ON UNIQUE ERGODICITY

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1. Introduction

A homeomorphism U of a compact metric space onto itself is said to be uniquely ergodic if it possesses a unique invariant Borel probability measure μ_U. For an introduction to the theory of unique ergodicity, we refer the reader to J. Oxtoby [11]. A point x in the shift space $\Omega^{\mathbf{Z}}$, where Ω is a finite state space, is called a uniquely ergodic sequence if the shift S, $(Sx)_i = x_{i+1}$, where $i \in \mathbf{Z}$, $x \in \Omega^{\mathbf{Z}}$, is a uniquely ergodic homeomorphism of the orbit closure $\mathcal{O}_x = \overline{\{S^i x : i \in \mathbf{Z}\}}$ of x. We denote the shift invariant probability measure of a uniquely ergodic sequence x by μ_x.

S. Kakutani [7], M. Keane [8], and K. Jacobs and M. Keane [5] have constructed a variety of uniquely ergodic sequences and investigated their measure theoretic properties. The first examples of weakly mixing uniquely ergodic systems were given by Jacobs [3]. F. Hahn and Y. Katznelson [2] constructed uniquely ergodic sequences with arbitrarily high entropy and Ch. Grillenberger [1] produced uniquely ergodic sequences in $\Omega^{\mathbf{Z}}$ whose entropy is arbitrarily close to $\log |\Omega|$. Further constructions of uniquely ergodic sequences were given by W. Veech (Section 3 of [12]).

We shall prove in Section 3 that for every ergodic shift invariant measure μ on $\Omega^{\mathbf{Z}}$ whose entropy $h(\mu)$ is less than $\log |\Omega|$, there exists a uniquely ergodic sequence $x \in \Omega^{\mathbf{Z}}$ such that the systems $(\Omega^{\mathbf{Z}}, \mu, S)$ and $(\mathcal{O}_x, \mu_x, S)$ are isomorphic and such that μ_x is in any given weak neighborhood of μ.

This result and the finite generator theorem for ergodic measure preserving transformations (see [9] and [10]) imply that every ergodic measure preserving invertible transformation T of a Lebesgue measure space with finite entropy $h(T)$ is isomorphic to a system $(\mathcal{O}_x, \mu_x, S)$, where x is a uniquely ergodic sequence in $\Omega^{\mathbf{Z}}$ and $\exp \{h(T)\} < |\Omega| \leqq \exp \{h(T)\} + 1$. In Section 4, we shall show that every ergodic invertible measure preserving transformation T of a Lebesgue measure space is isomorphic to a system (U, C, μ_U), where U is a uniquely ergodic homeomorphism of the Cantor discontinuum C. This was recently established by R. Jewett [6] under the additional assumption that T be weakly mixing, and conjectured by him to hold in the ergodic case. Our method of proof combines the basic idea of Jewett with the methods that were developed for the proof of the finite generator theorem for ergodic measure preserving transformations (see [9] and [10]). We require some tools that we develop in Section 2.

Jacobs has recently shown that every weakly mixing flow on a Lebesgue measure space is isomorphic to a flow of homeomorphisms of a compact metric space together with a unique invariant Borel measure [4].

2. Cesaro properties and admissible sequences

We introduce some notation. Let Ω be a finite state space. We denote the cylinder set of an $a \in \Omega^\theta$ for $\theta \subset \mathbf{Z}$ by

$$Z_a = \{x \in \Omega^{\mathbf{Z}}: (x_i)_{i \in \theta} = a\}. \tag{2.1}$$

Le* $I \in \mathbf{N}$ and let $\mathscr{W}_I$ be the set of probability measures on Ω^I. On $\mathscr{W}_I$, we use the metric

$$|\mu, \nu| = \max_{a \in \Omega^I} |\mu(a) - \nu(a)|, \qquad \mu, \nu \in \mathscr{W}_I. \tag{2.2}$$

