LOGARITHMIC POTENTIALS AND PLANAR BROWNIAN MOTION

SIDNEY C. PORT and CHARLES J. STONE
UNIVERSITY OF CALIFORNIA, LOS ANGELES

In this paper we continue our discussion of the connection between potential theory and Brownian motion begun in "Classical Potential Theory and Brownian Motion" that also appears in this Symposium volume. Throughout this paper, we will be dealing with a two dimensional Brownian motion process. We will continue numbering the sections from where we left off in the previous paper.

8. Planar Brownian motion

In Section 5, we saw that for a Brownian motion process in $n \geqq 3$ dimensions, $P_x(\lim_{t\to\infty} |X_t| = \infty) = 1$ for all x. In sharp contrast to this situation, a planar Brownian motion is certain to hit any nonpolar set.

THEOREM 8.1. *Let B be a Borel set. Then $P_x(V_B < \infty)$ is either identically* 1 *or identically* 0.

PROOF. A simple computation shows that for any $x \in R^2$, $\int_0^t p(s, x)\, ds \uparrow \infty$ as $t \uparrow \infty$. Thus, for any nonnegative function f having nonzero integral,

$$\lim_{t\to\infty} \int_0^t P^s f(x)\, ds = \infty. \tag{8.1}$$

Let $\varphi(x) = P_x(V_B < \infty)$. Then for any $h > 0$,

$$0 \leqq \int_0^t P^s(\varphi - P^h\varphi)\, ds = \int_0^h P^s\varphi\, ds - \int_t^{t+h} P^s\varphi\, ds \leqq 2h. \tag{8.2}$$

Letting $t \uparrow \infty$, we see that

$$0 \leqq \int_0^\infty P^s(\varphi - P^h\varphi)\, ds \leqq 2h. \tag{8.3}$$

But then it must be that $\varphi = P^h\varphi$ a.e. Since $P^t\varphi \uparrow \varphi$ as $t \downarrow 0$ and $P^t(P^h\varphi) \uparrow P^h\varphi$ as $t \downarrow 0$, it follows that $\varphi(x) = P^h\varphi(x)$ for all x. Using Proposition 2.3, we see that $\varphi(x) \equiv \alpha$ for some constant α. Now

$$P_x(t < V_B < \infty) = \int_{R^2} q_B(t, x, y)\varphi(y)\, dy = \alpha P_x(V_B > t). \tag{8.4}$$

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Letting $t \uparrow \infty$, we see that $\alpha P_x(V_B = \infty) = 0$. Thus, either $P_x(V_B = \infty) \equiv 0$ or $\alpha = 0$. In the first case $\varphi(x) \equiv 1$, while in the second case $\varphi(x) \equiv 0$. This establishes the proposition.

The difference between planar Brownian motion and Brownian motion in $n \geqq 3$ dimensions has its analytical counterpart in potential theory. We will now show that the potentials associated with planar Brownian motion are logarithmic potentials.

Let 1 denote the point $(1, 0)$ and let $a^\lambda(x) = g^\lambda(1) - g^\lambda(x)$. Using (2.30), we see that for $x \neq y$,

$$a^\lambda(y - x) = \int_{\bar{B}} \Pi_B^\lambda(x, dz)a^\lambda(y - z) - g_B^\lambda(x, y) + L_B^\lambda(x), \tag{8.5}$$

where

$$L_B^\lambda(x) = g^\lambda(1)[1 - E_x(\exp\{-\lambda V_B\})]. \tag{8.6}$$

Now

$$a^\lambda(x) = \int_0^\infty e^{-\lambda t}[p(t, 1) - p(t, x)]\, dt. \tag{8.7}$$

If $|x| \geqq 1$, then $p(t, 1) - p(t, x) \geqq 0$; so for $|x| \geqq 1$, $a^\lambda(x)$ is increasing. On the other hand, for $|x| < 1$, $p(t, 1) - p(t, x) < 0$ so $-a^\lambda(x)$ is increasing. In either case

$$\lim_{\lambda \downarrow 0} a^\lambda(x) = \int_0^\infty [p(t, 1) - p(t, x)]\, dt = \frac{1}{\pi} \log |x|, \tag{8.8}$$

and the convergence is uniform on any compact set not containing 0. For simplicity, we set $a(x) = (1/\pi) \log |x|$.

Our principle result in this section will be to establish the following theorem.

THEOREM 8.2. *Let B be a nonpolar set. Then $g_B(x, y) < \infty$ for $x \neq y$ and $\lim_{\lambda \downarrow 0} L_B^\lambda(x) = L_B(x)$ exists and is finite for all x. Moreover, for $x \neq y$,*

$$a(y - x) - \int_{\bar{B}} \Pi_B(x, dz)a(y - z) = -g_B(x, y) + L_B(x). \tag{8.9}$$

Before getting on with the proof, we observe first if $E_x V_B < \infty$ for all x then $L_B(x) \equiv 0$. Indeed,

$$L_B^\lambda(x) = \lambda g^\lambda(1) \left[\frac{1 - E_x(\exp\{-\lambda V_B\})}{\lambda}\right]; \tag{8.10}$$

so if $E_x V_B < \infty$, then the expression in the brackets converges to $E_x V_B < \infty$, while $\lambda g^\lambda(1) \to 0$ as $\lambda \to 0$. In particular by Proposition 2.2, $L_B(x) \equiv 0$ whenever B^c is relatively compact.

The proof of Theorem 8.2 is long and will be divided into several lemmas.

