ASYMPTOTIC DISTRIBUTION OF THE MOMENT OF FIRST CROSSING OF A HIGH LEVEL BY A BIRTH AND DEATH PROCESS

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1. Statement of the problem

In many applications of probability theory an essential role is played by birth and death processes, which is the name given to homogeneous Markov processes with a finite or countable number of states, which we denote by $0, 1, \cdots, n, \cdots$, in which an instantaneous transition is only possible between adjacent states. The probabilities $P_n(t) = P\{\xi(t) = n\}$ of these states satisfy the system of differential equations (see [2])

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) \tag{1.1}$$

$n = 0, 1, \cdots$, where $\lambda_{-1} = \mu_0 = 0$.

If the number of states is finite and equals N, then $\lambda_N = \mu_{N+1} = 0$. It is also assumed that all the other parameters λ_n and μ_n are positive. Let us consider the random variable $\tau_{k,n}$, $k < n$, the passage time from state k to state n:

$$\tau_{k,n} = \inf\{t: \xi(t) = n, t > 0 | \xi(0) = k\}. \tag{1.2}$$

The random variables $\tau_{k,n}$ are of considerable interest in reliability theory, where birth and death processes describe the behavior of storage systems with replace ments. If the states $0, 1, \cdots, n - 1$, correspond to functioning states of a system, and other states correspond to nonfunctioning states of a system, then the random variable $\tau_{k,n}$ may be regarded as the length of time that the system works without a failure, if it starts in state k. Most often the state $\xi(t)$ is taken to be the number of nonfunctioning elements, at time t, in some system, and it is assumed that at the starting time the system was completely functioning, that is, $\xi(0) = 0$. Therefore, the study of the random variables $\tau_{0,n}$ is of greatest interest. Let us assume that our process has a stationary distribution. As is known [2], for this it is necessary and sufficient that the following conditions be satisfied:

$$\sum_{n=0}^{\infty} \theta_n < \infty, \qquad \sum_{n=0}^{\infty} \frac{1}{\lambda_n\theta_n} = \infty, \tag{1.3}$$

where

$$\theta_0 = 1, \qquad \theta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \tag{1.4}$$

(for the case of a finite number of states, a stationary distribution always exists).

For such a process the time intervals $\tau_1^+, \tau_2^+, \cdots, \tau_m^+, \cdots$, for which $\xi(t) < n$ will alternate with the intervals $\tau_1^-, \tau_2^-, \cdots, \tau_m^-, \cdots$ for which $\xi(t) \geqq n$. Since the process is Markovian, the lengths of all these intervals are independent, and from the existence of a stationary distribution it follows that τ_m^+ and τ_m^- are proper variables. Further it is not hard to see that at the beginning of every interval τ_m^+, $m \geqq 2$, the process will be in state $n - 1$, and therefore the distribution of any one of the variables τ_m^+, $m = 2$, coincides with the distribution of the variable $\tau_{n-1,n}$. Thus, in reliability problems $\tau_{n-1,n}$ can be regarded as the length of time during which the system works without failure or, as engineers say, the work per failure.

Let us assume now that at the initial moment the process is in a stationary regime, that is, $P\{\xi(0) = k\} = p_k$, where the $p_k = \theta_k p_0$ are the stationary probabilities. For reliability theory another random variable, which we denote by τ_n, is of interest:

$$\tau_n = \inf \{t: \xi(t) \geqq n, t \geqq 0\}. \tag{1.5}$$

This variable can be interpreted as the amount of time that the system works without failure in a stationary regime. Since the intervals τ_m^+ and τ_m^- form an extended alternating renewal process [1], the distribution of the variable τ_n is related to the distribution of the variable $\tau_{n-1,n}$ in the following way:

$$P\{\tau_n > t | \tau_n > 0\} = \frac{\int_t^\infty P\{\tau_{n-1,n} > x\}\, dx}{\int_0^\infty P\{\tau_{n-1,n} > x\}\, dx}. \tag{1.6}$$

Using ordinary methods, it is not hard to find the precise distributions of the variables $\tau_{k,n}$ and τ_n, which will be done below. However, computation with the exact formulas is very cumbersome. On the other hand, in applications the attainment of the level n denotes, as a rule, an undesirable event (an absence of demand in queuing theory, and a failure of the system in reliability theory), and therefore the parameters λ_i and μ_i and the level n are usually such that a crossing of the level n is "infrequent," that is, the level n is high. In such a situation, it is natural to investigate the asymptotic behavior of the variables $\tau_{k,n}$ and, as a consequence, to obtain approximate formulas for their distributions. In [5] it is shown that for fixed λ_i and μ_i, as $n \to \infty$ the distribution function of the appropriately normalized variable $\tau_{k,n}$ converges to the function

$$1 - ae^{-x}, \qquad x > 0, \quad 0 < a \leqq 1. \tag{1.7}$$

However, this result fails to satisfy us for two reasons.

First, in problems of reliability theory the level n (which is usually the number of elements in reserve) is almost never large. On the other hand, the parameters λ_i and μ_i most often depend upon the number n. For example, for the case of an immediate reserve with one maintenance unit [4] $\lambda_i = (n - i)\lambda$, $\mu_i = \mu$. Therefore, the following formulation of the problem is more natural and more general. Suppose that the parameters λ_i and μ_i and the level n vary in an arbitrary way; find conditions under which the distribution of the normalized variables $\tau_{k,n}$ converges to the distribution (1.7).

