a linear subspace of $\mathscr{D}(T)$. Suppose that for each $f \in \mathscr{D}(T)$ there are a sequence $\{f_n\}$ in $\mathscr{M}$ and an element $g \in \mathscr{B}$ such that $w \lim f_n = f$ and $w \lim Tf_n = g$. Then, $\mathscr{M}$ is a core of T.

3. Potential operators for Markov process semigroups

Let S be a locally compact Hausdorff space with a countable base. Let $C_0(S)$ be the Banach space of real valued continuous functions on S vanishing at infinity if S is not compact, or the Banach space of real valued continuous functions on S if S is compact. We denote the collection of continuous functions with compact supports by $C_K(S)$ or C_K, and the collection of nonnegative functions in C_K by C_K^+. Let $\{T_t;\, t \geqq 0\}$ be a strongly continuous semigroup of positive linear operators on $C_0(S)$ with norm $\|T_t\| \leqq 1$. There corresponds to $\{T_t\}$ a right continuous, time homogeneous Markov process on S with transition probability $P(t, x, dy)$ such that

$$T_t f(x) = \int_S f(y) P(t, x, dy). \tag{3.1}$$

We call the Markov process (or the semigroup) *recurrent* if

$$\int_0^\infty P(t, x, U)\, dt = \infty \tag{3.2}$$

for all x and all open neighborhoods U of x, and *transient* if

$$\int_0^\infty P(t, x, K)\, dt < \infty \tag{3.3}$$

for all x and all compact K. It is *null recurrent* if it is recurrent and

$$\lim_{t \to \infty} P(t, x, K) = 0 \tag{3.4}$$

for all x and all compact K, and *positive recurrent* if

$$\liminf_{t \to \infty} P(t, x, U) > 0 \tag{3.5}$$

for all x and all open neighborhood U of x. We will show the relations of these notions with the existence of a potential operator. In applying the theorems in

Section 2, note that $C_0^*(S)$ is the space of signed measures with bounded variation normed by the total variation, and that $\{f_n\}$ converges weakly to f in $C_0(S)$ if and only if $f_n(x)$ converges to $f(x)$ pointwise and $\sup_n \|f_n\| < \infty$.

THEOREM 3.1. *If $\{T_t\}$ is transient, then it admits a potential operator.*

PROOF. Since C_K is dense, it suffices to show that $t^{-1}\int_0^t T_s f(x)\,ds$ tends to 0 pointwise as $t \to \infty$ for $f \in C_K$ (Theorem 2.2). But this is easily seen because $\int_0^\infty T_s|f|(x)\,ds$ is finite.

THEOREM 3.2. *If $\{T_t\}$ is null recurrent or, more generally, if (3.4) holds, then it admits a potential operator V and*

$$Vf = w \lim_{t\to\infty} \int_0^t T_s f\,ds. \tag{3.6}$$

PROOF. It follows from (3.4) that

$$w \lim_{t\to\infty} T_t f = 0 \tag{3.7}$$

for all $f \in C_K$, hence for all $f \in C_0(S)$. The theorem is then obtained from Theorems 2.2 and 2.4.

THEOREM 3.3. *If $\{T_t\}$ is positive recurrent, or if the process has a finite invariant measure, then the potential operator does not exist.*

PROOF. Property (3.5) implies that condition (f) of Theorem 2.2 does not hold. Existence of a finite invariant measure φ contradicts condition (i) of the same theorem since $T^*\varphi = \varphi \neq 0$.

REMARK. If S is a countable set with discrete topology and all points communicate with each other, then the following three conditions for the process are equivalent: to admit a potential operator; to be transient or null recurrent; to have no finite invariant measure. In fact, transience, null recurrence, and positive recurrence cover all possibilities in this case, and the process is positive recurrent if and only if it has a finite invariant measure [2].

If the process is transient, C_K is not necessarily contained in $\mathscr{D}(V)$. But we have:

THEOREM 3.4. *Suppose that $\{T_t\}$ admits a potential operator V and that $C_K \subset \mathscr{D}(V)$. Then, it is transient, C_K is a core of V, and*

$$Vf = s \lim_{t\to\infty} \int_0^t T_s f\,ds. \tag{3.8}$$

PROOF. If $f \in C_K^+$, then

$$\int_0^\infty T_t f(x)\,dt = \lim_{\lambda\to 0} \int_0^\infty e^{-\lambda t} T_t f(x)\,dt = Vf(x) < \infty. \tag{3.9}$$

Hence, the process is transient and we have (3.8) for C_K^+ by applying Dini's theorem to the one point compactification of S. Let V_0 be the smallest closed extension of $V \mid C_K$. We have to prove $V_0 = V$. First, let us show that if $f \in C_K^+$, then $J_\lambda f \in \mathscr{D}(V_0)$. In fact, let $u = J_\lambda Vf$ and let $\{g_n\}$ be an increasing sequence in

C_K^+ which converges strongly to $J_\lambda f$. Since we have

$$(3.10)\qquad \int_0^\infty T_t J_\lambda f(x)\,dt = \int_0^\infty \int_0^\infty e^{-\lambda s} T_{t+s} f(x)\,ds\,dt = \int_0^\infty e^{-\lambda s} T_s V f(x)\,ds = u,$$

