LIMIT THEOREMS FOR RANDOM WALKS WITH BOUNDARIES

A. A. BOROVKOV
INSTITUTE OF MATHEMATICS
NOVOSIBIRSK, U.S.S.R.

1. Introduction

In this review, we consider boundary problems for random walks generated by sums of independent items and some of their generalizations.

Let $\xi_1, \xi_2, \cdots$ be identically distributed independent random variables with distribution frunction $F(x)$. Let $S_0 = 0$, $S_n = \Sigma_{k=1}^n \xi_k$ with $n = 1, 2, \cdots$. We shall study the properties of the random trajectory formed by the sums $S_0, S_1, S_2, \cdots$. Let n be an integer parameter and let $g_n^{\pm}(t)$ be two functions on the real axis with the following properties:

$$g_n^+(0) > 0 > g_n^-(0), \qquad g_n^+(t) > g_n^-(t), \qquad t \geqq 0. \tag{1.1}$$

We shall denote by G_n the part of the halfplane $(t \geqq 0, x)$ which lies between these two curves. In the same halfplane (t, x), let us consider the trajectory formed by the points

$$\left(\frac{k}{n}, S_k\right), \qquad k = 0, 1, 2, \cdots \tag{1.2}$$

One of the main boundary functionals of trajectory (1.2) is the time η_G at which it leaves the region G_n:

$$\eta_G = \min \left\{\frac{k}{n} : \left(\frac{k}{n}, S_k\right) \notin G_n\right\}. \tag{1.3}$$

We shall define the value of the first jump χ_G across the boundary of the region G_n by the equalities

$$\chi_G = S_{\eta_G} - g_n^+(\eta_G) \quad \text{or} \quad \chi_G = S_{\eta_G} - g_n^-(\eta_G), \tag{1.4}$$

depending on whether trajectory (1.2) crosses the upper or lower boundary of the region G_n. Note that in general the random variables η_G and χ_G are not defined on the whole space of elementary events. We put $\eta_G = \infty$, where η_G remains undefined. We shall not define the random value χ_G on the set $\{\eta_G = \infty\}$.

Problems variously connected with distributions of the functionals η_G and χ_G will be called boundary problems for random walks. It is well known that these problems play an important part in mathematical statistics (in sequential analysis, nonparametric methods, and so forth) in queueing theory, and in other similar fields of mathematics. For instance, the classical problem of sequential analysis leads to the elucidation of probabilities of the type

$$P(\chi_G > 0; \eta_G < \infty), \tag{1.5}$$

for boundaries $g_n^{\pm}$ which are straight line boundaries. This is the probability of the event that the trajectory (1.2) crosses the upper boundary earlier than the lower one. The distribution of the maximum, $\bar{S}_n = \max_{0 \leqq k \leqq n} S_k$, is of great interest in queueing theory. It is evident that this distribution is also connected with boundary problems because

$$P(\bar{S}_n > x) = P(\eta_G < 1), \tag{1.6}$$

where the region G_n is formed by the straight lines $g_n^+(t) = x$ and $g_n^-(t) = -\infty$. One could give examples of still more complicated applied and theoretical problems leading to boundary problems.

Considering limit theorems for boundary problems (that is, methods of the approximate calculation of distribution η_G, χ_G, $\bar{S}_n$ when $n \to \infty$), we shall distinguish rather conditionally between the regions of *normal* and *large* deviations. If the limit values of the probabilities in question are nondegenerate (different from 0 and 1), then we shall refer to all these cases as problems about normal deviations. The remaining cases will be referred to as problems on large deviations.

2. Normal deviations

The most important problem occurs when

$$E\xi_k = 0, \qquad D\xi_k < \infty, \qquad g_n^{\pm}(t) = \sqrt{n}\, g^{\pm}(t). \tag{2.1}$$

Here the $g^{\pm}(t)$ possess the properties (1.1) and are continuous. Without restricting generality, we may assume $\mathscr{D}\xi_k$ to be equal to 1. It is well known that in this case

$$\begin{aligned} \lim_{n\to\infty} P(\eta_G < v) &= p(v, g^+, g^-) \\ &= 1 - P\big(g^-(t) < w(t) < g^+(t); 0 \leqq t \leqq v\big), \end{aligned} \tag{2.2}$$

where $w(t)$ is a standard Brownian motion process.