Second, any limit theorem which does not contain an estimate of the rate of convergence cannot, strictly speaking, be used for approximate calculations. Therefore, for applications it is highly essential to obtain constructive bounds on the rate of convergence, which do not contain the symbols $o(\cdot)$ and $O(\cdot)$. If F is the exact distribution, and Φ is the limit distribution, then a bound of the following type would seem to be ideal:

$$\|F - \Phi\| \leqq \varepsilon(\lambda_i, \mu_i, n) \text{ and } \|F - \Phi\| \sim \varepsilon(\lambda_i, \mu_i, n) \quad \text{as } \varepsilon(\lambda_i, \mu_i, n) \to 0. \tag{1.8}$$

An unimprovable bound of this kind gives simultaneously necessary and sufficient conditions for the convergence of F to Φ. It is estimates of just this kind which will be obtained in this paper. We will restrict ourselves to the study of the asymptotic behavior of the variables $\tau_{0,n}$, $\tau_{n-1,n}$ and τ_n since, as was shown, they are the variables of greatest interest in reliability theory.

2. The exact distributions of the variables $\tau_{0,n}$, $\tau_{n-1,n}$ and τ_n

We introduce the notation

$$\begin{aligned} P\{\tau_{0,n} < t\} &= F_n(t), \\ P\{\tau_{n-1,n} < t\} &= \Phi_n(t), \\ P\{\tau_n < t\} &= G_n(t). \end{aligned} \tag{2.1}$$

The exact distribution of the variable $\tau_{0,n}$ is found in [4]:

$$F_n(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{z\Delta_n(z)}\,dz, \qquad a > 0, \tag{2.2}$$

where the polynomials $\Delta_n(z) = 1 + \Delta_{n,1}z + \Delta_{n,2}z^2 + \cdots + \Delta_{n,n}z^n$ are determined from the recursion relation

$$\Delta_{k+1}(z) = \left(1 + \frac{\mu_k}{\lambda_k} + \frac{z}{\lambda_k}\right)\Delta_k(z) - \frac{\mu_k}{\lambda_k}\Delta_{k-1}(z). \tag{2.3}$$

Since $\tau_{0,n} = \tau_{0,1} + \cdots + \tau_{n-1,n}$ and since the terms are independent as a result of the process being Markovian, using elementary properties of the Laplace transform, we obtain

$$\Phi_n(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Delta_{n-1}(z)e^{zt}}{z\Delta_n(z)}\,dz, \qquad a > 0. \tag{2.4}$$

From formulas (2.2), (2.3), and (2.4), it is not hard to determine the expectation of $\tau_{0,n}$ and $\tau_{n-1,n}$:

$$\begin{aligned} M\tau_{0,n} &= T_{0,n} = \Delta_{n,1} = \sum_{k=0}^{n-1} \frac{1}{\lambda_k \theta_k} \sum_{s=0}^{k} \theta_s \\ M\tau_{n-1,n} &= T_{n-1,n} = T_{0,n} - T_{0,n-1} = \frac{1}{\lambda_{n-1}\theta_{n-1}} \sum_{k=0}^{n-1} \theta_k. \end{aligned} \tag{2.5}$$

To find the distribution of τ_n, we use formula (1.6):

$$1 - G_n(t) = P\{\tau_n > t\} = \frac{\sum_{k=0}^{n-1} p_k}{T_{n-1,n}} \int_t^{\infty} [1 - \Phi_n(x)]\,dx, \tag{2.6}$$

where the p_k are the stationary probabilities of the states.

Hence, again using the properties of the Laplace transform, we obtain

$$\begin{aligned} G_n(t) &= 1 - \mathscr{P}_n + \frac{\mathscr{P}_n}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Delta_n(z) - \Delta_{n-1}(z)}{z^2\Delta_n(z)T_{n-1,n}}\, e^{zt}\,dz \\ &= 1 - \mathscr{P}_n + \frac{\mathscr{P}_n}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\delta_n(z)\, e^{zt}}{z\Delta_n(z)}\,dz, \qquad a > 0, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \mathscr{P}_n &= \sum_{k=0}^{n-1} p_k, \\ \delta_n(z) &= \frac{\Delta_n(z) - \Delta_{n-1}(z)}{zT_{n-1,n}} = 1 + \delta_{n,1}z + \cdots + \delta_{n,n-1}z^{n-1}. \end{aligned} \tag{2.8}$$

3. Properties of the polynomials $\Delta_n(z)$ and $\delta_n(z)$

LEMMA 3.1. *The roots of the polynomials* $\Delta_n(z)$ *and* $\delta_n(z)$ *have the following properties:*

(i) *they are simple and negative;*

(ii) *between any two adjacent roots of* $\Delta_n(z)$ *there lies a root of* $\Delta_{n-1}(z)$*;*

(iii) *between any two adjacent roots of* $\Delta_n(z)$ *there lies a root of* $\delta_n(z)$.

PROOF. The first and second assertions concerning the polynomials $\Delta_n(z)$ follow from the theory of orthogonal polynomials [6]. Let $-z'_1, -z'_2, \cdots, -z'_{n-1}$ be the roots of $\Delta_{n-1}(z)$, and $-z''_1, -z''_2, \cdots, -z''_n$ be the roots of $\Delta_n(z)$, numbered in the order of increasing modulus. Since

$$0 < z_1'' < z_1' < z_2'' < z_2' < \cdots < z_{n-1}' < z_n'', \tag{3.1}$$

the polynomial

$$\delta_n(z) = \frac{\Delta_n(z) - \Delta_{n-1}(z)}{zT_{n-1,n}} \tag{3.2}$$

is positive at the points $-z_1''$, $-z_3''$, $-z_5''$, $\cdots$, and negative at the points $-z_2''$, $-z_4''$, $-z_6''$, $\cdots$, from which it follows that all the roots of $\delta_n(z)$ are negative, and that between any two roots of $\Delta_n(z)$ there lies a root of $\delta_n(z)$. This proves the lemma.