Vg_n tends strongly to u by Dini's theorem, and hence $J_\lambda f \in \mathscr{D}(V_0)$. Also, we have

$$(3.11)\qquad \lambda V_0 J_\lambda f + J_\lambda f = V_0 f$$

for $f \in C_K$. It follows from (3.11) that $\mathscr{R}(V_0)$ is dense. We see that if $f \in \mathscr{D}(V_0)$, then $J_\lambda f \in \mathscr{D}(V_0)$ and (3.11) holds. Thus, $\mathscr{R}(\lambda V_0 + 1)$ contains $\mathscr{R}(V_0)$, and hence is dense. If $g_n = (\lambda V_0 + 1)f_n \to g$ strongly, then $f_n = g_n - \lambda J_\lambda g_n \to g - \lambda J_\lambda g$ and hence $g \in \mathscr{R}(\lambda V_0 + 1)$ by closedness of V_0. The whole space is thus $\mathscr{R}(\lambda V_0 + 1)$. On the other hand, $\lambda V + 1$ is a one to one mapping and an extension of $\lambda V_0 + 1$, whence $V = V_0$. If $f \in C_K$, then it follows from (3.8) and Theorem 2.4 that $T_t V f \to 0$ strongly as $t \to \infty$. Since $V(C_K)$ is dense, $T_t g \to 0$ strongly for all $g \in C_0$, and hence we have (3.8), completing the proof.

If the process is conservative (that is, $P(t, x, S) = 1$ for all t and x), then V is unbounded. In fact, if V is bounded, then $0 \leqq J_\lambda f_n \leqq V f_n \leqq \|V\|$ for $0 \leqq f_n \leqq 1$ and, letting $f_n(x)$ increase to 1, we should get $\lambda^{-1} \leqq \|V\|$ for any $\lambda > 0$, which is absurd. This is in contrast to the fact that there are many bounded infinitesimal generators.

4. Potential operators for processes with stationary independent increments

Let $X_t(\omega)$, $t \geqq 0$, be a right continuous stochastic process on R^N starting at the origin with stationary independent increments defined on a probability space $(\Omega, \mathscr{F}, P)$. The process $x + X_t(\omega)$ is a Markov process starting at x. Its transition operator carries $C_0(R^N)$ into itself and forms a strongly continuous semigroup T_t, which commutes with any translation L_y defined by $L_y f(x) = f(x + y)$. Conversely, every strongly continuous positive semigroup T_t on $C_0(R^N)$ with norm $\|T_t\| = 1$ which commutes with translations is induced in this way. We denote the totality of infinitesimal generators of such semigroups on $C_0(R^N)$ by $\mathbf{G}_N$.

THEOREM 4.1. *The infinitesimal generator $A \in \mathbf{G}_N$ admits a potential operator, except if A is the zero operator.*

This fact, which generalizes [18], is a consequence of Section 3, since the process is transient or null recurrent. But we will prove this theorem from an estimation of $\|T_t\|$ (Theorem 4.3). Theorem 4.2 gives the representation of infinitesimal generators $\mathbf{G}_N$, and our proof of Theorem 4.3 which makes use of Theorem 4.2 and decomposition of semigroups may be of some interest. In the following, $D_i = \partial/\partial x_i$, $D_{i,j} = \partial^2/\partial x_i \partial x_j$, and C_K^∞ is the set of C^∞ functions with compact supports; $S(f)$ is the support of function f.

THEOREM 4.2. *Let $A \in \mathbf{G}_N$. Then,*
(i) $C_K^\infty \subset \mathscr{D}(A)$;
(ii) C_K^∞ *is a core of* A;
(iii) *for any* $u \in C_K^\infty$, Au *is of the form*

$$(4.1)\qquad Au(x) = \sum_{i,j=1}^{N} a_{i,j}D_{i,j}u(x) + \sum_{i=1}^{N} b_iD_iu(x) + \int_{R^N\setminus\{0\}} \left[u(x+y) - u(x) - \chi_U(y)\sum_{i=1}^{N} y_iD_iu(x)\right] n(dy),$$

where $a_{i,j}$ and b_i are constants, $(a_{i,j})$ is a symmetric positive semidefinite matrix, χ_U is the indicator function of the open unit ball U, and n is a measure on $R^N\setminus\{0\}$ satisfying

$$(4.2)\qquad n(R^N\setminus U) < \infty, \qquad \int_{U\setminus\{0\}} |y|^2 n(dy) < \infty.$$

Conversely, if A satisfies (iii) *and $\mathscr{D}(A) = C_K^\infty$, then A is closable in $C_0(R^N)$ and the smallest closed extension of A is a member of $\mathbf{G}_N$.*

The measure n is called *Lévy measure*. A proof is found in [11]. Hunt [4] has a similar theorem.

THEOREM 4.3. *Let $A \in \mathbf{G}_N$.*
(i) *If $a_{i,j} \neq 0$ for some i, j, then*

$$(4.3)\qquad \|T_tf\| \leqq (4\pi\alpha t)^{-1/2}\|f\| \operatorname{diam} S(f)$$

for any $f \in C_K$ and $t > 0$, where α is the maximum eigenvalue of $(a_{i,j})$.

(ii) *If the Lévy measure n does not identically vanish, then*

$$(4.4)\qquad \|T_tf\| \leqq e(2\pi\beta t)^{-1/2}\|f\|[1 + \varepsilon^{-1} \operatorname{diam} S(f)]$$

for any $f \in C_K$, $t > 0$, and $\varepsilon > 0$, where

$$(4.5)\qquad \beta = \beta(\varepsilon) = \max_{1 \leqq i \leqq N} \max\{n(\{y; y_i \geqq \varepsilon\}), n(\{y; y_i \leqq -\varepsilon\})\}.$$

In the proof, we use the following lemma, which is easily proved. This is closely connected with Theorem 1 of Trotter [14].