LEMMA 8.1. *Suppose B is relatively compact. Then*

$$\lim_{\lambda \downarrow 0} \int_{\bar{B}} \Pi_B^\lambda(x, dz)a^\lambda(y - z) = \int_{\bar{B}} \Pi_B(x, dz)a(y - z). \tag{8.11}$$

PROOF. If B is polar, there is nothing to prove since both Π_B^λ and Π_B are the zero measure, so suppose B is nonpolar. If $y \notin \bar{B}$, then as $\lambda \downarrow 0$, $a^\lambda(y - z)$ converges to $a(y - z)$ uniformly in $z \in \bar{B}$, and thus (8.11) holds in this case. Suppose $y \in \bar{B}$. Let D_ε be the open disk of center y and radius $\varepsilon < 1$. We can write

$$(8.12)\qquad \int_{\bar{B}} \Pi_B^\lambda(x, dz)a^\lambda(y - z) = \int_{\bar{B}\cap D_\varepsilon^c} \Pi_B^\lambda(x, dz)a^\lambda(y - z) + \int_{\bar{B}\cap D_\varepsilon} \Pi_B^\lambda(x, dz)a^\lambda(y - z).$$

Since

$$(8.13)\qquad \int_{\bar{B}\cap D_\varepsilon^c} \Pi_B^\lambda(x, dz)a^\lambda(y - z) = E_x[\exp\{-\lambda V_B\}\, a^\lambda(y - X_{V_B})1_{D_\varepsilon^c}(X_{V_B})]$$

and $a^\lambda(y - z)$ converges to $a(y - z)$ uniformly for $z \in \bar{B} \cap D_\varepsilon^c$, we see that

$$(8.14)\qquad \lim_{\lambda\downarrow 0} \int_{\bar{B}\cap D_\varepsilon^c} \Pi_B^\lambda(x, dz)a^\lambda(y - z) = E_x[a(y - X_{V_B})1_{D_\varepsilon^c}(X_{V_B})].$$

On the other hand,

$$(8.15)\qquad -\int_{\bar{B}\cap D_\varepsilon} \Pi_B^\lambda(x, dz)a^\lambda(y - z) = E_x[\exp\{-\lambda V_B\}[-a^\lambda(y - X_{V_B})]1_{D_\varepsilon}(X_{V_B}).$$

Since

$$(8.16)\qquad -\exp\{-\lambda V_B\}a^\lambda(y - X_{V_B})1_{D_\varepsilon}(X_{V_B}) \uparrow -a(y - X_{V_B})1_{D\varepsilon}(X_{V_B}),$$

monotone convergence shows that

$$(8.17)\qquad \lim_{\lambda\downarrow 0} \int_{\bar{B}\cap D_\varepsilon} \Pi_B^\lambda(x, dz)a^\lambda(y - z) = E_x[a(y - X_{V_B})1_{D\varepsilon}(X_{V_B})] \geqq -\infty.$$

As

$$(8.18)\qquad -\infty < E_x[a(y - X_{V_B})1_{D_\varepsilon^c}(X_{V_B}) < +\infty,$$

we see from (8.14) and (8.17) that (8.11) holds. This establishes the lemma.

LEMMA 8.2. *Suppose B is a nonpolar relatively compact set. Let A be a compact set of positive measure such that $A \cap \bar{B} = \varnothing$. Then*

$$(8.19)\qquad \lim_{\lambda\downarrow 0} \int_A g_B^\lambda(x, y)\, dy = E_x \int_0^{V_B} 1_A(X_t)\, dt < \infty.$$

PROOF. For a given x_0 there is a $t_0 > 1$ such that

$$(8.20)\qquad P_{x_0}(X_s \in B \text{ for some } s \in (1, t_0)) = \int_{R^2} p(1, y - x_0)P_y(V_B \leqq t_0 - 1)\, dy > 0,$$

since otherwise

$$\int_{R^n} p(1, y - x_0) P_y(V_B < \infty)\, dy = P_{x_0}(V_B < \infty) = 0. \tag{8.21}$$

Thus, there must be a compact set F of positive measure such that $P_y(V_B \leqq t_0 - 1) > 0$ for all $y \in F$. As $p(1, x)$ is a strictly positive continuous function, it follows that

$$\inf_{x \in A} P_x(V_B \leqq t_0) \geqq \inf_{x \in A} \int_F p(1, y - x) P_y(V_B \leqq t_0 - 1) = \delta > 0. \tag{8.22}$$

Let $I_j = [jt_0, (j + 1)t_0)$ and let $C = \{t: X_t \in A, V_B > t\}$. Define the index set Γ by $j \in \Gamma$ if and only if $I_j \cap C \neq \varnothing$, and enumerate Γ increasing order by $j_1 < j_2 < \cdots$. Define the times $T_1 < T_2 < \cdots$ as follows:

$$T_1 = \inf\{t: t \in C\} \quad (= \infty \text{ if there is no such } t) \tag{8.23}$$

and

$$T_{n+1} = \inf\{t: t \in C \cap [j_n t_0, \infty\} \quad (= \infty \text{ if there is no such } t). \tag{8.24}$$

Let $N \leqq \infty$ denote the number of indices in Γ. Then

$$\begin{aligned} P_x(N > n, N \leqq n + 2) &= P_x(T_n < \infty, T_{n+2} = \infty) \\ &\geqq P_x(T_n < \infty, V_B \leqq T_n + t_0) \\ &= \int_A P_x(T_n < \infty, X_{T_n} \in dz) P_z(V_B \leqq t_0) \\ &\geqq \delta P_x(T_n < \infty) = \delta P_x(N > n), \end{aligned} \tag{8.25}$$

so $P_x(N > n + 2) \leqq (1 - \delta) P_x(N > n)$. Thus, $E_x N < \infty$, and hence

$$E_x \int_0^{V_B} 1_A(X_t)\, dt = E_x|C| \leqq E_x \Big| \bigcup_{j \in \Gamma} I_j \Big| = t_0 E_x N < \infty, \tag{8.26}$$

as desired. This establishes Lemma 8.2.

We can now prove Theorem 8.2 when B is a relatively compact set.