Let us now introduce the normalized polynomials

$$\begin{aligned} \Delta_n\left(\frac{z}{\Delta_{n,1}}\right) &= 1 + z + a_{n,2}z^2 + \cdots + a_{n,n}z^n, \\ \delta_n\left(\frac{z}{\Delta_{n,1}}\right) &= 1 + b_{n,1}z + b_{n,2}z^2 + \cdots + b_{n,n-1}z^{n-1}. \end{aligned} \tag{3.3}$$

It follows from (2.3) that

$$\Delta_{n+1}(z) - \Delta_n(z) = z\frac{\sum_{k=0}^{n} \theta_k \Delta_k(z)}{\lambda_n\theta_n}, \tag{3.4}$$

from which we obtain the following recursion relation for the coefficients $\Delta_{n,k}$:

$$\Delta_{n,k} = \sum_{k=0}^{n-1} \frac{1}{\lambda_k\theta_k} \sum_{s=0}^{k} \theta_s\Delta_{s,k-1}. \tag{3.5}$$

Hence, the coefficients $a_{n,k}$ and $b_{n,k}$ of the normalized polynomials are given by

$$a_{n,k} = \frac{\Delta_{n,k}}{\Delta_{n,1}^k}, \qquad b_{n,k} = \frac{\Delta_{n,k+1} - \Delta_{n-1,k+1}}{\Delta_{n,1} - \Delta_{n-1,1}}. \tag{3.6}$$

Let us write these polynomials in the form

$$\begin{aligned} \Delta_n\left(\frac{z}{\Delta_{n,1}}\right) &= (1 + \alpha_0 z)(1 + \alpha_1 z) \cdots (1 + \alpha_{n-1}z), \\ \delta_n\left(\frac{z}{\Delta_{n,1}}\right) &= (1 + \beta_1 z)(1 + \beta_2 z) \cdots (1 + \beta_{n-1}z). \end{aligned} \tag{3.7}$$

If we assume that the numbers α_i and β_i are numbered in decreasing order, then as follows from Lemma 3.1

$$\alpha_0 > \beta_1 > \alpha_1 > \beta_2 > \alpha_2 > \cdots > \beta_{n-1} > \alpha_{n-1}. \tag{3.8}$$

We now make some estimates of the α_i and β_i.

LEMMA 3.2. *If we put* $\alpha = 1 - \alpha_0 = \alpha_1 + \cdots + \alpha_{n-1}$ *and* $\beta = \beta_1 + \beta_2 + \cdots + \beta_{n-1}$, *then*

$$\alpha < \beta, \tag{3.9}$$

and for $a_{n,2} < \frac{1}{4}$, *we have*

$$1 - (1 - 2a_{n,2})^{1/2} < \alpha < \frac{1 - (1 - 4a_{n,2})^{1/2}}{2}. \tag{3.10}$$

PROOF. Property (3.9) follows from (3.8). From (3.7), it follows that

$$\begin{aligned} \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} &= 1, \\ \alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{n-1}^2 &= 1 - 2a_{n,2}; \end{aligned} \tag{3.11}$$

whence

$$1 - 2a_{n,2} < \alpha_0(\alpha_0 + \cdots + \alpha_{n-1}) = 1 - \alpha, \tag{3.12}$$

that is, $\alpha < \frac{1}{2}$. Further,

$$1 - 2a_{n,2} < \alpha_0^2 + (\alpha_1 + \cdots + \alpha_{n-1})^2 = \alpha^2 + (1 - \alpha)^2 \tag{3.13}$$

or

$$\alpha^2 - \alpha + a_{n,2} > 0. \tag{3.14}$$

Solving this inequality and keeping in mind that $\alpha < \frac{1}{2}$, we obtain

$$\alpha < \tfrac{1}{2}[1 - (1 - 4a_{n,2})^{1/2}]. \tag{3.15}$$

On the other hand,

$$\alpha_0^2 = (1 - \alpha)^2 < 1 - 2a_{n,2}, \tag{3.16}$$

that is, $\alpha > 1 - (1 - 2a_{n,2})^{1/2}$. This proves the lemma.

COROLLARY 3.1. *It follows from Lemma* 3.2 *that for* $a_{n,2} \to 0$

$$\alpha = a_{n,2} + O(a_{n,2}^2). \tag{3.17}$$

LEMMA 3.3. *For any* $k \geqq 1$ *the following inequalities hold:*

$$a_{n,k} \leqq \frac{\alpha^{k-1}}{(k-1)!}, \tag{3.18}$$

$$b_{n,k} \leqq \frac{\beta^k}{k!}. \tag{3.19}$$

PROOF. We shall say that the series $A(z) = \Sigma_{n=0}^{\infty} a_n z^n$ is a majorant of the series $B(z) = \Sigma_{n=0}^{\infty} b_n z^n$, if $|b_n| \leqq a_n$ for every n. We will write this relation as $B(z) \ll A(z)$. It is easily seen that the following holds. If $B_1(z) \ll A_1(z)$ and