LEMMA 4.1. *Let $A^{(1)}, A^{(2)} \in \mathbf{G}_N$ and let $T_t^{(1)}$ and $T_t^{(2)}$ be respective generated semigroups. Then, $T_t^{(1)}$ and $T_s^{(2)}$ commute for any t and s. Let $T_t^{(3)} = T_t^{(2)}T_t^{(1)}$. Then $T_t^{(3)}$ is a strongly continuous semigroup with infinitesimal generator $A^{(3)} \in \mathbf{G}_N$ and $A^{(3)}u = A^{(1)}u + A^{(2)}u$ for $u \in C_K^\infty$.*

PROOF OF THEOREM 4.3.

(i) By an orthogonal transformation of R^N, we can and shall assume that the matrix $(a_{i,j})$ is diagonal and $a_{1,1} = \alpha$. Let $A^{(1)}$ and $A^{(2)}$ be members of $\mathbf{G}_N$ such that $A^{(1)}u = \alpha D_{1,1}u$ and $A^{(2)}u = Au - A^{(1)}u$ for $u \in C_K^\infty$ (Theorem 4.2). By using

Lemma 4.1, we have $\|T_t f\| = \|T_t^{(2)} T_t^{(1)} f\| \leqq \|T_t^{(1)} f\|$, which implies (4.3) since

$$T_t^{(1)} f(x) = \int_{R^1} \frac{\exp\{-y_1^2/4\alpha t\}}{(4\pi\alpha t)^{1/2}} f(x_1 + y_1, x_2, \cdots, x_N)\, dy_1. \tag{4.6}$$

(ii) Let us consider the case where $\beta = n(\{y; y_1 \geqq \varepsilon\})$. The other cases are treated in the same manner. Let $n^{(1)}$ be a measure defined by $n^{(1)}(E) = n(E \cap \{y; y_1 \geqq \varepsilon\})$, let $A^{(1)}$ be a bounded operator defined by

$$A^{(1)} u(x) = \int [u(x + y) - u(x)] n^{(1)}(dy), \tag{4.7}$$

and let $A^{(2)} = A - A^{(1)}$. By virtue of Lemma 4.1, it is enough to prove the estimate (4.4) with T_t replaced by $T_t^{(1)}$ generated by $A^{(1)}$. Let $Y_t(\omega)$, $t \geqq 0$, be a Poisson process with paths being right continuous step functions and $EY_t = \beta t$. Let $\{Z_k(\omega); k = 1, 2, \cdots\}$ be independent identically distributed R^N valued random variables independent of the process $\{Y_t; t \geqq 0\}$, each Z_k having distribution $\beta^{-1} n^{(1)}$. Let $S_0 = 0$ and $S_k(\omega) = \Sigma_{j=1}^k Z_j(\omega)$. Then, $S_{Y_t(\omega)}(\omega)$ is a process with stationary independent increments which induces the semigroup $T_t^{(1)}$. We have

$$T_t^{(1)} f(x) = \sum_{k=0}^{\infty} P(Y_t = k) E f(x + S_k), \tag{4.8}$$

and hence

$$|T_t^{(1)} f(x)| \leqq e^{-\beta t} \left(\max_{k \geqq 0} \frac{(\beta t)^k}{k!} \right) \|f\| \sum_{k=0}^{\infty} P(S_k \in S(f) - x). \tag{4.9}$$

We have

$$\sum_{k=0}^{\infty} P(S_k \in S(f) - x) \leqq E \sum_{k=0}^{\infty} \chi_{K_1 - x_1}(S_{k,1}) \leqq 1 + \varepsilon^{-1} \operatorname{diam} S(f), \tag{4.10}$$

where K_1 is the projection of $S(f)$ into the first coordinate, and $S_{k,1} = \Sigma_{j=1}^k Z_{j,1}$, $Z_{j,1}$ being the first coordinate of Z_j. Note that $Z_{j,1} \geqq \varepsilon$ with probability one. Let $\rho(k) = (k!)^{-1} e^{-k} (k + 1)^{k+1/2}$. Since we have

$$\sup_{k \geqq 0} \rho(k) = \lim_{k \to \infty} \rho(k) = e(2\pi)^{-1/2}, \tag{4.11}$$

by elementary calculus (it will suffice to recall the proof of Stirling's formula), we have

$$e^{-\beta t} \left(\max_{k \geqq 0} \frac{(\beta t)^k}{k!} \right) = e^{-\beta t} \frac{(\beta t)^{[\beta t]}}{[\beta t]!} \leqq (\beta t)^{-1/2} \rho([\beta t]) \leqq e(2\pi\beta t)^{-1/2}, \tag{4.12}$$

which, combined with (4.9) and (4.10), proves (4.4). The proof is complete.

PROOF OF THEOREM 4.1. In the cases described in Theorem 4.3, $T_t f$ strongly converges to 0 as $t \to \infty$ for every $f \in C_0(R^N)$, which implies the existence of a potential operator by Theorem 2.2. There remains the case where $Au = \Sigma_{i=1}^{N} b_i D_i u$ for $u \in C_K^\infty$ and $b_i \neq 0$ for some i. In this case, the semigroup is $T_t f(x) = f(x_1 + b_1 t, \cdots, x_N + b_N t)$, and hence $T_t f$ weakly converges to 0 as $t \to \infty$ for every f. This also suffices for our conclusion by Theorem 2.2.