LEMMA 8.3. *Suppose B is a nonplanar relatively compact set. Then Theorem 8.2 holds.*

PROOF. Let A be as in Lemma 8.2. Since

$$\int_A a^\lambda(y - x)\, dy \to \int_A a(y - x)\, dy \tag{8.27}$$

uniformly in x on compacts, we see that

$$\lim_{\lambda \downarrow 0} \int_{\bar{B}} \Pi_B^\lambda(x, dz) \int_A a^\lambda(y - z)\, dy = \int_{\bar{B}} \Pi_B(x, dz) \int_A a(y - z)\, dy. \tag{8.28}$$

By Lemma 8.2,

$$\int_A g_B^\lambda(x, y)\, dy \uparrow \int_A g_B(x, y)\, dy < \infty. \tag{8.29}$$

By (8.5)

$$|A| L_B^\lambda(x) = \int_A a^\lambda(y - x)\, dy - \int_{\bar{B}} \Pi_B^\lambda(x, dz) \int_A a^\lambda(y - z)\, dy + \int_A g_B^\lambda(x, y)\, dy. \tag{8.30}$$

Since the right side has a finite limit as $\lambda \downarrow 0$, we see that $L_B^\lambda(x)$ must have a finite limit as $\lambda \downarrow 0$. Call this limit function $L_B(x)$. By (8.5),

$$-g_B^\lambda(x, y) = a^\lambda(y - x) - \int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) - L_B^\lambda(x), \tag{8.31}$$

and thus for $x \neq y$,

$$\lim_{\lambda \downarrow 0} \left[-g_B^\lambda(x, y) + \int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) \right] = a(y - x) - L_B(x) \tag{8.32}$$

is finite. Now $0 \leqq g_B^\lambda(x, y) \uparrow g_B(x, y) \leqq +\infty$ and by Lemma 8.1

$$\lim_{\lambda \downarrow 0} \int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) = \int_{\bar{B}} \Pi_B(x, dz) a(y - z), \tag{8.33}$$

and

$$-\infty \leqq \int_{\bar{B}} \Pi_B(x, dz) a(y - z) < +\infty. \tag{8.34}$$

Thus for $x \neq y$, we see that $g_B(x, y) < +\infty$, $\int_{\bar{B}} \Pi_B(x, dz) a(y - z) > -\infty$, and that (8.9) holds.

To handle the unbounded case we need two additional lemmas.

LEMMA 8.4. *Suppose $A \subset B$. Then $g_A(x, y) \geqq g_B(x, y)$ for all x, y.*

PROOF. Since $A \subset B$, $V_A \geqq V_B$ and thus for any Borel set F,

$$P_x(V_A > t, X_t \in F) \geqq P_x(V_B > t, X_t \in F). \tag{8.35}$$

Hence (in the notation of Section 2) for a.e. y,

$$q_A(t, x, y) \geqq q_B(t, x, y). \tag{8.36}$$

But then

$$\int_{R^2} q_A(t - \varepsilon, x, z) p(\varepsilon, y - z)\, dz \geqq \int_{R^2} q_B(t - \varepsilon, x, z) p(\varepsilon, y - z)\, dz, \tag{8.37}$$

and thus letting $\varepsilon \downarrow 0$ (see Section 2), it follows that (8.36) holds for all y.

Integrating on t, we see that

$$g_A(x, y) = \int_0^\infty q_A(t, x, y)\, dt \geqq \int_0^\infty q_B(t, x, y)\, dt = g_B(x, y) \tag{8.38}$$

as desired.

LEMMA 8.5. *Let B be any Borel set. Then for $x \neq y$,*

$$\lim_{\lambda \downarrow 0} \int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) = \int_{\bar{B}} \Pi_B(x, dz) a(y - z) > -\infty. \tag{8.39}$$

PROOF. If B is polar, there is nothing to prove so suppose B is nonpolar. Suppose $y \in (\bar{B})^c$. Let D_r be a disk of center 0 and radius $r > 1 + |y|$. Then $|y - z| > 1$ for $z \in D_r^c$. Hence, by monotone convergence

$$\begin{aligned} \int_{\bar{B} \cap D_r^c} \Pi_B^\lambda(x, dz) a^\lambda(y - z) &= E_x[\exp\{-\lambda V_B\} a^\lambda(y - X_{V_B}) 1_{D_r^c}(X_{V_B})] \\ &\uparrow E_x[a(y - X_{V_B}) 1_{D_r^c}(X_{V_B})] = \int_{\bar{B} \cap D_r^c} \Pi_B(x, dz) a(y - z) \geqq 0. \end{aligned} \tag{8.40}$$

On the other hand, $a^\lambda(y - z) \rightarrow a(y - z)$ uniformly in $z \in \bar{B} \cap D_r$ so

$$\lim_{\lambda \downarrow 0} \int_{\bar{B} \cap D_r} \Pi_B^\lambda(x, dz) a^\lambda(y - z) = \int_{\bar{B} \cap D_r} \Pi_B(x, dz) a(y - z), \tag{8.41}$$

and

$$-\infty < \int_{\bar{B} \cap D_r} \Pi_B(x, dz) a(y - z) < \infty. \tag{8.42}$$

Thus, (8.39) holds for $y \notin \bar{B}$. Suppose $y \in \bar{B}$ and let A_ε be the disk of center y and radius $\varepsilon < 1$. We can write

$$\begin{aligned} \int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) &= \int_{D_r \cap A_\varepsilon} \Pi_B^\lambda(x, dz) a^\lambda(y - z) \\ &\quad + \int_{D_r \cap A_\varepsilon^c} \Pi_B^\lambda(x, dz) a^\lambda(y - z) \\ &\quad + \int_{D_r^c} \Pi_B^\lambda(x, dz) a^\lambda(y - z). \end{aligned} \tag{8.43}$$