Similar limits also exist for probabilities of the more general type,

$$P\left(\eta_G < v, \chi_G > 0, \frac{S_{[nv]}}{\sqrt{n}} \in \Delta\right). \tag{2.3}$$

These limits are equal to the probabilities $p_1(v, g^+, g^-, \Delta)$ that the trajectory $w(t)$ during time v left the band

$$G = \{(t, x): g^-(t) < x < g^+(t)\} \tag{2.4}$$

across the upper boundary and that the value $w(v)$ belongs to the interval Δ.

It is also possible to show that when $y > 0$,

$$\lim_{n\to\infty} P(\eta_G < v, \chi_G > y) = p_1(v, g^+, g^-, R)P(y), \tag{2.5}$$

where $R = (0, \infty)$ and $P(y) = \lim_{n\to\infty} P(\chi_{\bar{G}_n} > y)$ for the regions $\bar{G}_n$ generated by the straight lines $g_n^+(t) = n$ and $g_n^-(t) = -\infty$. The latter is the distribution of the first jump of sequence $\{S_k\}$ across the infinitely remote positive barrier.

Assertion (2.2) was obtained by Kolmogorov in 1931. Since that time, it has become clear that the problem of estimating the rate of convergence of distributions (2.2) and (2.3) to their limits is very complicated. The following result by S. V. Nagaev [1] is the most general in this direction.

THEOREM 2.1. *Let conditions* (2.1) *hold, let* $c_3 = E|\xi_k|^3 < \infty$, *and assume that the functions* $g^\pm(t)$ *satisfy the Lipschitz condition with constant* L. *Then an absolute constant* A *exists such that*

$$|P(\eta_G < v) - p(v, g^+, g^-)| < \frac{Ac_3^2(L+1)}{\sqrt{n}}. \tag{2.6}$$

The method of proving this theorem allows one to obtain the same estimates for the rate of convergence to their limits of the probabilities (2.3).

Obtaining the asymptotic expansions in powers of $1/\sqrt{n}$, requires more special assumptions concerning the distribution $F(x)$ and the form of the boundaries $g^\pm(t)$. One may find a rather comprehensive review of the achievements obtained in this field before 1964 in the paper by A. A. Borovkov and V. S. Koroljuk [3]. For instance, as asymptotic expansion of $P(\eta_G < v)$ of the form

$$\sum_{k=0}^{s} p_k n^{-k/2} + o(n^{-s/2}) \tag{2.7}$$

is possible if $E|\xi|^{2s+6} < \infty$, and the density of distribution $F(x)$ and the boundaries $g^\pm(t)$ have a sufficiently high degree of smoothness (Koroljuk).

The assumptions for one straight boundary, $g_n^+(t) = x = x(n)$, are more economical. If $F(t)$ has an absolutely continuous component and $E\exp\{\lambda\xi_k\} < \infty$ for $|\lambda| < \varepsilon$ for some $\varepsilon > 0$, then probabilities (2.3) admit the full asymptotic expansion in powers of x/n and $1/\sqrt{n}$ (Borovkov). If, in the last assertion, instead of finiteness of $E\exp\{\lambda\xi_k\}$, we demand only the existence of a finite number of moments, then the following result obtained recently by Nagaev will hold [4].

THEOREM 2.2. *Let* $F(t)$ *have an absolutely continuous component and suppose that* $c_s = E|\xi_k|^s < \infty$, *for* $s > 3$. *Then* $(g^+(t) = x, g^-(t) = -\infty)$

$$\begin{aligned} P(\eta_G > 1) &= P(\bar{S}_n < x\sqrt{n}) \\ &= \sqrt{\frac{2}{\pi}} \int_0^x \exp\{-\tfrac{1}{2}t^2\}\, dt + \exp\{-\tfrac{1}{2}x^2\} \sum_{j=1}^{s-3} \Pi_j(x) n^{-j/2} \\ &\qquad + O(\min[n^{-1/2}, (1 + x^{1-s})n^{-(s-2)/2}\log^2 n]), \end{aligned} \tag{2.8}$$

where $\Pi_j(x)$ *are some polynomials in* x.

A similar but somewhat weaker result would hold if we required that

$$\limsup_{|\lambda|\to\infty} |E \exp\{i\lambda\xi_k\}| < 1, \tag{2.9}$$

instead of the existence of an absolutely continuous component.

The nature of the coefficients of polynomials Π_j is rather complicated ([4], [5]).