We give some results on cores of the potential operator V of $A \in \mathbf{G}_N$.

THEOREM 4.4. *The collection $\mathscr{M}$ of $f \in \mathscr{D}(V)$ such that f and Vf are integrable is a core of V. If $f \in \mathscr{M}$, then*

$$\int_{R^N} f(x)\, dx = 0. \tag{4.13}$$

PROOF. It is easy to see that if $g \in C_0$ is integrable, then $J_\lambda g$ is integrable and

$$\lambda \int J_\lambda g(x)\, dx = \int g(x)\, dx. \tag{4.14}$$

If $f \in \mathscr{M}$, then we get (4.13) from (4.14) by letting $g = f + \lambda V f$, $f = g - \lambda J_\lambda g$. Since C_K^∞ is a core of A (Theorem 4.2), the set $\mathscr{M}_1$ of $f \in \mathscr{D}(V)$ such that Vf belongs to C_K^∞ is a core of V. We claim that Au is integrable if $u \in C_K^\infty$, which will imply $\mathscr{M}_1 \subset \mathscr{M}$ and completes the proof. Let $K = S(u)$. In the expression (4.1) of Au, the first two terms have compact supports, and the integral

$$\iint_{R^N \times R^N} \left| u(x+y) - u(x) - \chi_U(y) \sum_{i=1}^{N} y_i D_i u(x) \right| dx\, n(dy) \tag{4.15}$$

is finite because the integral over each of the following four sets is finite: $x \in K$ and $y \in U$; $x \in K$ and $y \in R^N \backslash U$; $x + y \in K$ and $y \in U$; $x + y \in K$ and $y \in R^N \backslash U$.

THEOREM 4.5. *If $n(R^N \backslash B_a) = 0$ for some $B_a = \{x; |x| \leqq a\}$, then $C_K \cap \mathscr{D}(V)$ is a core of V.*

PROOF. The condition implies $Au \in C_K$ if $u \in C_K^\infty$. Hence, the set $\mathscr{M}_1$ defined in the preceding proof is a subset of $C_K \cap \mathscr{D}(V)$.

An open question is whether $C_K \cap \mathscr{D}(V)$ is a core of V without any assumption on Lévy measure.

It should be noted that a paper by Port and Stone [10] includes the following result: let Σ be the collection of points x such that for each open neighborhood U of x there is a $t > 0$ satisfying $P(X_t \in U) > 0$, assume that the closed group generated by Σ is R^N, and consider transient cases. In case $N = 1$ and X_t has finite nonzero mean, a function f in C_K belongs to $\mathscr{D}(V)$ if and only if it satisfies (4.13). Otherwise all functions in C_K belong to $\mathscr{D}(V)$.

5. Examples

EXAMPLE 5.1. *Brownian motion.* Let $A \in \mathbf{G}_N$ be such that

$$Au(x) = \sum_{i=1}^{N} D_{i,i} u(x), \qquad u \in C_K^\infty, \tag{5.1}$$

and V be the corresponding potential operator.

Let $N = 1$. *Then, a function* f *in* $C_0(R^1)$ *belongs to* $\mathscr{D}(V)$ *if and only if*

$$\int_{-\infty}^{\infty} f(x)\,dx = 0, \tag{5.2}$$

and

$$\lim_{a,b\to\infty}\left[\int_{-a}^{b} xf(x)\,dx - a\int_{-\infty}^{-a} f(x)\,dx + b\int_{b}^{\infty} f(x)\,dx\right] = 0. \tag{5.3}$$

If $f \in \mathscr{D}(V)$, *then*

$$Vf(x) = \lim_{a\to\infty} 2^{-1}\left[-\int_{-a}^{a} |y - x| f(y)\,dy + a\int_{-a}^{a} f(y)\,dy\right]. \tag{5.4}$$

Here integrals with infinite endpoints are understood as Riemann improper integrals.

COROLLARY 5.1. *Consider the case where* $xf(x) \in L^1$. *A necessary and sufficient condition for* $f \in \mathscr{D}(V)$ *is*

$$\int_R xf(x)\,dx = \int_R f(x)\,dx = 0. \tag{5.5}$$

If $f \in \mathscr{D}(V)$, *then*

$$Vf(x) = -\frac{1}{2}\int_R |y - x| f(y)\,dy. \tag{5.6}$$

Note that there are functions f in $\mathscr{D}(V)$ for which $xf(x)$ is not an L^1 function. Consider, for example, $f = u''$, letting $u(x)$ equal $x^\alpha \sin x$, $-1 \leqq \alpha < 0$, for large x.

COROLLARY 5.2. *Let* I *be a closed interval and let* $f \in \mathscr{D}(V)$. *Then,* $S(f) \subset I$ *if and only if* $S(Vf) \subset I$.