Choose r so large that $B \cap D_r$ is nonpolar and such that $|y - z| > 1$ for $z \in D_r^c$. The second term on the right of (8.43) converges to

$$-\infty < \int_{D_r \cap A_\varepsilon^c} \Pi_B(x, dz) a(y - z) < \infty, \tag{8.44}$$

as $\lambda \downarrow 0$, since $a^\lambda(y - z) \to a(y - z)$ uniformly for $z \in \bar{B} \cap D_r \cap A_\varepsilon^c$. Let $B_r = B \cap D_r$, and note that $\Pi_{B_r}(x, dz) \geqq \Pi_B(x, dz)$ for $z \in D_r$. Since $-a^\lambda(y - z) \uparrow -a(y - z)$, $z \in A_\varepsilon$, it follows from the monotone convergence theorem that

$$(8.45) \qquad \begin{aligned} -\int_{D_r \cap A_\varepsilon} & \Pi_B^\lambda(x, dz) a^\lambda(y - z) \\ &= E_x(\exp\{-\lambda V_B\}[-a^\lambda(y - X_{V_B})1_{A_\varepsilon}(X_{V_B})1_{D_r}(X_{V_B})) \\ &\qquad \uparrow E_x[-a(y - X_{V_B})1_{A_\varepsilon}(X_{V_B})1_{D_r}(X_{V_B})] \\ &= -\int_{D_r \cap A_\varepsilon} \Pi_B(x, dz) a(y - z). \end{aligned}$$

Since $a(y - z) \leqq 0$ for $z \in A_\varepsilon$, we see that

$$(8.46) \qquad \int_{D_r \cap A_\varepsilon} \Pi_B(x, dz) a(y - z) \geqq \int_{D_r \cap A_\varepsilon} \Pi_{B_r}(x, dz) a(y - z).$$

By Lemma 8.3,

$$(8.47) \qquad \int_{\bar{B}_r} \Pi_{B_r}(x, dz) a(y - z) > -\infty,$$

and it is clear that

$$(8.48) \qquad \infty > \int_{\bar{B}_r \cap (D_r \cap A_\varepsilon)^c} \Pi_{B_r}(x, dz) a(y - z) > -\infty.$$

Hence,

$$(8.49) \qquad \lim_{\lambda \downarrow 0} \int_{D_r \cap A_\varepsilon} \Pi_B^\lambda(x, dz) a^\lambda(y - z) = \int_{D_r \cap A_\varepsilon} \Pi_B(x, dz) a(y - z) > -\infty.$$

Finally, as in the case when $y \in (\bar{B})^c$, monotone convergence shows that

$$(8.50) \qquad \lim_{\lambda \downarrow 0} \int_{D_r^c} \Pi_B^\lambda(x, dz) a^\lambda(y - z) = \int_{D_r^c} \Pi_B(x, dz) a(y - z) \geqq 0.$$

Thus, using (8.43), we see that (8.39) holds for $y \in \bar{B}$. This completes the proof.

We may now easily establish Theorem 8.2.

PROOF OF THEOREM 8.2. Since B is nonpolar some relatively compact subset $A \subset B$ must be nonpolar. But then, by Lemma 8.3, $g_B(x, y) \leqq g_A(x, y) < \infty$ for $x \neq y$. Also

$$(8.51) \qquad \begin{aligned} L_B^\lambda(x) &= g^\lambda(1)[1 - E_x(\exp\{-\lambda V_B\})] \\ &\leqq g^\lambda(1)[1 - E_x(\exp\{-\lambda V_A\})] = L_A^\lambda(x), \end{aligned}$$

so

$$(8.52) \qquad \limsup_{\lambda \downarrow 0} L_B^\lambda(x) \leqq L_A(x) < \infty.$$

Using (8.5), we see that for $x \neq y$,

$$\lim_{\lambda \downarrow 0} \left[\int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) + L_B^\lambda(x) \right] = a(y - x) + g_B(x, y) \tag{8.53}$$

has a finite limit.

Using (8.52) and Lemma 8.5, we see that

$$\lim_{\lambda \downarrow 0} \int_{\bar{B}} \Pi_B^\lambda(x, dz) a^\lambda(y - z) = \int_{\bar{B}} \Pi_B(x, dz) a(y - z), \tag{8.54}$$

must be finite for $x \neq y$. Thus, it must be that $\lim_{\lambda \downarrow 0} L_B^\lambda(x) = L_B(x)$ exists and that (8.9) is satisfied. This establishes the theorem.

One of the main applications of Theorem 8.2 is to show that $g_B(x, y) < \infty$, $x \neq y$, $x, y \in B^c$, when B is a compact nonpolar set. As we will see in the next section, g_B restricted to $B^c \times B^c$ (for B a closed set) is just the Green function of B^c. Now if one is interested only in showing that $g_B(x, y) < \infty$ for $x \neq y$, $x, y \in B^c$ and in showing that (8.9) is valid for such x, y when B is a nonpolar compact set then the proof of the theorem can be considerably shortened. In fact, all one needs is Lemma 8.2 and the simple parts of Lemmas 8.1 and 8.3.

A simple consequence of Theorem 8.2 is the following result.

THEOREM 8.3. *Let B be a nonpolar Borel set. Then*

$$\lim_{t \to \infty} \log(t) P_x(V_B > t) = 2\pi L_B(x). \tag{8.55}$$

PROOF. One easily checks that

$$\lambda g^\lambda(1) \sim \frac{1}{2\pi} \lambda \log\left(\frac{1}{\lambda}\right), \qquad \lambda \downarrow 0. \tag{8.56}$$

The theorem follows from this and the fact that

$$\lim_{\lambda \downarrow 0} \lambda g^\lambda(1) \left[\frac{1 - E_x(\exp\{-\lambda V_B\})}{\lambda} \right] \equiv L_B(x) \tag{8.57}$$

exists by well-known Tauberian theorems.

Of course Theorem 8.3 is uninteresting when $E_x V_B < \infty$ since then $L_B(x) \equiv 0$. When B is relatively compact, however, $L_B(x) > 0$ for all $|x|$ sufficiently large. In Section 10, we will see that $L_B(x) = \lim_{|y| \to \infty} g_B(x, y)$ whenever B is relatively compact.

9. Green function

Now that we have Theorem 8.2 at our disposal, we can easily derive the properties of $g_B(x, y)$ for a nonpolar set B.