In the assertion mentioned above, we considered the event $\{\eta_G > v\}$ for $v = 1$. In this regard, we remark that the generality achieved by considering the events $\{\eta_G > v\}$ for arbitrary finite v is illusory. One can restrict oneself to the value $v = 1$. Indeed, $n\eta_G$ is an integer, but for integer nv, the event $\{\eta_G > v\}$ can be written as $\{\eta_G > 1\}$ for a new value of parameter $\tilde{n} = nv$ and new functions $\tilde{g}^{\pm}(t) = g^{\pm}(tv)v^{-1/2}$.

3. Large deviations

We arrive at the problem of large deviations if, for example, $E\xi_k = 0$, $D\xi_k = 1$, $g_n^{\pm}(t) = x(n)g^{\pm}(t)$, $x(n)/\sqrt{n} \to \infty$ with $n \to \infty$. In real problems, one can usually reduce determination of the asymptotic behavior of probabilities of the form

$$P\left(\eta_G < 1, \chi_G < 0, \frac{S_n}{x} \in \Delta\right), \qquad x = x(n), \tag{3.1}$$

or

$$P\left(\eta_G < 1, \chi_G > 0, \frac{S_n}{x} \in \Delta\right), \qquad x = x(n), \tag{3.2}$$

to the same boundary problems but with one boundary. For example, if $g^+(t) > 0$ and $g^-(t) < 0$, it is easy to see that

$$\begin{aligned} P(\eta_G < 1) \sim\ & P\left\{\max_{k\leqq n}\left\{S_k - xg^+\left(\frac{k}{n}\right)\right\} > 0\right\} \\ & + P\left\{\min_{k\leqq n}\left\{S_k - xg^-\left(\frac{k}{n}\right)\right\} < 0\right\}. \end{aligned} \tag{3.3}$$

Here, the relationship $a_n \sim b_n$ means that $a_n/b_n \to 1$ for $n \to \infty$. Accordingly, we shall restrict ourselves in this section to the case of one boundary when $g^-(t) = -\infty$.

Depending on the type of function $g^+(t) = g(t)$, when evaluating $P(\eta_G < 1)$ asymptotically, we arrive at two qualitatively different types of problems.

The first type of problem arises when $g(t) > 0$. In this case $P(\eta_G \leqq 1) \to 0$ with $n \to \infty$. Asymptotics of this probability are investigated in detail in [6]. The so called *level curves* play the most important part in their description. In order to describe these curves, we introduce the *deviation function*

(3.4) $$\Lambda(\alpha) = -\inf(-\lambda\alpha + \log\varphi(\lambda)), \qquad \varphi(\lambda) = E\exp\{\lambda\xi_k\}.$$

The function $\Lambda(\alpha) \geqq 0$ is defined for all real α. We denote

(3.5) $$\begin{aligned} \lambda_+ &= \sup\{\lambda: \varphi(\lambda) < \infty\}, & \lambda_- &= \inf\{\lambda: \varphi(\lambda) < \infty\}, \\ \alpha_+ &= \lim_{\lambda\uparrow\lambda_+}[\log\varphi(\lambda)]', & \alpha_- &= \lim_{\lambda\downarrow\lambda_-}(\log\varphi(\lambda))'. \end{aligned}$$

If $\lambda(\alpha)$ is a point where $\inf(-\lambda\alpha + \log\varphi(\lambda))$ is achieved, then $\Lambda(\alpha)$ may be expressed as

(3.6) $$\Lambda(\alpha) = \int_a^\alpha \lambda(u)\,du, \qquad a = E\xi_k.$$

From this it easily follows that $\Lambda(\alpha)$ is a convex function, achieving its minimum equal to 0 at $\alpha = a$. In the regions $(-\infty, \alpha_-)$, (α_+, ∞), the function $\lambda(\alpha)$ is constant and equals $\lambda_\mp$ respectively. Thus, the function $\Lambda(\alpha)$ is analytic in each of three regions, $(-\infty, \alpha_-)$, (α_-, α_+), (α_+, ∞). Discontinuity, for example, at the point α_+ is possible only if $\lambda_+ = \infty$, $\alpha_+ < \infty$ (the variable ξ_k is bounded from above by the value α_+) and $P(\xi_k = \alpha_k) > 0$. If the $\lambda_\pm$ are finite, then $\Lambda(\alpha)$ together with its first derivatives will be continuous at the points $\alpha_\pm$.