PROOF. It is known that $\mathscr{D}(A)$ is the collection of $u \in C_0(R^1)$ such that u is of class C^2 and $u'' \in C_0(R^1)$. We have $Au = u''$ for all $u \in \mathscr{D}(A)$. Let $f \in \mathscr{D}(V)$. It follows that $u = Vf \in \mathscr{D}(A)$, $f = -u''$, and $u' \in C_0(R^1)$, whence (5.2). Equation (5.3) follows from

$$\int_{-a}^{b} xf(x)\,dx = a\int_{-\infty}^{-a} xf(x)\,dx - b\int_{b}^{\infty} xf(x)\,dx + u(-a) - u(b). \tag{5.7}$$

In order to prove the converse, let f be a function in $C_0(R^1)$ satisfying (5.2) and (5.3). Let

$$g_1(x) = \int_{x}^{\infty} f(y)\,dy, \qquad g_2(x) = \int_{-\infty}^{x} f(y)\,dy, \tag{5.8}$$

$$h_1(x) = \lim_{b\to\infty}\left[\int_{x}^{b} yf(y)\,dy + b\int_{b}^{\infty} f(y)\,dy\right], \tag{5.9}$$

$$h_2(x) = \lim_{a\to\infty}\left[\int_{-a}^{x} yf(y)\,dy - a\int_{-\infty}^{-a} f(y)\,dy\right]. \tag{5.10}$$

All of these exist and $g_1 + g_2 = 0$, $h_1 + h_2 = 0$. Define u by the right side of (5.4). Then we have

$$u(x) = xg_1(x) - h_1(x) = -xg_2(x) + h_2(x). \tag{5.11}$$

Since $xg_1(x) - h_1(x)$ is

$$\int_{x_0}^{x} yf(y)\,dy + x\int_{x}^{\infty} f(y)\,dy - \lim_{b\to\infty}\left[\int_{x_0}^{b} yf(y)\,dy + b\int_{b}^{\infty} f(y)\,dy\right] \tag{5.12}$$

for fixed x_0, we have $xg_1(x) - h_1(x) \to 0$ as $x \to \infty$. Similarly, $xg_2(x) - h_2(x) \to 0$ as $x \to -\infty$, and hence $u \in C_0$. Since $u'' = -f$, it follows that $u \in \mathscr{D}(A)$ and $Au = -f$.

For higher dimensions, the following are classical results. For $N = 2$, a function f in C_K belongs to $\mathscr{D}(V)$ if and only if f has integral null. If $f \in C_K \cap \mathscr{D}(V)$, then

$$Vf(x) = -\frac{1}{2\pi}\int_{R^2} f(y)\log|y - x|\,dy. \tag{5.13}$$

For $N \geqq 3$, C_K is contained in $\mathscr{D}(V)$, and

$$Vf(x) = \frac{\Gamma(N/2)}{2(N-2)\pi^{N/2}}\int_{R^N} \frac{f(y)}{|y - x|^{N-2}}\,dy \tag{5.14}$$

for $f \in C_K$. We do not know the complete description of $\mathscr{D}(V)$ for $N \geqq 2$, but we have the following partial result for $N = 2$. Let $\mathscr{M}$ be the collection of $f(x)$ in $C_0(R^2)$ which depends only on $|x|$. Let B_a be the closed disc with radius a centered at the origin. Then, $f \in \mathscr{D}(V)$ if and only if

$$\lim_{a\to\infty}\left[-\int_{B_a} f(y)\log|y|\,dy + (\log a)\int_{B_a} f(y)\,dy\right] \tag{5.15}$$

exists and is finite, provided that $f \in \mathscr{M}$. If $f \in \mathscr{D}(V) \cap \mathscr{M}$, then

$$Vf(x) = \lim_{a\to\infty}\frac{1}{2\pi}\left[-\int_{B_a}\log|y - x| f(y)\,dy + \log a\int_{B_a} f(y)\,dy\right]. \tag{5.16}$$

The relations $S(f) \subset B_a$ and $S(Vf) \subset B_a$ are equivalent, provided that $f \in \mathscr{D}(V) \cap \mathscr{M}$. The proof, which makes use of Green functions for discs, is omitted. In any dimension, $C_K \cap \mathscr{D}(V)$ is a core of V by Theorem 4.5.

In the sequel, we denote by $\mathscr{M}_0$ the collection of functions in $C_K(R^1)$ with integral null.

EXAMPLE 5.2. *Brownian motion with drift.* Suppose $A \in \mathbf{G}_1$ and $Au = u'' + bu'$, $b \neq 0$, for $u \in C_K^\infty$. The process is transient. Suppose $b > 0$ for simplicity. Then, it is easy to see that $C_K \cap \mathscr{D}(V) = \mathscr{M}_0$, that $\mathscr{M}_0$ is a core of V, and that

$$Vf(x) = \frac{1}{b}\left[\int_{-\infty}^{x} f(y) \exp\{b(y - x)\}\, dy + \int_{x}^{\infty} f(y)\, dy\right] \tag{5.17}$$

for $f \in \mathscr{M}_0$.

EXAMPLE 5.3. *Deterministic motion.* Suppose $A \in \mathbf{G}_1$ and $Au = bu'$, $b \neq 0$, for $u \in C_K^\infty$. It is trivial that $C_K \cap \mathscr{D}(V) = \mathscr{M}_0$, which is a core of V, and that

$$Vf(x) = \frac{1}{b}\int_{x}^{\infty} f(y)\, dy \tag{5.18}$$

for $f \in \mathscr{M}_0$, provided that $b > 0$.