THEOREM 9.1. *Let B be nonpolar. Then*

(i) $0 \leqq g_B(x, y) < \infty$ *for* $x \neq y$,
(ii) $g_B(x, y) = g_B(y, x)$,

(iii) $g_B(x, y) + a(y - x)$ *is harmonic in* y *on* $(\bar{B})^c$,

(iv) $g_B(x, y)$ *is harmonic in* y *on* $(\bar{B})^c - \{x\}$,

(v) $g_B(x, y)$ *is subharmonic in* y *and upper semicontinuous in* y *on* $R^2 - \{x\}$,

(vi) $\lim_{y \to y_0} g_B(x, y) = g_B(x, y_0) = 0$, $y_0 \in B^r$.

PROOF. Part (i) is part of Theorem 8.2. Part (ii) follows from the fact that it is true for g_B^λ and letting $\lambda \downarrow 0$. Parts (iii) to (v) follow from (8.9) and the fact that $a(y - x)$ is harmonic in $y \neq x$ and that for any finite measure μ, $\int_B \log |y - x| \mu(dx)$ is an upper semicontinuous and subharmonic function that is harmonic in $(\bar{B})^c$. Finally, (vi) follows from the upper semicontinuity and the fact that $g_B^\lambda(x, y_0) = 0$ for all λ and x.

An open set G *is called Greenian if* there exists a function $g(x, y)$ on $G \times G$ such that $g(x, y) + a(y - x)$ is harmonic in y on G. If G is Greenian the smallest such function is called the Green function of G.

From our work in Section 5, we know that in dimension $n \geqq 3$ every open set is Greenian and that g_{G^c} restricted to $G \times G$ is its Green function. Based on this, and Theorem 9.1, we would fully expect that the following holds in the planar case.

THEOREM 9.2. *An open set* $G \subset R^2$ *is Greenian if and only if* G^c *is nonpolar. In that case* g_{G^c} *restricted to* $G \times G$ *is the Green function of* G.

PROOF. Suppose first that G is a bounded open set such that each point of ∂G is regular for G^c. Then clearly G^c is nonpolar so Theorem 9.1 shows that g_{G^c} restricted to $G \times G$ is Greenian. To see that it is the smallest of the Greenian functions, suppose g is another such function. Then by property (vi) of Theorem 9.1,

$$\liminf_{y \to y_0} [g(x, y) - g_{G^c}(x, y)] = \liminf_{y \to y_0} g(x, y) \geqq 0. \tag{9.1}$$

Thus, by the minimum principle, $g(x, y) - g_{G^c}(x, y) \geqq 0$ on G.

Suppose now that G is any open subset of R^2. By Corollary 3.1, we can find bounded open sets $G_1 \subset \bar{G}_1 \subset G_2 \subset \cdots$, $\bigcup_n G_n = G$, such that each point of ∂G_n is regular for G_n^c and such that $P_x(V_{\partial G_n} \uparrow V_{\partial G}) = 1$ for all $x \in G$.

By Lemma 8.4, the sequence $g_{G_n^c}$ is increasing and bounded above by $g_{G^c}(x, y)$. Since $g_{G_n^c}(x, y) + a(y - x)$ is harmonic in $y \in G$ for fixed $x \in G_n$, Harnack's theorem tells us that the limit function $g^*(x, y) \leqq g_{G^c}(x, y)$ is such that in a given component of G it is either identically infinite or a harmonic function on $G - \{x\}$. Let $f \geqq 0$ be arbitrary. Then for any $x \in G$,

$$E_x \int_0^{V_{\partial G_n}} f(X_t)\, dt = E_x \int_0^{V_{G_n^c}} f(X_t)\, dt \uparrow E_x \int_0^{V_{G^c}} f(X_t)\, dt, \qquad n \uparrow \infty. \tag{9.2}$$

But

$$E_x \int_0^{V_{G_n^c}} f(X_t)\, dt = \int_{R^2} g_{G_n^c}(x, y) f(y)\, dy, \tag{9.3}$$

so monotone convergence gives that

$$\int_{R^2} g_{G_n^c}(x, y)f(y)\,dy \uparrow \int_{R^2} g^*(x, y)f(y)\,dy, \tag{9.4}$$

as $n \uparrow \infty$. Thus,

$$\int_{R^2} g^*(x, y)f(y)\,dy = \int_{R^2} g_{G^c}(x, y)f(y)\,dy. \tag{9.5}$$

Suppose G^c is nonpolar. Then as $g^*(x, y) \leqq g_{G^c}(x, y)$, $g^*(x, y) < \infty$ for $x \neq y$, and thus $g^*(x, y) + a(y - x)$ is harmonic in y on G. Since the function f in (9.5) is any nonnegative function, (9.5) implies that $g^*(x, y) = g_{G^c}(x, y)$ a.e. y. Since $g_{G^c}(x, y) + a(y - x)$ is harmonic in y, we see that $g^* = g_{G^c}$ on $G \times G$. Suppose g is any function having the required properties. Then this function restricted to G_n also has the required properties so by what has already been proved, $g(x, y) \geqq g_{G_n^c}(x, y)$, $x, y \in G_n$. Thus,

$$g_{G^c}(x, y) = \lim_n g_{G_n^c}(x, y) \leqq g(x, y). \tag{9.6}$$

Hence, g_{G^c} restricted to $G \times G$ is the Green function of G. Finally, suppose G^c is polar. Then G cannot be Greenian. Indeed, if it were Greenian, then let g be a function with the required properties. But then, as argued above

$$g^*(x, y) \leqq g(x, y) < \infty, \qquad x \neq y. \tag{9.7}$$

But (9.4) shows that $g^*(x, y) = \infty$ for a.e. y, a contradiction. This completes the proof.

10. Logarithmic potentials

Let μ be a bounded measure having compact support k. The function $\varphi_\mu(x) = -\int_k a(y - x)\mu(dy)$ is called *the potential* of μ. One easily verifies that $\varphi_\mu(x)$ is a lower semicontinuous function that is superharmonic on R^2 and harmonic on k^c.