It is not difficult to find the expansion of the function $\Lambda(\alpha)$ valid in the neighborhood of the point $\alpha = a$, with coefficients of $(\alpha - a)^k$ defined by k semi-invariants of the distribution ξ.

For example, for the normal distribution $\varphi(\lambda) = \exp\{\frac{1}{2}\lambda^2\}$, $\alpha_\pm = \pm\infty$ and $\lambda(\alpha) = \alpha$.

For the Bernoulli scheme, $\varphi(\lambda) = pe^\lambda + qe^{-\lambda}$,

(3.7) $$\alpha_\pm = \pm 1, \qquad \lambda(\alpha) = \tfrac{1}{2}\log\frac{q(1+\alpha)}{p(1-\alpha)}.$$

For the centered Poisson distribution with parameter μ,

(3.8) $$\alpha_+ = \infty, \qquad \alpha_- = -\mu, \qquad \lambda(\alpha) = \log\frac{\mu+\alpha}{\mu}.$$

The probabilistic meaning of the deviation function is given by the equality

(3.9) $$\Lambda(\alpha) = -\lim_{\Delta_\alpha\to 0}\lim_{n\to\infty}\frac{1}{n}\log P\left(\frac{S_n}{n}\in\Delta_\alpha\right),$$

where Δ_α is a shrinking neighborhood of the point α.

The "inversion formula,"

(3.10) $$\varphi(\lambda) = \exp\left\{-\int_0^\lambda \theta(t)\,dt\right\},$$

is also valid, where $\theta(t)$ is the inverse function of

(3.11) $$t = \lambda(\theta) = \frac{\partial\Lambda(\theta)}{\partial\theta}.$$

In what follows, we assume that $a = E\xi_k = 0$, $D\xi_k = 1$, $\lambda_+ > 0$, $\lambda_- < 0$. We call the positive solution (if it exists) for the functional equation

$$t\Lambda\left(\frac{a}{t}\right) = \Lambda(\tau), \qquad 0 < t \leqq 1, \tag{3.12}$$

the *level curve* $a_\tau(t)$, depending on parameter τ.

If ξ is such that $\Lambda(\alpha_+) = \infty$, then this equation has for each τ a unique solution $\alpha_\tau(t)$, which is a convex increasing function.

If $\Lambda(\alpha_+) < \infty$, then $a_\tau(t)$, possesses the indicated properties only if

$$t \in (t_\tau, 1), \qquad t_\tau = \frac{\Lambda(\tau)}{\Lambda(\alpha_+)}. \tag{3.13}$$

When $t \in (0, t_\tau)$, the function $a_\tau(t)$ is to be defined as a segment of the straight line connecting the point $(0, 0)$ with the end of the curve $a_\tau(t)$ at the point t_τ.

The functions $\alpha_\tau(t)$ are also increasing functions of τ. For τ small, $a_\tau(t) \sim \tau\sqrt{t}$. (The exact equality, $a_\tau(t) = \tau\sqrt{t}$, is true only for the normal distribution law.)

We may return now to the asymptotic description of $P(\eta_G < 1)$. To simplify the formulations of the results, we put $x(n) = n$ (if $x = \varepsilon n$, one may consider the function $g^*(t) = \varepsilon g(t)$ and $x^*(n) = n$). The main role is played by the maximal value of the parameter $\tau = \tau_g$ at which the curve $a_\tau = a_{\tau_g}$ first intersects the curve $g(t)$ as τ increases. The important fact is how this contact happened.

The following theorem is true.

THEOREM 3.1. *Let the set B of the points t, where the values $a_{\tau_g}(t)$ and $g(t)$ coincide, be contained in the interval $(t_{\tau_g}, 1)$ and be such that* mes $B > 0$. *Then*

$$P(\eta_G < 1, \chi_G < y) \sim \phi_1(F, y, B)\sqrt{n}\exp\{-n\Lambda(\tau_g)\}, \tag{3.14}$$

where $\phi_1(F, y, B)$ is a functional of known form.