EXAMPLE 5.4. *Stable processes.* Let X_t be a one dimensional stable process with index α. Excluding deterministic motions and normalizing time scale, we have the characteristic function $E \exp\{i\zeta X_t\}$ of the form

$$\exp\{-t\zeta^2\}, \tag{5.19}$$

if $\alpha = 2$,

$$\exp\left\{-t|\zeta|^\alpha\left(1 + i\beta \tan\frac{\pi\alpha}{2} \operatorname{sgn} \zeta\right)\right\}, \qquad -1 \leqq \beta \leqq 1, \tag{5.20}$$

if $0 < \alpha < 1$ or $1 < \alpha < 2$, and

$$\exp\{-t|\zeta|(1 - i\gamma \operatorname{sgn} \zeta)\}, \qquad -\infty < \gamma < \infty, \tag{5.21}$$

if $\alpha = 1$. Equation (5.19) is the Brownian motion examined above. The process is recurrent if $1 \leqq \alpha \leqq 2$, and transient if $0 < \alpha < 1$.

If $1 < \alpha < 2$, *then* $C_K \cap \mathscr{D}(V) = \mathscr{M}_0$, $\mathscr{M}_0$ *is a core of* V, *and*

$$Vf(x) = \int_R k(y - x) f(y)\, dy, \qquad f \in C_K \cap \mathscr{D}(V), \tag{5.22}$$

$$k(x) = \frac{|x|^{\alpha - 1}(1 - \beta \operatorname{sgn} x)}{2(1 + h^2)\Gamma(\alpha) \cos\frac{\pi\alpha}{2}}, \qquad h = \beta \tan\frac{\pi\alpha}{2}. \tag{5.23}$$

Indeed, the fact $C_K \cap \mathscr{D}(V) = \mathscr{M}_0$ and the expressions (5.22), (5.23) are proved by Port [9]. His proof can be simplified by systematic use of Theorem 2.4 (ii). The proof that $\mathscr{M}_0$ is a core of V is as follows. First, note that if $u \in C_K^\infty$, then $Au(x) = O(|x|^{-\alpha-1})$ as $|x| \to \infty$, and hence Au has integral null by Theorem 4.4, because the Lévy measure has density $c_1 x^{-\alpha-1}$ for $x > 0$ and $c_2|x|^{-\alpha-1}$ for $x < 0$ with some nonnegative constants c_1, c_2. Therefore, since $\{Au; u \in C_K^\infty\}$ is a core of V, it suffices to show that for each function $f \in C_0$ with integral null satisfying

$f(x) = O(|x|^{-\alpha-1})$ as $|x| \to \infty$ there exists a sequence $\{f_n\}$ in $\mathscr{M}_0$ such that $f_n \to f$ weakly and $Vf_n \to u$ weakly for some $u \in C_0$. Let

$$u(x) = \int k(y - x)f(y)\,dy = \int [k(y - x) - k(-x)]f(y)\,dy. \tag{5.24}$$

Since there is an estimate

$$|k(y - x) - k(-x)| \leqq \text{const}\,|y|^{\alpha-1} \tag{5.25}$$

([9], p. 146) and since $k(y - x) - k(x)$ tends to 0 as $|x| \to \infty$, u belongs to C_0. Choose a function $g \in C_K^+$ such that $g(x) = 1$ on $[-1, 1]$, and let $f_n(x) = (a_n + f(x))g(x/n)$, choosing a_n in such a way that f_n has integral null. It follows that $a_n = O(n^{-\alpha-1})$. Hence, $\|f_n - f\| \to 0$ as $n \to \infty$ and $f_n(x) = O(|x|^{-\alpha-1})$ uniformly in n as $|x| \to \infty$. Using (5.22) for Vf_n, we see that $\|Vf_n - u\| \to 0$, completing the proof.

If $\alpha = 1$, then we have similarly $C_K \cap \mathscr{D}(V) = \mathscr{M}_0$ and (5.22) *with*

$$k(x) = \frac{-\log|x| + 2^{-1}\pi\gamma\,\operatorname{sgn} x}{\pi(1 + \gamma^2)} \tag{5.26}$$

and $\mathscr{M}_0$ is a core of V.

To prove this, let $p(t, x)$ be the distribution density (Cauchy) of X_t and let

$$q(t, x) = \int_0^t \big(p(s, x) - p(s, 1)\big)\,ds. \tag{5.27}$$

We have

$$q(t, x) = \frac{1}{2\pi(1 + \gamma^2)}\left(\log\frac{t^2 + (x - \gamma t)^2}{t^2 + (1 - \gamma t)^2} - 2\log|x|\right) + g(t, x), \tag{5.28}$$

$$\lim_{t\to\infty} q(t, x) = k(x) - \gamma[2(1 + \gamma^2)]^{-1} \tag{5.29}$$

with $k(x)$ defined by (5.26) and $g(t, x)$ a bounded function. Hence, we have estimate

$$|q(t, x)| \leqq c_1\big|\log|x|\big| + c_2, \tag{5.30}$$

$$|q(t, x + y) - q(t, x)| \leqq c_3\big|\log|1 + y/x|\big| + c_4 \tag{5.31}$$

for $x \neq 0, -y$ with c_i independent of x, y, t. Let $f \in \mathscr{M}_0$ and let u be defined by (5.24). Then we see that $u \in C_0$. Using (5.30) and (5.31), we can prove that

$$\begin{aligned}\int_0^t T_s f(x)\,ds &= \int_R q(t, y - x)f(y)\,dy \\ &= \int_R \big(q(t, y - x) - q(t, -x)\big)f(y)\,dt,\end{aligned} \tag{5.32}$$

and that (5.32) is bounded in t and x and tends pointwise to $u(x)$ as $t \to \infty$. Hence, $f \in \mathscr{D}(V)$ and $Vf = u$ by Theorem 2.4 (ii). Conversely, if $f \in C_K \cap \mathscr{D}(V)$, then $\int_0^t T_s f(x)\, ds$ is bounded by the same theorem, and hence f must have integral null because

$$\int_0^t T_s f(0)\, ds = \int q(t, y) f(y)\, dy + \int f(y)\, dy \int_0^t p(s, 1)\, ds. \tag{5.33}$$

The fact that $\mathscr{M}_0$ is a core of V is proved by the same idea as in case $1 < \alpha < 2$. Details are omitted.