$B_2(z) \ll A_2(z)$, then

$$B_1(z) \cdot B_2(z) \ll A_1(z) \cdot A_2(z). \tag{3.20}$$

This property will be applied in order to estimate the coefficients $a_{n,k}$ and $b_{n,k}$:

$$\Delta_n\left(\frac{z}{\Delta_{n,1}}\right) = (1 + \alpha_0 z)(1 + \alpha_1 z) \cdots (1 + \alpha_{n-1} z) \ll (1 + \alpha_0 z)e^{\alpha z}. \tag{3.21}$$

Hence,

$$a_{n,k} \leqq \frac{\alpha^k}{k!} + \frac{(1 - \alpha)\alpha^{k-1}}{(k - 1)!} \leqq \frac{\alpha^{k-1}}{(k - 1)!}. \tag{3.22}$$

It is simpler yet to bound the $b_{n,k}$:

$$\delta_n\left(\frac{z}{\Delta_{n,1}}\right) = (1 + \beta_1 z) \cdots (1 + \beta_{n-1} z) \ll e^{\beta z}, \tag{3.23}$$

that is, $b_{n,k} \leqq \beta^k/k!$. This proves the lemma.

COROLLARY 3.2. *From inequality* (3.18) *and relation* (3.17) *it follows that for* $a_{n,2} \to 0$, *we have*

$$a_{n,k} = O\left(a_{n,2}^{k-1}\right). \tag{3.24}$$

4. Asymptotic behavior of the variable $\tau_{0,n}$

We will consider the normalized variable $\xi_n = \tau_{0,n}/T_{0,n}$. It follows from (2.2) that

$$P\{\xi_n < x\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{zx}\, dz}{z(1 + \alpha_0 z)(1 + \alpha_1 z) \cdots (1 + \alpha_{n-1} z)}. \tag{4.1}$$

THEOREM 4.1. *For* $a_{n,2} < \frac{1}{4}$, *we have the inequality*

$$\max_{0 \leqq t < \infty} \left| F_n(t) - 1 + \exp\left\{-\frac{t}{T_{0,n}}\right\} \right| \leqq \frac{\alpha}{1 - \alpha} \leqq \frac{1 - (1 - 4a_{n,2})^{1/2}}{1 + (1 - 4a_{n,2})^{1/2}}. \tag{4.2}$$

PROOF. It is evident from (4.1) that $\xi_n = \eta_0 + \eta_1 + \cdots + \eta_{n-1}$, where the η_i are independent and distributed according to an exponential law

$$P\{\eta_i > x\} = \exp\left\{-\frac{x}{\alpha_i}\right\}, \qquad i = 0, 1, \cdots, n - 1. \tag{4.3}$$

Let $P\{\eta_1 + \cdots + \eta_{n-1} > x\} = f_n(x)$. Then

$$\begin{aligned}(4.4)\qquad \varepsilon_n(x) &= F_n(T_n x) - 1 + e^{-x}\\ &= \frac{1}{\alpha_0}\int_0^x \exp\left\{-\frac{x-u}{\alpha_0}\right\}[1 - f_n(u)]\,du - 1 + e^{-x}\\ &= e^{-x} - \exp\left\{-\frac{x}{\alpha_0}\right\} - \frac{1}{\alpha_0}\int_0^x \exp\left\{-\frac{x-u}{\alpha_0}\right\} f_n(u)\,du.\end{aligned}$$

We bound this quantity from above and below:

$$(4.5)\qquad \begin{aligned}\varepsilon_n(x) &\leqq e^{-x} - \exp\left\{-\frac{x}{\alpha_0}\right\} \leqq \frac{\alpha}{e(1-\alpha)},\\ \varepsilon_n(x) &\geqq -\frac{1}{\alpha_0}\int_0^x \exp\left\{-\frac{x-u}{\alpha_0}\right\} f_n(u)\,du \geqq -\frac{1}{\alpha_0}\int_0^\infty f_n(u)\,du\\ &= \frac{1}{\alpha_0} M(\eta_1 + \cdots + \eta_{n-1}) = -\frac{\alpha}{1-\alpha}.\end{aligned}$$

The assertion of the theorem follows from these bounds and inequality (3.10).

COROLLARY 4.1. *In order that*

$$(4.6)\qquad \lim P\left\{\frac{\tau_{0,n}}{T_{0,n}} > x\right\} = e^{-x},$$

it is necessary and sufficient that $a_{n,2} \to 0$.

The necessity is evident from formula (4.1), and the sufficiency follows from inequality (4.2).

REMARK 4.1. Estimating the difference $\varepsilon_n(x)$, for example, at the point $x = \sqrt{\alpha}$, it is not hard to show that for $a_{n,2} \to 0$

$$(4.7)\qquad \max_{0 \leqq t < \infty}\left|F_n(t) - 1 + \exp\left\{-\frac{t}{T_{0,n}}\right\}\right| \sim a_{n,2},$$

and thus inequality (4.2) is asymptotically exact. We also note that from (3.6) it is easy to find an explicit expression for $a_{n,2}$:

$$(4.8)\qquad a_{n,2} = \frac{1}{T_{0,n}^2}\sum_{k=0}^{n-1}\frac{1}{\lambda_k\theta_k}\sum_{s=0}^{k}\theta_s T_{0,s},$$

and therefore inequality (4.2) enables us to easily and precisely estimate the rate of convergence in (4.6).