Now let the point of contact v be a single one and in the neighborhood of this contact let the function $g(t)$ be p times differentiable. Let q be the number of initial derivatives of the functions $g(t)$ and $a_{\tau_g}(t)$ which are identical. Assume $p > q + 1$, then

$$P(\eta_G < 1, \chi_G < y) \sim \phi_2(F, y, v, g)n^{1/2 - 1/(q+1)}\exp\{-n\Lambda(\tau_g)\}, \tag{3.15}$$

where $\phi_2 > 0$ is also a known functional. However, unlike ϕ_1, the functional ϕ_2 has a local character with respect to g, since it depends only on $q + 1$ derivatives of the function $g(t)$ at the point v. When $x = o(n)$, the points of contact v of the functions $(x/n)g(t)$ and $a_{\tau_{xg/n}}(t)$ are, in general, mobile (they change with n) and the formulations of the results become more delicate, although the character of the dependence of the asymptotics on the value τ_g and on the type of contact remains as before.

The case of several points of contact is easily reduced to the case of a single point, considered in Theorem 3.1.

It is possible to make the analogous asymptotic analysis for the probabilities

$$P\left(\eta_G < 1, \chi_G < y, \frac{S_n}{x} \in \Delta\right). \tag{3.16}$$

The main factor here is the mutual disposition of curve $g(t)$ and the level curves; however, the level curves themselves are defined differently and already depend on two parameters rather than on one (see [6]).

4. Large deviations, special case

Consider the special case with a straight line boundary $g(t) = d + bt$. With the help of transformation of reversion it is always possible here to reduce the problem on distribution η_G to the investigation of the properties of $\bar{S}_n = \max_{0 \leqq k \leqq n} S_k$. The interesting case is that in which it happens that $E\xi_k = a < 0$ after the reversion, and therefore $\bar{S} = \bar{S}_\infty < \infty$ with probability 1. As mentioned above, a complete analysis of the asymptotic properties of the $\bar{S}_n$ distribution is in [5]. To those properties we add the following remarks (see [5], [7]). (The symbols c denotes different constants.)

THEOREM 4.1. *Suppose that $F(x)$ has an absolutely continuous component, and that there exists a root $q > 0$ of the equation $\varphi(\lambda) = 1$. Further, let $x/n \sim \alpha$, $\alpha_0 = \varphi'(q)$. Then if $\alpha < \alpha_0$, for a certain $\varepsilon > 0$,*

$$P(\bar{S}_n > x) \sim P(\bar{S} > x) = c \exp\{-qx\}(1 + o(\exp\{-\varepsilon x\})). \tag{4.1}$$

If $\alpha_0 < \alpha < \alpha_+$, then

$$P(\bar{S}_n > x) \sim c_1(\alpha) P(S_n > x) \sim \frac{c_2(\alpha)}{\sqrt{n}} \exp\left\{-n\Lambda\left(\frac{x}{n}\right)\right\}. \tag{4.2}$$

If $\alpha = \alpha_0(\Lambda(\alpha_0) = q)$, then the transition from one sort of asymptotics to another occurs with the help of the normal distribution function ϕ: for $u = (\sqrt{n}/\sigma)(x/n - a_0) = o(n^{1/6})$,

$$P(\bar{S}_n > x) = P(\bar{S} > x)\left[\phi(u) + \exp\{-\tfrac{1}{2}u^2\} \sum_{j=1}^{\infty} \frac{\Pi_{3j-1}(x)}{n^{j/2}}\right], \tag{4.3}$$

where σ is the variance of the distribution $F_q(A) = \int_A \exp\{qx\}\, dF(x)$, and $\Pi_k(u)$ are polynomials of degree k.

Since the asymptotic behavior of $P(S_n > x)$ has been studied well, to describe the asymptotic behavior of $P(\bar{S}_n > x) > P(S_n > x)$ in the general case (when the conditions of Theorem 4.1 are not satisfied), it is sufficient to know (1) the rate of the convergence of distributions $\bar{S}_n$ and $\bar{S}$, and (2) the asymptotic behavior of $\bar{S}$. The answer to the first question is in Theorem 4.2.

THEOREM 4.2. *For any λ such that $\varphi(\lambda) \leqq 1$,*

$$P(\bar{S} > x) - P(\bar{S}_n > x) \leqq \exp\{-\lambda x\}\varphi^n(\lambda). \tag{4.4}$$

(*One can write* $\exp\{-n\Lambda(x/n)\}$ *on the right side of* (4.4) *if* $\varphi[\lambda(x/n)] \leqq 1$.)