If $0 < \alpha < 1$, *then* $C_K \subset \mathscr{D}(V)$, C_K *is a core of* V, *and we have* (5.22) *with* $k(x)$ *defined by* (5.23).

In fact, we have

$$\int_0^\infty p(t, x)\, dt = k(x), \qquad x \neq 0 \tag{5.34}$$

for distribution density $p(t, x)$ of X_t and so $\int_0^t T_s f\, ds$ converges weakly to the right side of (5.22) as $t \to \infty$ if $f \in C_K$. Thus, we can apply Theorem 2.4 (ii). Finally, C_K is a core by Theorem 3.4.

EXAMPLE 5.5. *Poisson process.* If $Au(x) = u(x + \ell) - u(x)$ with $\ell \neq 0$, then

$$Vf(x) = \sum_{k=0}^{\infty} f(x + k\ell), \qquad f \in C_K \cap \mathscr{D}(V), \tag{5.35}$$

and $C_K \cap \mathscr{D}(V)$ is the collection of $f \in C_K$ such that $\int f(x)\varphi(x)\, dx = 0$ for all $\varphi \in \Phi$, where Φ is the set of continuous periodic functions with period ℓ.

EXAMPLE 5.6. *Symmetrized Poisson process.* If $Au(x) = u(x + \ell) + u(x - \ell) - 2u(x)$ with $\ell \neq 0$, then

$$Vf(x) = -\frac{1}{2} \sum_{k=-\infty}^{\infty} |k| f(x + k\ell), \qquad f \in C_K \cap \mathscr{D}(V), \tag{5.36}$$

and $C_K \cap \mathscr{D}(V)$ is the set of $f \in C_K$ such that

$$\int f(x)\varphi(x)\, dx = \int f(x) x \varphi(x)\, dx = 0 \tag{5.37}$$

for all $\varphi \in \Phi$, where Φ is the same as in 5.5.

EXAMPLE 5.7. *Brownian motion on* $[0, \infty)$ *with reflecting boundary condition.*

This is null recurrent and the potential operator V in $C_0([0, \infty))$ is

$$Vf(x) = -\frac{1}{2} \int_0^\infty (|y - x| + |y + x|) f(y)\, dy \tag{5.38}$$

for $f \in C_K([0, \infty)) \cap \mathscr{D}(V)$. The function $f \in C_K([0, \infty))$ belongs to $\mathscr{D}(V)$ if and only if f has null integral. No other condition appears.

EXAMPLE 5.8. *Brownian motion with reflecting boundary condition on a strip* $S = R^1 \times [0, 1]$. This is also null recurrent. Kimio Kazi [6a] found the potential operator V for this process. If $f \in C_K \cap \mathscr{D}(V)$, then

$$(5.39)\qquad Vf(x_1, x_2) = \int_S k(x_1, x_2; y_1, y_2) f(y_1, y_2)\, dy_1 dy_2,$$

$$(5.40)\qquad \begin{aligned} &k(x_1, x_2; y_1, y_2) \\ &\quad = -2^{-1}|y_1 - x_1| + k_1(x_1, x_2; y_1, y_2) + k_1(x_1, -x_2; y_1, y_2), \end{aligned}$$

$$(5.41)\qquad \begin{aligned} &k_1(x_1, x_2; y_1, y_2) \\ &\quad = -(4\pi)^{-1} \log\big(1 - 2\exp\{-\pi|y_1 - x_1|\}\cdot\cos \pi|y_2 - x_2| \\ &\qquad\qquad + \exp\{-2\pi|y_1 - x_1|\}\big). \end{aligned}$$

A function $f \in C_K(S)$ belongs to $\mathscr{D}(V)$ if and only if

$$(5.42)\qquad \int_S f(x_1, x_2)\, dx_1 dx_2 = \int_S x_1 f(x_1, x_2)\, dx_1 dx_2 = 0.$$

The kernel k is approximately equal to that of the one dimensional Brownian motion if $|y_1 - x_1|$ is large, but has logarithmic singularity when (x_1, x_2) and (y_1, y_2) are close. We do not know such a nice expression of potential kernel for the similar process on $R^2 \times [0, 1]$.

6. Some properties of V and V^*

Let $\mathscr{B}$ be a Banach lattice (see [1] or [16] for definition). The symbols f^+, f^-, and $|f|$ mean $f \vee 0$, $-(f \wedge 0)$, and $f \vee (-f)$, respectively. The adjoint space $\mathscr{B}^*$ becomes a Banach lattice in the naturally induced order where $\varphi \leqq \psi$ means that $(\varphi, f) \leqq (\psi, f)$ for all $f \in \mathscr{B}$ such that $f \geqq 0$ ([1], p. 368). We call $\{T_t\}$ an *M semigroup* on $\mathscr{B}$ if it is a strongly continuous semigroup of positive linear operators with norm $\leqq 1$. We will show characteristic properties of potential operators of M semigroups and their adjoint operators.