An even more precise approximation for the distribution $F_n(t)$ yields:

THEOREM 4.2. *For* $a_{n,2} < \frac{1}{4}$, *we have the equality*

$$(4.9)\qquad F_n(t) = 1 - \frac{\lambda'}{\lambda}\exp\left\{-\frac{\lambda t}{T_{0,n}}\right\} + \theta\frac{2a_{n,2}}{1 - 4a_{n,2}}\exp\left\{-\frac{t}{2a_{n,2}T_{0,n}}\right\},$$

where $0 < \theta < 1$ *and*

$$\lambda = \frac{1}{\alpha_0} = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{d^{m-1}}{dw^{m-1}} [\varphi_n^m(-w)]_{w=1}, \tag{4.10}$$

$$\lambda' = \frac{T_{0,n}}{\Delta_n'\left(-\dfrac{\lambda}{T_{0,n}}\right)} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dw^m} [\varphi_n^m(-w)]_{w=1} \tag{4.11}$$

and

$$\varphi_n(z) = \Delta_n\left(\frac{z}{T_{0,n}}\right) - 1 - z. \tag{4.12}$$

PROOF. We move the path of integration in the integral in (4.1) to the left so as to separate out the residues at the points $z = 0$ and $z = -1/\alpha_0 = -\lambda$:

$$F_n(T_{0,n}x) = 1 - \frac{e^{-\lambda x}}{\dfrac{\lambda}{T_{0,n}} \Delta_n'\left(-\dfrac{\lambda}{T_{0,n}}\right)} + R_n, \tag{4.13}$$

where

$$R_n = \frac{1}{2\pi i} \int_{-b-i\infty}^{-b+i\infty} \frac{e^{zx}\, dz}{z(1 + \alpha_0 z) \cdots (1 + \alpha_{n-1} z)}, \quad \frac{1}{\alpha_0} < b < \frac{1}{\alpha_1}. \tag{4.14}$$

Bounding R_n in terms of the maximum modulus of the integrand, we obtain

$$\begin{aligned} |R_n| &\leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-bx}\, ds}{(b^2 + s^2)^{1/2} [(b\alpha_0 - 1)^2 + \alpha_0^2 s^2]^{1/2} (1 - \alpha b)} \\ &= \frac{e^{-bx}}{2\pi(b\alpha_0 - 1)(1 - b\alpha)} \int_{-\infty}^{\infty} \frac{d(s/b)}{[1 + (s/b)^2]^{1/2}[1 + \alpha_0^2 s^2/(b\alpha_0 - 1)^2]^{1/2}} \\ &\leqq \frac{e^{-bx}}{2(b\alpha_0 - 1)(1 - b\alpha)}. \end{aligned} \tag{4.15}$$

Putting $b = [2\alpha(1 - \alpha)]^{-1}$, we find that

$$|R_n| \leqq \frac{2\alpha\alpha_0 \exp\left\{-\dfrac{x}{2\alpha\alpha_0}\right\}}{1 - 4\alpha\alpha_0} < \frac{2a_{n,2}}{1 - 4a_{n,2}} \exp\left\{-\frac{x}{2a_{n,2}}\right\} \tag{4.16}$$

since $\alpha\alpha_0 < a_{n,2}$. We now write the residue at the point $z = -1/\alpha_0 = -\lambda$ in the form

$$\frac{1}{2\pi i} \int_{|z+\lambda|=\rho} \frac{e^{zx}\, dx}{z[1 + z + \varphi_n(z)]}, \tag{4.17}$$

where ρ is sufficiently small, and

$$\varphi_n(z) = a_{n,2}z^2 + a_{n,3}z^3 + \cdots + a_{n,n}z^n. \tag{4.18}$$

One could expand the integrand in powers of $\varphi_n(z)$ and obtain the corresponding expansion for the residue. It has the form $\Sigma_{k=0}^{\infty} \pi_k(x)e^{-x}$, where $\pi_k(x)$ is a polynomial of degree k. By virtue of the inequalities (3.18), this series will be an asymptotic one. However, for the residue that we are considering, one can obtain a more convenient representation. We have

$$\frac{1}{2\pi i}\int_{|z+\lambda|=\rho} \frac{e^{zx}\,dz}{z[1 + z + \varphi_n(z)]} = \frac{\lambda'}{\lambda}e^{-\lambda x}, \tag{4.19}$$

where λ satisfies the equation $1 = \lambda - \varphi_n(-\lambda)$, and $\lambda' = [1 + \varphi_n'(-\lambda)]^{-1}$. Now consider the function $w = z - \varphi_n(-z)$, and denote its inverse by $z = \psi(w)$ It is easily seen that $\lambda = \psi(1)$ and $\lambda' = \psi'(1)$. We expand the function $\psi'(w)$ in a series:

$$\begin{aligned}\psi'(w) &= \frac{1}{2\pi i}\int_C \frac{d\psi(\zeta)}{\zeta - w} = \frac{1}{2\pi i}\int_{C_1} \frac{dz}{z - w - \varphi_n(-z)} \\ &= \sum_{m=0}^{\infty} \frac{1}{2\pi i}\int_{C_1} \frac{\varphi_n^m(-z)}{(z - w)^{m+1}}\,dz = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{d^m}{dw^m}[\varphi_n^m(-w)].\end{aligned} \tag{4.20}$$

Hence,

$$\psi(w) = w + \sum_{m=1}^{\infty} \frac{1}{m!}\frac{d^{m-1}}{dw^{m-1}}[\varphi_n^m(-w)]. \tag{4.21}$$

Letting $w = 1$ in these series, we obtain the desired expansions (4.10), (4.11). This proves the theorem.