If $E\xi_k = a < 0$, $D\xi_k = 1$, *and* $c_m^+ = E[\max(0, \xi)]^m < \infty$ *for* $m \geqq 2$, *then* $\bar{c}_{m-1} \equiv E\bar{S}^{m-1} < \infty$ *and for all* $n \geqq n_0$,

$$P(\bar{S} > x) - P(\bar{S}_n > x) = \frac{2^{m+1} c_m^+ e}{|a|^m n^{m-1}} + \frac{\bar{c}_{m-1}}{\left(x + \frac{1}{2}|a|n\right)^{m-1}}, \tag{4.5}$$

where $n_0 = n_0(m, c_m)$ *is known explicitly.*

Concerning the asymptotic properties of the distribution of $\bar{S}$, it is well known that if $\lambda_+ > 0$, $\varphi(\lambda_+) \geqq 1$, then $P(\bar{S} > x) \sim c_1 e^{-qx}$ with $x \to \infty$ (at $\lambda_+ = q$ we also suppose that $\varphi'(\lambda_+) < \infty$.) Supplement this assertion by the following one.

THEOREM 4.3. *If* $\varphi(\lambda_+) < 1$ *or* $\lambda_+ = 0$, $\varphi'(0) = a > -\infty$, *then* $P(\bar{S} > x) \sim c_2 H(x)$, *where* $c_2 = |a|^{-1}$ *for* $\lambda_+ = 0$, *and where* $H(x) = \int_x^\infty (1 - F(t))\, dt$.

It is assumed here that for any $0 < h \leqq 1$ *and* $t \to \infty$,

$$\exp\{h\lambda_+\} \frac{H(t + h)}{H(t)} \to 1, \qquad 0 < \frac{\exp\{tb\lambda_+\} H(tb)}{\exp\{t\lambda_+\} H(t)} < c(b) < \infty. \tag{4.6}$$

Concerning the distribution of $\bar{S}$, we shall also note that there exists a class of distributions $\mathscr{R}$ everywhere dense in the sense of weak convergence in the set of all distributions, and such that for $F \in \mathscr{R}$ the distribution of $\bar{S}$ can be found in explicit form. This fact is used in conjunction with the "continuity theorem" showing when the nearness of F_1 and F_2 (in the sense of weak convergence) implies the nearness of corresponding suprema. The latter will hold if the value

$$\int_{-\infty}^{\infty} a(t) |F_1(t) - F_2(t)|\, dt \tag{4.7}$$

is small for a function $a(t) > 0$ such that $a(t) = 1$ when $t > 0$, $\int_{-\infty}^0 a(t)\, dt < \infty$. By itself, weak convergence, $F_1 \Rightarrow F_2$, is not sufficient for the convergence of suprema distributions.

The class $\mathscr{R}$, mentioned above, contains all the distributions for which either $\varphi_+(\lambda) = E(e^{\lambda\xi}; \xi \geqq 0)$ or $\varphi_-(\lambda) = E(e^{\lambda\xi}; \xi \leqq 0)$ are rational functions.

Results close to the latter were also obtained by H. J. Rossberg [8], [9] under some special conditions.

5. The second type of problem: problems on large deviations

Now let us return to the general boundary problems for large deviations. In Section 3, we considered the case where $g(t) > 0$, $t \in [0, 1]$. If $\inf_{(0,1)} g(t) < 0$, then $P(\eta_G \leqq 1) \to 1$, when $n \to \infty$ (here again $E\xi_k = 0$) and we shall investigate the rate of convergence to zero of the complementary probability $P(\eta_G > 1)$. The asymptotic nature of this probability appears to be quite different and significantly more complicated. Here one can find only the asymptotic behavior for the *logarithm* of this probability $P(\eta_G > 1)$.

Suppose $g(t)$ has no discontinuities of the second kind and for every t, $g(t) = \min\big(g(t-0), g(t+0)\big)$. Let t_0 be the point where $\inf_{(0,1)} g(t) = g(t_0)$ and let B_0 be the set of points $(0, 0)$, $\big(t, y > g(t)\big)$ for $0 \leqq t \leqq t_0$. We denote by $h(t)$ the lower boundary of the convex closure of the set B_0. The curve $h(t)$ evidently realizes the shortest path from point $(0, 0)$ to point $\big(t_0, g(t_0)\big)$ which does not intersect B_0. This function will be convex and absolutely continuous. If we put $h(t) = g(t_0)$ on $[t_0, 1]$, then $h(t)$ will keep this property. For such functions $h(t)$, the functional

$$W(h) = \int_0^1 \Lambda\left(\frac{dh}{dt}\right) dt = \int_0^{t_0} \Lambda\left(\frac{dh}{dt}\right) dt \tag{5.1}$$

is defined. First let $x = x(n) \sim n$ when $n \to \infty$. (As already noted, one can reduce the case $x \sim \varepsilon n$ to $x \sim n$ by changing the boundary.)