Let B be a nonpolar relatively compact set. Since $a(y - x) - a(y) \to 0$ as $|y| \to \infty$, uniformly in x on compacts, it follows at once from (8.9) that, uniformly in x on compacts,

$$\lim_{|y| \to \infty} g_B(x, y) = L_B(x). \tag{10.1}$$

As $g_B(x, y) = g_B(y, x)$, we see that

$$\lim_{|x| \to \infty} g_B(x, y) = L_B(y). \tag{10.2}$$

Let D_r be the closed disk of center 0 and radius r. The hitting measure, $\Pi_{D_r}(x, dy)$, is just the unit mass at x for $x \in D_r$. For $x \in D_r^c$ and φ a continuous function on ∂D_r, $\Pi_{D_r}\varphi(x) = \Pi_{\partial D_r}\varphi(x)$ is the unique bounded solution to the

Dirichlet problem for D_r^c with boundary function φ. As is well known from texts on complex variables or partial differential equations, the solution to this Dirichlet problem is provided by the Poisson integral. Thus,

$$\Pi_{D_r}(x, dy) = \frac{[|x|^2 - r^2]}{|y - x|^2}\,\sigma_r(0, dy), \qquad x \in D_r^c. \tag{10.3}$$

It follows from (10.3) that

$$\lim_{|x|\to\infty} \Pi_{D_r}(x, dy) = \sigma_r(0, dy), \tag{10.4}$$

in the sense of strong convergence of measures.

Let B be any nonpolar relatively compact set and let D_r be a disk of center 0 and radius r that contains $\bar{B}$ in its interior. Then for any bounded function φ

$$\Pi_B\varphi(x) = \int_{\partial D_r} \Pi_{D_r}(x, dz)\Pi_B\varphi(z), \qquad x \in D_r^c. \tag{10.5}$$

Hence,

$$\lim_{|x|\to\infty} \Pi_B\varphi(x) = \int_{\partial D_r} \Pi_B\varphi(z)\sigma_r(0, dz). \tag{10.6}$$

Let

$$\mu_B(dy) = \int_{\partial D_r} \Pi_B(z, dy)\sigma_r(0, dz). \tag{10.7}$$

Equation (10.6) shows that $\mu_B(dy) = \lim_{|x|\to\infty} \Pi_B(x, dy)$ in the sense of strong convergence of measures. We have thus proved the following important result.

THEOREM 10.1. *Let B be a nonpolar relatively compact set. Then $g_B(x, y) \to L_B(y)$ as $|x| \to \infty$, and $\Pi_B(x, dy) \to \mu_B(dy)$ as $|x| \to \infty$ in the sense of strong convergence.*

The measure μ_B has the obvious probabilistic significance as the hitting probability of B starting from infinity. We will show that μ_B should be considered in potential theoretic terms as the equilibrium measure of B.

THEOREM 10.2. *Let B be a nonpolar relatively compact set. Then*

$$\lim_{|x|\to\infty} [L_B(x) - a(x)] = k(B) \tag{10.8}$$

exists and is finite. Moreover,

$$\varphi_{\mu_B}(x) = k(B) - L_B(x). \tag{10.9}$$

PROOF. Suppose $y \notin \bar{B}$. Then $a(y - z)$ is bounded for $z \in \bar{B}$, and thus by Theorem 10.2,

$$\lim_{|x|\to\infty} \int_{\bar{B}} \Pi_B(x, dz)a(y - z) = \int_{\bar{B}} \mu_B(dz)a(y - z) \tag{10.10}$$

exists and is finite. Using (8.9), we see that

$$a(y - x) - a(x) - \int_{\bar{B}} \Pi_B(x, dz)a(y - z) + g_B(x, y) = L_B(x) - a(x). \tag{10.11}$$

Using (10.10) and (10.2), we see that the left side converges to $\varphi_{\mu_B}(y) + L_B(y)$ as $|x| \to \infty$. Hence, the right side must have a finite limit. This establishes (10.8) and (10.9) for $y \in (\bar{B})^c$. Suppose $y \in \bar{B}$. Then (8.9) shows

$$\lim_{|x| \to \infty} \int_{\bar{B}} \Pi_B(x, dz)a(y - z) = L_B(y) - k(B). \tag{10.12}$$

The function $g_B(\xi, y) + L_B(\xi) - a(y - \xi)$ is clearly bounded in ξ if $|\xi| > r$ for some sufficiently large r. But then by (8.9),

$$\int_{\bar{B}} \Pi_B(\xi, dz)a(y - z) = g_B(\xi, y) + L_B(\xi) - a(y - \xi) \tag{10.13}$$

is bounded in ξ for $|\xi| > r$. Now for any closed disk D_r of center 0 and radius r containing $\bar{B}$ in its interior,

$$\int_{\bar{B}} \Pi_B(x, dz)a(y - z) = \int_{\partial D_r} \Pi_{D_r}(x, d\xi) \int_{\bar{B}} \Pi_B(\xi, dz)a(y - z), \qquad x \notin \bar{D}. \tag{10.14}$$

Thus, by Theorem 10.1 and equation (10.7),

$$\begin{aligned} \lim_{|x| \to \infty} \int_{\bar{B}} \Pi_B(x, dz)a(y - z) &= \int_{\partial D_r} \sigma_r(0, d\xi) \int_{\bar{B}} \Pi_B(\xi, dz)a(y - z) \\ &= \int_{\bar{B}} \mu_B(dz)a(y - z). \end{aligned} \tag{10.15}$$

This establishes the theorem for $y \in \bar{B}$ and thereby completes the proof.

DEFINITION 10.1. *Let B be a nonpolar relatively compact set. The measure μ_B in Theorem* 10.1 *is called the* equilibrium measure *of B. The constant $k(B)$ in Theorem* 10.2 *is called the* Robin's constant *of B and the potential of μ_B is called the* equilibrium potential *of B.*

For a relatively compact polar set, we define $k(B) = +\infty$. Theorem 10.3 given below will show that this is the natural definition of $k(B)$ for a polar set.