REMARK 4.2. Using inequality (3.18), it is not hard to show by a simple estimate using the maximum modulus that the mth terms of these series do not exceed the quantity $2[2(e^{2\alpha} - 1)]^m$, from which, taking (3.10) into account, it follows that the series (4.10), (4.11) converge for $a_{n,2} < 0.2$.

5. The asymptotic behavior of the variables τ_n

THEOREM 5.1. *For $\beta < (e - 1)/e$ one has the inequality*

$$\max_{0 \leq t < \infty} \left| G_n(t) - 1 + \mathscr{P}_n \exp\left\{-\frac{t}{T_{0,n}}\right\}\right| \leq \beta\mathscr{P}_n. \tag{5.1}$$

PROOF. As follows from (2.7),

$$\begin{aligned}G_n(T_{0,n}x) &= P\left\{\frac{\tau_n}{T_{0,n}} < x\right\} \\ &= 1 - \mathscr{P}_n + \frac{\mathscr{P}_n}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{(1 + \beta_1 z)\cdots(1 + \beta_{n-1}z)e^{zx}}{z(1 + \alpha_0 z)\cdots(1 + \alpha_{n-1}z)}\,dz\end{aligned} \tag{5.2}$$

$$= 1 - \mathscr{P}_n \exp\left\{-\frac{x}{\alpha_0}\right\} + \mathscr{P}_n \frac{1}{\alpha_0}\int_0^x \exp\left\{-\frac{x-u}{\alpha_0}\right\}[g_n(u) - 1]\,du,$$

where

$$g_n(u) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{(1+\beta_1 z)\cdots(1+\beta_{n-1}z)e^{zu}}{z(1+\alpha_1 z)\cdots(1+\alpha_{n-1}z)}\,dz = 1 + \sum_{k=1}^{n-1}\frac{A_k}{\alpha_k}\exp\left\{-\frac{u}{\alpha_k}\right\}. \tag{5.3}$$

From inequalities (3.8), we easily obtain

$$A_k = \frac{\left(1-\dfrac{\beta_1}{\alpha_k}\right)\cdots\left(1-\dfrac{\beta_{n-1}}{\alpha_k}\right)}{-\dfrac{1}{\alpha_k}\left(1-\dfrac{\alpha_1}{\alpha_k}\right)\cdots\left(1-\dfrac{\alpha_{k-1}}{\alpha_k}\right)\left(1-\dfrac{\alpha_{k+1}}{\alpha_k}\right)\cdots\left(1-\dfrac{\alpha_{n-1}}{\alpha_k}\right)} > 0. \tag{5.4}$$

Consequently,

$$\left|\frac{1}{\alpha_0}\int_0^x \exp\left\{-\frac{x-u}{\alpha_0}\right\}[g_n(u)-1]\,du\right| < \frac{1}{\alpha_0}\int_0^\infty [g_n(u)-1]\,du = \frac{1}{\alpha_0}\sum_{k=1}^{n-1} A_k = \frac{\beta-\alpha}{1-\alpha}. \tag{5.5}$$

We can now bound the difference:

$$0 < G_n(T_0 x) - 1 + \mathscr{P}_n e^{-x} = \mathscr{P}_n\left\{e^{-x} - e^{-x/\alpha} + \frac{1}{\alpha_0}\int_0^x \exp\left\{-\frac{x-u}{\alpha_0}\right\}[g_n(u)-1]\,du\right\} < \left[\frac{\alpha}{e(1-\alpha)} + \frac{\beta-\alpha}{1-\alpha}\right]\mathscr{P}_n. \tag{5.6}$$

If $\beta < (e-1)/e = 0.63\cdots$, then the last expression does not exceed $\beta\mathscr{P}_n$, from which we obtain the assertion of the theorem.

COROLLARY 5.1. *From the theorem just proved, it follows that*

$$\lim_{\beta\to 0} P\left\{\frac{\tau_n}{T_{0,n}} > x \,\middle|\, \tau_n > 0\right\} = e^{-x}. \tag{5.7}$$

REMARK 5.1. It is not hard to show that for $x = 1$

$$G_n(T_{0,n}) - 1 + \mathscr{P}_n e^{-1} \sim \frac{\beta}{e} \mathscr{P}_n, \tag{5.8}$$

from which it follows that for $\beta \to 0$ the bound (5.1) is exact, as concerns order.

REMARK 5.2. Proceeding as in Theorem 4.2, it is not difficult to obtain a more precise bound. For $\beta < \frac{1}{2}$

$$G_n(t) = 1 - \mathscr{P}_n \frac{\lambda''}{\lambda} \exp\left\{-\frac{\lambda t}{T_{0,n}}\right\} + \theta \frac{\beta}{1 - 2\beta} \exp\left\{-\frac{t}{\beta T_{0,n}}\right\}, \quad |\theta| < 1, \tag{5.9}$$

where λ is defined by the series (4.10) and

$$\lambda'' = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dw^m} \left[\delta_n\left(-\frac{w}{T_{0,n}}\right) \varphi_n^m\left(-\frac{w}{T_{0,n}}\right)\right]_{w=1}. \tag{5.10}$$

It is easy to show that the mth term of this series is of order β^m.