THEOREM 5.1. *If $h'(0) > -\infty$ and the interval $\big(h'(0), 0\big)$ contains no point of discontinuity of the function $\Lambda(\alpha)$, then $\log P(\eta_G > 1) \sim -nW(h)$.*

To describe the asymptotic form of the distribution η_G in the problem with a fixed terminal value, that is, the probability of the event $\{\eta_G > 1, S_n/x \in \Delta_b\}$ (where Δ_b is the neighborhood of the point b), it is necessary to construct the shortest path $h_b(t)$ connecting the points $(0, 0)$ with $(1, b)$ and not intersecting the set B_1 of the points $\big(t, y > g(t)\big)$, $0 \leqq t \leqq 1$. (This path will evidently coincide with $h(t)$ in the segment $[0, t_0]$ if $bt_0 \geqq g(t_0)$.)

THEOREM 5.2. *If Δ_b is a shrinking neighborhood of the point b,*

$$\lim_{\Delta_b \to 0} \lim_{n \to \infty} \frac{1}{n} \log P\left(\eta_G > 1, \frac{S_n}{x} \in \Delta_b\right) = -W(h_b). \tag{5.2}$$

If ξ_k is lattice-like and $P(S_n = bx) > 0$, then from the very beginning we may take a single point b as a Δ_b.

Now let us consider the case $x = o(n)$. Introduce the functional

$$V(h) = \int_0^1 \left(\frac{dh}{dt}\right)^2 dt = \int_0^1 \frac{dh}{dt}\, dh(t). \tag{5.3}$$

THEOREM 5.3. *If $x/n \to 0$, $x(n \log n)^{-1/2} \to \infty$, with $|\lambda_\pm| > 0$, then*

$$\log P(\eta_G > 1) \sim -\frac{x^2}{2n} V(h). \tag{5.4}$$

The analogous result is true for $P(\eta_G > 1, S_n/x \in \Delta_b)$.

These theorems, obtained in [10], show what the *exponential* part in the probability $P(\eta_G > 1)$ is. As for the *power* factor, we may conclude on the basis of different examples that it may be of any form depending on the corresponding function $g(t)$. It is possible to calculate this factor only for very special kinds of boundaries $g(t)$, for example, for the broken lines consisting of a finite number of straight line segments.

6. Generalization

It is natural to consider the next generalization of these problems. For example, some asymptotic problems of testing theory for statistical hypotheses lead to this. Denote by $S_n(t)$ a random broken line connecting the points $(k/n, S_k/x)$ for $k = 0, 1, \cdots, n$. In the previous parts, the asymptotics of

$$\log P(S_n(t) \in G) \tag{6.1}$$

were investigated when G is the set of functions $f(t) < g(t)$, $t \in [0, 1]$.

The question concerns the behavior of (6.1) when $n \to \infty$ if G is the arbitrary open set in the space $C(0, 1)$.

The following theorems hold here [10]. Denote by $\bar{G}$ the closure of G in the metric of space $C(0, 1)$ and by $E \subset C(0, 1)$ the set of all absolute continuous functions $f(t)$, $f(0) = 0$, for which there exists a finite number of intervals $\Delta_1, \Delta_2, \cdots, \Delta_r; \cup \bar{\Delta}_j = [0, 1]$, where $f'(t)$ is monotone and bounded. Obviously, $\Lambda(f'(t))$ is Riemann integrable for $f \in E$. Further, let

$$\begin{aligned} W(G) &= \inf_{f \in E \cap G} W(f) = \inf_{f \in E \cap G} \int_0^1 \Lambda(f'(t))\, dt, \\ V(G) &= \inf_{f \in E \cap G} V(f) = \inf_{f \in E \cap G} \int_0^1 (f'(t))^2\, dt. \end{aligned} \tag{6.2}$$