We define for $\mu \in \mathscr{W}_I$,

$$\tilde{h}(\mu) = \sum_{a \in \Omega^{I-1}} \sum_{\alpha \in \Omega} \mu(a_1, \cdots, a_{I-1}, \alpha) \log \Big(\sum_{\alpha \in \Omega} \mu(a_1, \cdots, a_{I-1}, \alpha)\Big) - \sum_{a \in \Omega^I} \mu(a) \log \mu(a). \tag{2.3}$$

Every shift invariant probability measure μ on $\Omega^{\mathbf{Z}}$ furnishes a measure $\mu^{(I)} \in \mathscr{W}_I$,

$$\mu^{(I)}(a) = \mu(Z_a), \qquad a \in \Omega^I. \tag{2.4}$$

To every $b \in \Omega^K$, $K \geqq I$, there is a $\lambda^{(I)}[b] \in \mathscr{W}_I$ assigned by

$$\lambda^{(I)}[b](a) = (K - I + 1)^{-1} \sum_{i=1}^{K-I+1} \delta_{a,(b_j)_{j=i}^{i+I-1}}, \qquad a \in \Omega^I. \tag{2.5}$$

Now let $\mu \in \mathscr{W}_I$ and $\varepsilon > 0$. Also let $K, L, M \in \mathbf{N}$, $L \geqq 3M(K + 1)$. Then $\mathscr{C}^{(L)}_{M,K}(\mu, \varepsilon)$ will denote the set of all $a \in \Omega^L$ that have the following property: for all i, where $1 \leqq i \leqq L - 3M(K + 1) + 1$, there exist $j(k) \in \mathbf{N}$, $1 \leqq k \leqq K$, such that

$$\begin{aligned} &i \leqq j(k) < j(k + 1) \leqq L, && 1 \leqq k < K, \\ &j(1) - i < 3M, \qquad M \leqq j(k + 1) - j(k) \leqq 3M, && 1 \leqq k < K, \end{aligned} \tag{2.6}$$

and

$$|\lambda^{(I)}[(a_i)_{i=j(k)+1}^{j(k+1)}], \mu| < \varepsilon, \qquad 1 \leqq k < K. \tag{2.7}$$

For $a \in \Omega^m$, and $b \in \Omega^n$, $m, n \in \mathbf{N}$, we denote by $a + b$ the element $c \in \Omega^{m+n}$ that is given by

$$(c_i)_{i=1}^m = a, \qquad (c_i)_{i=m+1}^{m+n} = b. \tag{2.8}$$

Our first lemma will not be proved here.

LEMMA 2.1. *Let* $L, L' \geqq 3M(K + 1)$, *and* $a \in \mathscr{C}^{(L)}_{M,K}(\mu, \varepsilon)$, $a' \in \mathscr{C}^{(L')}_{M,K}(\mu, \varepsilon)$, *and let*

$$(a_i)_{i=L-3M(K+1)+1}^{L} + (a'_i)_{i=1}^{3M(K+1)} \in \mathscr{C}^{(6M(K+1))}_{M,K}(\mu, \varepsilon). \tag{2.9}$$

Then $a + a' \in \mathscr{C}^{(L+L')}_{M,K}(\mu, \varepsilon)$.

Next, we note a relation between the sets

$$\mathscr{D}_{M,K}(\mu, \varepsilon) = \bigcap_{j=-\infty}^{\infty} \{x \in \Omega^{\mathbf{Z}}: (x_i)_{i=j}^{j+3M(K+1)-1} \in \mathscr{C}^{(3M(K+1))}_{M,K}(\mu, \varepsilon)\} \tag{2.10}$$

and the sets

$$(2.11)\qquad \mathscr{U}_N(\mu, \varepsilon) = \bigcap_{j=-\infty}^{\infty} \bigcap_{N'=N}^{\infty} \{x \in \Omega^{\mathbf{Z}} : |\lambda^{(I)}[(x_i)_{i=j}^{j+N'-1}], \mu| < \varepsilon\}, \qquad N \in \mathbf{N}.$$