6. The asymptotic behavior of the variables $\tau_{n-1,n}$

THEOREM 6.1. *For any $t \geqq 0$, we have the inequality*

$$\left|\Phi_n(t) - 1 + \frac{T_{n-1,n}}{T_{0,n}} \exp\left\{-\frac{t}{T_{0,n}}\right\}\right| \leqq \left[1 - \frac{T_{n-1,n}(1 - \alpha - \beta)}{T_{0,n}(1 - \alpha)}\right] \exp\left\{-\frac{t}{\alpha T_{0,n}}\right\} + \left|\frac{\beta}{1 - \alpha}\right|. \tag{6.1}$$

PROOF. It follows from (2.4) that

$$\begin{aligned} P\left\{\frac{\tau_{n-1,n}}{T_{0,n}} > x\right\} &= 1 - \Phi_n(T_{0,n}x) \\ &= \frac{T_{n-1,n}}{T_{0,n}} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\delta_n\left(\frac{z}{T_{0,n}}\right) e^{zx}}{\Delta_n\left(\frac{z}{T_{0,n}}\right)} dz, \qquad a > 0. \end{aligned} \tag{6.2}$$

We consider the integral

$$\begin{aligned} h_n(x) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\delta_n\left(\frac{z}{T_{0,n}}\right) e^{zx}}{\Delta_n\left(\frac{z}{T_{0,n}}\right)} dz \\ &= B_0 e^{-x/\alpha_0} + B_1 e^{-x/\alpha_1} + \cdots + B_{n-1} e^{-x/\alpha_{n-1}}. \end{aligned} \tag{6.3}$$

Making use of the inequalities (3.8), it is easily shown that $B_k > 0$. Moreover,

$$\frac{1}{1-\alpha} > B_0 = \frac{\left(1 - \frac{\beta_1}{\alpha_0}\right)\cdots\left(1 - \frac{\beta_{n-1}}{\alpha_0}\right)}{\alpha_0\left(1 - \frac{\alpha_1}{\alpha_0}\right)\cdots\left(1 - \frac{\alpha_{n-1}}{\alpha_0}\right)} > \frac{1-\alpha-\beta}{1-\alpha}. \tag{6.4}$$

It is not hard to see that

$$B_0 + B_1 + \cdots + B_{n-1} = h_n(0) = \frac{T_{0,n}}{T_{n-1,n}}. \tag{6.5}$$

Further,

$$\begin{aligned} |h_n(x) - e^{-x}| &\leqq |B_0 e^{-x/\alpha_0} - e^{-x}| + |h_n(x) - B_0 e^{-x/\alpha_0}| \\ &\leqq |B_0 - 1| + (B_1 + \cdots + B_{n-1})\, e^{-x/\alpha_1} \\ &\leqq \frac{\beta}{1-\alpha} + \left[\frac{T_{0,n}}{T_{n-1,n}} - \frac{1-\alpha-\beta}{1-\alpha}\right] e^{-x/\alpha}, \end{aligned} \tag{6.6}$$

from which we obtain the assertion of the theorem.

COROLLARY 6.1. *From the theorem just proved, it follows that for any* $x > 0$

$$\lim_{\beta \to 0} \left[\Phi_n(T_{0,n}x) - 1 + \frac{T_{n-1,n}}{T_{0,n}} e^{-x}\right] = 0; \tag{6.7}$$

however, this convergence, generally speaking, is not uniform in the neighborhood of zero. In order that the limit in (6.7) be uniform, it is necessary and sufficient that

$$\lim_{\beta \to 0} \frac{T_{n-1,n}}{T_{0,n}} = 1. \tag{6.8}$$

7. A class of infinitely divisible laws

We have earlier obtained conditions under which the distributions of the suitably normalized variables $\tau_{0,n}$, τ_n and $\tau_{n-1,n}$ converge to an exponential distribution (possibly with a jump at zero). It would be natural to go further and attempt to find other limiting distributions for these variables. If, passing to the limit, one varies the level n as well as the parameters λ_k and μ_k, then the problem is empty, since by choosing these parameters appropriately we can always make the roots of the polynomials $\Delta_n(z)$ equal to any preassigned negative numbers. Therefore, we assume that the parameters λ_k and μ_k are fixed, and that the level $n \to \infty$.

For this statement of the problem, the following holds.

THEOREM 7.1. *Suppose that the parameters* λ_k *and* μ_k *satisfy the conditions*

$$\lim_{n \to \infty} T_{0,n} = +\infty, \tag{7.1}$$

$$\lim_{n\to\infty} \frac{T_{0,n+1}}{T_{0,n}} = 1, \tag{7.2}$$

$$\lim_{n\to\infty} \frac{T_{0,n}}{\lambda_n T_{n,n+1}^2} = \gamma, \qquad 0 \leqq \gamma \leqq \infty. \tag{7.3}$$

Then

$$\lim_{n\to\infty} P\left\{\frac{\tau_{0,n}}{T_{0,n}} < x\right\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zx}}{z\Phi(z,\gamma)}\,dz, \tag{7.4}$$

where

$$\Phi(z,\gamma) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma)(\gamma z)^k}{k!\,\Gamma(k+\gamma)}. \tag{7.5}$$

PROOF. We shall prove that for any k, $\lim_{n\to\infty} a_{n,k} = a_k$ exists. From formula (3.6) it is not difficult to find an explicit expression for $a_{n,k}$:

$$a_{n,k} = \frac{\Delta_{n,k}}{\Delta_{n,1}^k} = \frac{1}{T_{0,n}^k} \sum_{\ell=0}^{n-1} \frac{1}{\lambda_\ell \theta_\ell} \sum_{s=0}^{\ell} \theta_s a_{s,k-1}\, T_{0,s}^{k-1}. \tag{7.6}$$