PROPOSITION 10.1. *Let A and B be two relatively compact sets such that $A \subset B$. Then $k(A) \geqq k(B)$.*

PROOF. If B is polar then A must also be polar. In this case $k(A) = k(B) = +\infty$. Suppose B is nonpolar. If A is polar, then $k(A) = +\infty$ and $k(B) < \infty$ so the proposition is valid. Suppose A is also nonpolar. Then $L_A \geqq L_B$ (since $L_A^\lambda \geqq L_B^\lambda$) and (10.8) shows that $k(A) \geqq k(B)$.

THEOREM 10.3. *Let B be a compact set. Then*

$$k(B) = \sup \{k(U): U \text{ open}, U \supset B \text{ and } \bar{U} \text{ compact}\}. \tag{10.16}$$

On the other hand, if U is an open relatively compact set, then

$$k(U) = \inf\{k(A): A \text{ compact}, A \subset U\}. \tag{10.17}$$

PROOF. Suppose B is compact. Let B_n, $n \geqq 1$ be relatively compact open sets such that $B_1 \supset \bar{B}_2 \supset B_2 \supset \cdots$, $\bigcap_n B_n = \bigcap_n \bar{B}_n = B$. Then as was shown in Proposition 3.2, $P_x(V_{B_n} \uparrow V_B) = 1$ for all $x \in B^c \cup B^r$. Suppose that B is polar. Let $f \geqq 0$ be continuous with compact support and have integral 1 and set $Af(z) = \int_{R^2} a(y - z)f(y)\, dy$. Then for any $x \in B^c$,

$$\int_{R_n} g_{B_n}(x, y)f(y)\, dy = E_x \int_0^{V_{B_n}} f(X_t)\, dt \uparrow \infty, \qquad n \to \infty. \tag{10.18}$$

Now $Af(x)$ is a continuous function, and thus for all n

$$\left| \int_{\bar{B}_n} \Pi_{B_n}(x, dz)Af(z) \right| \leqq \sup_{z \in \bar{B}_1} |Af(z)| = M < \infty. \tag{10.19}$$

Using (8.9), we see that

$$\begin{aligned} \left| -\int_{R^n} g_{B_n}(x, y)f(y)\, dy + L_{B_n}(x) \right| &= \left| Af(x) - \int_{\bar{B}_n} \Pi_{B_n}(x, dz)Af(z) \right| \\ &\leqq |Af(x)| + M < \infty. \end{aligned} \tag{10.20}$$

Thus, using (10.18), we see that $L_{B_n}(x) \uparrow \infty$ for each $x \in B^c$. By (10.9),

$$\int_{R^2} \varphi_{\mu_{B_n}}(x)f(x)\, dx + \int_{R^2} L_{B_n}(x)f(x)\, dx = k(B_n). \tag{10.21}$$

But

$$\left| \int_{R^2} \varphi_{\mu_{B_n}}(x)f(x)\, dx \right| = \left| \int_{R^2} Af(x)\mu_{B_n}(dx) \right| \leqq M < \infty, \tag{10.22}$$

and $\int_{R^2} L_{B_n}(x)f(x)\, dx \uparrow \infty$. Thus, $k(B_n) \uparrow \infty$. This establishes (10.16) when B is polar.

Suppose now that B is nonpolar. Let f and Af be as before. Then for $x \in B^c \cup B^r$,

$$\begin{aligned} \int_{R^n} g_{B_n}(x, y)f(y)\, dy &= E_x \int_0^{V_{B_n}} f(X_t)\, dt \uparrow E_x \int_0^{V_B} f(X_t)\, dt \\ &= \int_{R2} g_B(x, y)f(y)\, dy. \end{aligned} \tag{10.23}$$

Also as $P_x(X_{V_{B_n}} \in \bar{B}_1) = 1$ for all n.

$$\begin{aligned} \lim_{n \to \infty} \int_{\bar{B}_n} \Pi_{B_n}(x, dz)Af(z) &= \lim_{n \to \infty} E_x Af(X_{V_{B_n}}) \\ &= E_x Af(X_{V_B}) = \int_{\bar{B}} \Pi_B(x, dz)Af(z). \end{aligned} \tag{10.24}$$

Hence by (8.9), for $x \in B^c \cup B^r$,

$$(10.25)\quad \lim_{n\to\infty} L_{B_n}(x)$$

$$= \lim_{n\to\infty}\left[Af(x) - \int_{\bar{B}_n} \Pi_{B_n}(x, dz)Af(z) + \int_{R^2} g_{B_n}(x, y)f(y)\,dy\right]$$

$$= Af(x) - \int_{\bar{B}} \Pi_B(x, dz)Af(z) + \int_{R^2} g_B(x, y)f(y)\,dy = L_B(x).$$

Since $(B^r)^c \cap B$ has measure 0, we see that

$$(10.26)\qquad \int_{R^2} L_{B_n}(x)f(x)\,dx \uparrow \int_{R^2} L_B(x)f(x)\,dx.$$

Now if D is a disk of center 0 containing $\bar{B}$ in its interior,

$$(10.27)\qquad \lim_{n\to\infty} \int_{R^2} \mu_{B_n}(dx)Af(x) = \lim_{n\to\infty} \int_{\partial D} \sigma(0, d\xi)E_\xi Af(X_{V_{B_n}})$$

$$= \int_{\partial D} \sigma(0, d\xi)E_\xi Af(X_{V_B})$$

$$= \int_{R^2} \mu_B(dx)Af(x).$$

Hence, using this fact, (10.26), and (10.21), we see that $k(B_n) \uparrow k(B)$. This establishes (10.16). To prove (10.17), note that we can find compacts $A_n \subset U$ such that $A_1 \subset A_2 \subset \cdots$, $\cup_n A_n = U$. But then $P_x(V_{A_n} \downarrow V_U) = 1$ for all x. The remainder of the proof of (10.17) is similar to the proof of (10.16) for B a nonpolar set. We omit these details.