We use a functional on $\mathscr{B} \times \mathscr{B}$ defined by

$$(6.1)\qquad \rho(f, g) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}(\|(f + \varepsilon g)^+\| - \|f^+\|).$$

This functional is introduced in [12] with notation $\varphi_0(f, g)$, and shown to have several nice properties. We define $\rho(\varphi, \psi)$ on $\mathscr{B}^* \times \mathscr{B}^*$ in the same way.

THEOREM 6.1. *Let $\{T_t\}$ be a strongly continuous semigroup on $\mathscr{B}$ satisfying condition* (1.1) *and admitting a potential operator V. Then, the following conditions are equivalent:*

(a) *T_t is an M semigroup;*
(b) *$\rho(Vf, f) \geqq 0$ for all $f \in \mathscr{D}(V)$;*
(c) *$\|(Vf)^+\| \leqq \|(Vf + \varepsilon f)^+\|$ for all $f \in \mathscr{D}(V)$ and $\varepsilon > 0$;*

(d) $\rho(V^*\varphi, \varphi) \geqq 0$ *for all* $\varphi \in \mathscr{D}(V^*)$;

(e) $\|(V^*\varphi)^+\| \leqq \|(V^*\varphi + \varepsilon\varphi)^+\|$ *for all* $\varphi \in \mathscr{D}(V^*)$ *and* $\varepsilon > 0$.

PROOF. The family $\{T_t\}$ is an M semigroup if and only if J_λ is positive and has norm $\leqq \lambda^{-1}$ for all $\lambda > 0$. Hence, the equivalence of (a), (b), (c) is proved in [13]. Likewise, both (d) and (e) are equivalent to the fact that J_λ^* is positive with norm $\leqq \lambda^{-1}$ for all $\lambda > 0$. The last property is equivalent to the same property for J_λ.

If we rewrite the above conditions by using the infinitesimal generator A and its adjoint A^* instead of V and V^*, then the theorem is true even in case $\{T_t\}$ does not admit a potential operator.

In the case $\mathscr{B} = C_0(S)$, we have

$$\rho(f, g) = \max_{x \in K(f^+)} g(x), \qquad f^+ \neq 0, \tag{6.2}$$

where $K(f^+) = \{x; f^+(x) = \|f^+\|\}$. (If $f^+ = 0$, then $\rho(f, g) \geqq 0$ for all g by definition (6.1). Hence, the inequality in (b) is trivial if $Vf \leqq 0$. Similar remarks apply to the other conditions.) Therefore, condition (b) in $C_0(S)$ is the weak principle of positive maximum studied by Hunt [4]. We can prove

$$\rho(\varphi, \psi) = \psi_\varphi^c(S_\varphi^+) + \|(\psi_\varphi^s)^+\| \tag{6.3}$$

for all signed measures $\varphi, \psi \in C_0^*(S)$, where ψ_φ^c and ψ_φ^s are, respectively, the absolutely continuous part and the singular part of ψ with respect to $|\varphi|$, and S_φ^+ is the positive set in the Hahn decomposition of S relative to φ. Thus, condition (d) seems to be a new kind of maximum principle for adjoint potential operators.

In order that a given operator V be the potential operator of an M semigroup, some properties of V^* are decisive, as Yosida [20] suggests. Let us define

$$\tau(\varphi, \psi) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}(\|\varphi + \varepsilon\psi\| - \|\varphi\|). \tag{6.4}$$

The following theorem is a consequence of [20] and Theorem 6.1.

THEOREM 6.2. *Let V be a closed linear operator satisfying condition* (b) *of Theorem* 6.1 *with domain and range both dense in* $\mathscr{B}$. *Then the following are equivalent:*

(a) *there exists an M semigroup with potential* V;

(b) V^* *satisfies condition* (d) *of Theorem* 6.1;

(c) V^* *satisfies condition* (e) *of Theorem* 6.1;

(d) $\tau(V^*\varphi, \varphi) \geqq 0$ *for all* $\varphi \in \mathscr{D}(V^*)$;

(e) $\|V^*\varphi\| \leqq \|V^*\varphi + \varepsilon\varphi\|$ *for all* $\varphi \in \mathscr{D}(V^*)$ *and* $\varepsilon > 0$;

(*f*) $V^* + \varepsilon$ *is one to one for all* $\varepsilon > 0$.

Actually, (d) and (e) are equivalent in general [12]. If we replace *closed* in this theorem by *one to one*, V is closable ([12]) and the theorem remains true with V in (a) replaced by the smallest closed extension of V. In the case $\mathscr{B}^* =$

$C_0^*(S)$, we can prove

$$\tau(\varphi, \psi) = \psi_\varphi^c(S_\varphi^+) - \psi_\varphi^c(S_\varphi^-) + \|\psi_\varphi^s\|. \tag{6.5}$$

So, (d) is another kind of maximum principle for V^*.

If $\mathscr{B}$ is a sublattice of a vector lattice $\tilde{\mathscr{B}}$ and we require that T_t preserves e majorization (that is, $f \leqq e$ implies $T_t f \leqq e$) for a fixed $e \in \tilde{\mathscr{B}}$, then a property which replaces condition (b) of Theorem 6.1 is investigated in [8] and [12].

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