We will prove the existence of the limit by induction on k. For $k = 1$, we have $a_{n,1} = 1$ and the assertion is trivial. Let us assume that $\lim_{n\to\infty} a_{n,k-1} = a_{k-1}$ exists. Since $T_{0,n}\uparrow\infty$, to find the limit of the $a_{n,k}$, we can apply a theorem of Stolz [3] (the finite-difference analog of L'Hospital's rule):

$$\begin{aligned}\lim_{n\to\infty} a_{n,k} &= \lim_{n\to\infty} \frac{\sum_{\ell=0}^{n-1} \frac{1}{\lambda_\ell\theta_\ell} \sum_{s=0}^{\ell} \theta_s a_{s,k-1}\, T_{0,s}^{k-1}}{T_{0,n}^k} \\ &= \lim_{n\to\infty} \frac{\sum_{s=0}^{n} \theta_s a_{s,k-1}\, T_{0,s}^{k-1}}{k T_{0,n+1}^{k-1} \sum_{s=0}^{n} \theta_s} \\ &= \lim_{n\to\infty} \frac{\theta_n a_{n,k-1} T_{0,n}}{k[(k-1)T_{0,n}^{k-2} T_{n,n+1}^2 \lambda_n \theta_n + T_{0,n}^{k-1}\theta_n]} \\ &= \lim_{n\to\infty} \frac{a_{n,k-1}}{k\left[(k-1)\frac{\lambda_n T_{n,n+1}^2}{T_{0,n}} + 1\right]} = \frac{\gamma a_{k-1}}{k(k-1+\gamma)} = a_k.\end{aligned} \tag{7.7}$$

We have twice applied Stolz's theorem, using here the condition (7.1). Thus

$$a_k = \frac{\gamma a_{k-1}}{k(k-1+\gamma)} = \frac{\gamma^k \Gamma(\gamma)}{k!\,\Gamma(k+\gamma)}, \tag{7.8}$$

from which it follows, taking account of the inequality (3.18), that

$$\lim_{n\to\infty} \Delta_n\left(\frac{z}{T_{0,n}}\right) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma)(\gamma z)^k}{k!\,\Gamma(k+\gamma)} = \Phi(z,\gamma) \tag{7.9}$$

uniformly on any finite interval. Therefore,

$$\lim_{n\to\infty} P\left\{\frac{\tau_{0,n}}{T_{0,n}} < x\right\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zx}}{z\Phi(z,\gamma)}\,dz, \tag{7.10}$$

which proves the theorem.

REMARK 7.1. It follows from the hypothesis of the theorem that the terms in the sum $\tau_{0,n} = \tau_{0,1} + \tau_{1,2} + \cdots + \tau_{n-1,n}$ are uniformly small. Therefore, the distribution (7.4) is infinitely divisible.

REMARK 7.2. For the extreme values, we have

$$\Phi(z,0) = 1 + z, \qquad \Phi(z,\infty) = e^z, \tag{7.11}$$

which corresponds to the exponential and the identity distribution.

REMARK 7.3. In a similar way, one can show that if the conditions (7.1), (7.2), (7.3) are satisfied, then

$$\lim_{n\to\infty} \delta_n\left(\frac{z}{T_{0,n}}\right) = \Phi'_z(z,\gamma), \tag{7.12}$$

from which it follows that

$$\lim_{n\to\infty} P\left\{\frac{\tau_n}{T_{0,n}} < x\right\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Phi'_z(z,\gamma)}{z\Phi(z,\gamma)}\, e^{zx}\,dz, \tag{7.13}$$

and

$$\lim_{n\to\infty} P\left\{\frac{\tau_{n-1,n}}{T_{0,n}} > x\right\}\frac{T_{0,n}}{T_{n-1,n}} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Phi'_z(z,\gamma)}{\Phi(z,\gamma)}\, e^{zx}\,dz. \tag{7.14}$$

In concluding this paper, we note that the results of Section 4 (bounds on the distribution of $\tau_{0,n}$) are obviously valid also for an arbitrary Markov process with a finite number of states for which

$$M \exp\{-z\tau_{0,n}\} = \frac{1}{P(z)}, \tag{7.15}$$

where $P(z)$ is a polynomial, all of whose roots are negative. The equality (7.15) will, for example, be satisfied for a process in which instantaneous transitions upward can only occur by one state at a time. In the general case, where

$$M \exp\{-z\tau_{0,n}\} = \frac{Q(z)}{P(z)} \tag{7.16}$$

and the roots of the polynomial $P(z)$ are negative, the asymptotic behavior of the variables $\tau_{0,n}$ can be investigated by our methods, if certain restrictions are imposed upon the growth of the polynomial $Q(z)$.

REFERENCES

[1] D. R. Cox, *Renewal Theory*, London, Methuen; New York, Wiley, 1962.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, New York, Wiley, 1968 (3rd ed.).

[3] G. M. Fichtenholz, *Course in Differential and Integral Calculus*, Moscow and Leningrad, Fizmatgiz, 1963. (In Russian.)

[4] B. V. Gnedenko, Yu. K. Belyayev, and A. D. Soloviev, *Mathematical Methods of Reliability Theory* (edited by Richard Barlow), New York, Academic Press, 1969.

[5] S. Karlin and J. L. McGregor, "The differential equations of birth-and-death processes, and the Stieltjes moment problem," *Trans. Amer. Math. Soc.*, Vol. 85 (1957), pp. 489–546.

[6] I. P. Natanson, *Constructive Function Theory* (translated by J. R. Schulenberger), New York, Ungar, 1965.