THEOREM 6.1. *Let $x \sim n$ and $W(G) = W(\bar{G})$. If $|\lambda_{\pm}| = \infty$, then*

$$\log P(S_n(t) \in G) \sim -nW(G). \tag{6.3}$$

THEOREM 6.2. *Let $x/n \to 0$, $x(n \log n)^{-1/2} \to \infty$, $V(G) = V(\bar{G})$. If $|\lambda_{\pm}| > 0$, then*

$$\log P(S_n(t) \in G) \sim -\frac{x^2}{2n} V(G). \tag{6.4}$$

The condition $|\lambda_{\pm}| = \infty$ in Theorem 6.1 implies that $\varphi(\lambda)$ is an entire function. We are quite sure that this requirement is unnecessary and that the theorem will hold with finite $\lambda_{\pm}$, but this is only a conjecture. We can give here only one sufficient condition. Namely, if $W(G) = W(\bar{G})$ and there exists a compact K in $C(0, 1)$ such that when $n \to \infty$

$$\log P(S_n \in G \cap K) > \log P(S_n(t) \in G) + o\left(\frac{x^2}{n}\right), \tag{6.5}$$

then (6.3) is true.

Relations (6.3) and (6.4) remain valid in problems with a fixed terminal value (see Theorem 5.2), but it is necessary to take the infimum of the functionals W, V for $f \in E_b \cap G$, where $E_b \in E$ contains only the functions for which $f(1) = b$.

Many of the results given remain valid also for the processes with independent increments. B. A. Rogozin [11] extends to these cases the theorems on asymptotic expansions and large deviations for the maximum $\bar{S}_n(t)$, where

$$\bar{S}_n(t) = \sup_{0 \leq u \leq t} S_n(u), \qquad S_n(t) = \frac{X(nt)}{x}, \tag{6.6}$$

and $X(u)$ is a process with independent increments, satisfying rather weak conditions.

If $X(u)$ is the sum of a Wiener process and of a generalized Poisson process, that is, if

$$\psi(\lambda) = \log E \exp\{\lambda X(1)\} = \lambda\theta - \tfrac{1}{2}\sigma^2\lambda^2 + \beta \int (\exp\{\lambda u\} - 1)\, dN(u), \tag{6.7}$$

where $N(u)$ is a distribution function, $\beta \geqq 0$, then the assertions of Theorem 6.1 and 6.2 remain completely valid, where G denotes now the set from the space $D(0, 1)$ containing with f its ρ_c neighborhood $\rho_c(f, g) = \sup_{[0,1]}|f(t) - g(t)|$. The closure $\bar{G}$ is used in the sense of ρ_c convergence. Then if $W(G) = W(\bar{G})$, $\psi(\lambda)$ is an integral function, and for $x \sim n$, we have

$$\log P(S_n(t) \in G) \sim -nW(G). \tag{6.8}$$

The case where $x = o(n)$ is similar. From this theorem one can derive, in particular, the theorem of I. N. Sanov [12] on large deviations of empirical distribution functions. Since this theorem was obtained by us under somewhat different conditions, we shall give it here. Let $F_n(t)$ be an empirical function constructed by n independent observations of a random variable with the continuous distribution function $F(t)$. And let G be a measurable ρ_c open set in $D(0, 1)$. Put

$$W_F(G) = \inf_{f \subset G \cap E_F} \int \log \frac{df}{dF}\, df, \qquad V_F(G) = \inf_{g \in E_F \cap G} \int \frac{dg}{dF}\, dg, \tag{6.9}$$

where E_F is the set of distribution functions g absolutely continuous with respect to F, and such that there exists a finite number of intervals $\Delta_1, \cdots, \Delta_r, \cup \bar{\Delta}_j = [-\infty, \infty]$, where dg/dF is monotone and bounded; $\bar{G}$ denotes the ρ_c closure of G.

THEOREM 6.3. *If* $W_F(G) = W_F(\bar{G})$, *then* $\log P(F_n(t) \in G) \sim -nW_F(G)$.

The following affirmation is also valid.

THEOREM 6.4. *If* $x/n \to 0$, $x(n \log n)^{-1/2} \to \infty$, $V_F(G) = V_F(\bar{G})$, *then*

$$\log P\left(F_n(t) - F(t) \in \frac{x}{n} G\right) \sim \frac{x^2}{2n}(1 - V_F(G)). \tag{6.10}$$

Theorems of this kind are essential in mathematical statistics.

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