Let A and B be two Borel sets. Then,

$$(10.28)\qquad P_x(V_{A\cap B} \leqq t) \leqq P_x(V_A \leqq t, V_B \leqq t)$$

$$\leqq P_x(V_A \leqq t) + P_x(V_B \leqq t) - P_x(V_{A\cup B} \leqq t).$$

Thus, $P_x(V_{A\cap B} > t) \geqq P_x(V_A > t) + P_x(V_B > t) - P_x(V_{A\cup B} > t)$. It follows from this and (8.6) that

$$(10.29)\qquad L^\lambda_{A\cap B}(x) \geqq L^\lambda_A(x) + L^\lambda_B(x) - L^\lambda_{A\cup B}(x).$$

Letting $\lambda \downarrow 0$, we see that whenever A and B are nonpolar

$$(10.30)\qquad L_{A\cap B}(x) + L_{A\cup B}(x) \geqq L_A(x) + L_B(x).$$

If we take $L_k(x) \equiv \infty$ whenever k is polar, then (10.30) is valid for all sets. Using (10.8), we see that for relatively compact sets

$$(10.31)\qquad -k(A \cup B) + [-k(A \cap B)] \leqq -k(A) + (-k(B)).$$

Also as $L_A^\lambda(x) \geqq L_B^\lambda(x)$ for $A \subset B$, we see that $L_A(x) \geqq L_B(x)$, so for $A \subset B$,

$$-k(A) \leqq -k(B). \tag{10.32}$$

Define $k^*(B) = -k(B)$ if B is compact and define $k^*(U) = \sup\{k^*(B): B \subset U, B \text{ compact}\}$ if U is open. Then (10.16), (10.17), (10.31), and (10.32) show that $k^*(\cdot)$ is a Choquet capacity on the compact sets. The extension theorem of Choquet then implies that for any Borel set B,

$$\sup\{k^*(A): A \subset B, A \text{ compact}\} = \inf\{k^*(U): U \supset B, U \text{ open}\}, \tag{10.33}$$

and that the common value $k^*(B)$ is such that $k^*(A) \leqq k^*(B)$, $A \subset B$ and

$$k^*(A \cup B) + k^*(A \cap B) \leqq k^*(A) + k^*(B). \tag{10.34}$$

Let B be any relatively compact set. Then for any compact set $A \subset B$ and any open set $U \supset B$, $-k(A) \leqq -k(B) \leqq -k(U)$. Thus,

$$k^*(B) \leqq -k(B) \leqq k^*(B). \tag{10.35}$$

Equation (10.35) tells us that whenever B is relatively compact, $k(B) = -k^*(B)$. For a general Borel set, define $k(B)$ to be $-k^*(B)$ and define the capacity $C(B)$ of B to be $e^{-k(B)}$. Our discussion above has shown the following result.

THEOREM 10.4. *Let B be any relatively compact set. Then*

$$\inf\{k(A): A \subset B, A \text{ compact}\} = k(B) = \sup\{k(U): U \text{ open}, U \supset B\}. \tag{10.36}$$

COROLLARY 10.1. *A Borel set B has capacity 0 if and only if it is polar.*

PROOF. The set B has capacity 0 if and only if $k(B) = \infty$. We have already shown that for a relatively compact set B this is the case if and only if B is polar. If B is not relatively compact and $k(B) = \infty$, then (10.36) shows that $k(A) = \infty$ for every compact subset of B, and thus (again by (10.36)) $k(D) = \infty$ for every relatively compact subset D of B. But then every relatively compact subset of B is polar, and as B is a countable union of relatively compact sets, B must be polar. Conversely, if B is polar, every compact subset of B is polar, so for any compact subset $A \subset B$, $k(A) = \infty$. Thus, (10.36) shows that $k(B) = \infty$. This establishes the corollary.

If B is a relatively compact set in dimension $n \geqq 3$, then as was shown in Section 6, B has capacity 0 (that is, B is polar) if and only if the only finite measure μ having support on B with a bounded potential is the 0 measure. For potentials in the plane we have the following analog.

THEOREM 10.5. *A relatively compact set B is polar if and only if* $\sup_x \varphi_\mu(x) = +\infty$ *for every nonzero finite measure μ having support on B.*

PROOF. Let A be a relatively compact open set containing $\bar{B}$. Let $a_N(x) = a(x)$ if $a(x) \geqq -N$ and let $a_N(x) = -N$ if $a(x) < -N$, $N > 0$. Then clearly

$$-\int_{R^2} \mu(dx) \int_{R^2} a_N(y-x)\mu_A(dy) = -\int_{R^2} \mu_A(dy) \int_{R^2} a_N(y-x)\mu(dx). \tag{10.37}$$

By monotone convergence as $N \uparrow \infty$, we see that

$$\int_{\bar{B}} \mu(dx)\varphi_{\mu_A}(x) = \int_{\bar{A}} \mu_A(dx)\varphi_\mu(x). \tag{10.38}$$

Since $\bar{B} \subset A$ equation (10.9) shows that the left side is $\mu(\bar{B})k(A)$. Thus, $\mu(\bar{B})k(A) \leqq \sup_x \varphi_\mu(x)$, and hence by Theorem 10.4, $\mu(\bar{B})k(B) \leqq \sup_x \varphi_\mu(x)$. If B is polar then $k(B) = +\infty$. On the other hand, if $\sup_x \varphi_\mu(x) = +\infty$ for all nonzero μ supported on $\bar{B}$, then B must be polar. For, if B were nonpolar then the equilibrium measure μ_B of B would be a probability measure on B whose potential would be $\leqq k(B) < +\infty$ everywhere, a contradiction. This establishes the theorem.

REMARK. By using the maximum principle for potentials φ_μ (see, for example, Hille, *Analytic Function Theory*, Vol. II), we replace $\sup_x$ by $\sup_{x \in \bar{B}}$ in Theorem 10.5.