LEMMA 2.2. *Let $0 < \varepsilon < 1$, let $I \in \mathbf{N}$, let $\mu \in \mathscr{W}_I$, and let $K, M, N \in \mathbf{N}$ be such that*

$$(2.12)\qquad M^{-1}I < \varepsilon, \qquad K^{-1} < \varepsilon, \qquad N^{-1}M(K+1) \leqq \varepsilon.$$

Then $\mathscr{D}_{M,K}(\mu, \varepsilon) \subset \mathscr{U}_N(\mu, 2^5\varepsilon)$.

PROOF. Let $x \in \mathscr{D}_{M,K}(\mu, \varepsilon)$, let $i' \in \mathbf{Z}$, and let $i'' \geqq i' + N - 1$. Since $(x_i)_{i=i'}^{i''} \in \mathscr{D}_{M,K}(\mu, \varepsilon)$, there is an $R \in \mathbf{N}$ and $j(r, k) \in \mathbf{Z}$, for $1 \leqq r \leqq R$ and $1 \leqq k \leqq K$, such that

$$(2.13)\qquad \begin{aligned} &0 \leqq j(1,1) - i' < 3M, \\ &0 \leqq i'' - j(R, K) < 3M(K+1), \\ &0 < j(r+1, 1) - j(r, K) \leqq 3M, && 1 \leqq r < R, \\ &M \leqq j(r, k+1) - j(r, k) \leqq 3M, && 1 \leqq r \leqq R, \quad 1 \leqq k < K, \end{aligned}$$

and

$$(2.14)\qquad |\lambda^{(I)}[(x_i)_{i=j(r,k)+1}^{j(r,k+1)}], \mu| < \varepsilon, \qquad 1 \leqq r \leqq R, \quad 1 \leqq k < K.$$

This together with (2.12) implies that

$$(2.15)\qquad |\lambda^{(I)}[(x_i)_{i=i'}^{i''}], \mu| < 2^5\varepsilon,$$

which was to be proved.

Let $\nu \in \mathscr{W}_I$. We say that an $a \in \Omega^J$, $J \geqq I$, is ν admissible if $\nu((a_i)_{i=j}^{j+I-1}) > 0$ for $1 \leqq j \leqq J - I + 1$. The set of ν admissible sequences in Ω^J will be denoted by $\mathscr{E}(\nu, J)$. We say that an $a \in \mathscr{E}(\nu, m)$ and a $b \in \mathscr{E}(\nu, n)$ can be ν connected in k steps if there exists a $c \in \Omega^k$ such that $a + c \in \mathscr{E}(\nu, m + k)$ and $((a + c)_i)_{i=m+k-I+1}^{m+k} = b$.

Now let μ be an ergodic shift invariant probability measure on $\Omega^{\mathbf{Z}}$. The following four lemmas are well known from the theory of Markov chains. For $a \in \mathscr{E}(\mu^{(I)}, I)$, denote by $\pi_a(\mu)$ the smallest of all positive integers p with the property that a can be $\mu^{(I)}$ connected with itself in kp steps, provided that k is sufficiently large.

LEMMA 2.3. *Let $a \in \mathscr{E}(\mu^{(I)}, I)$ and let $b \in \mathscr{E}(\mu^{(I)}, n)$, where $n > I$, be such that $a = (b_i)_{i=1}^{I} = (b_i)_{i=n-I+1}^{n}$. Then $n - I$ is a multiple of $\pi_a(\mu)$.*

PROOF. Let q be the greatest common divisor of $\pi_a(\mu)$ and $n - I$, and let $M \in \mathbf{N}$ be such that, for all $M' \geqq M$, a can be $\mu^{(I)}$ connected with itself in $M'\pi_a(\mu)$ steps. If $\ell \in \mathbf{N}$, $1 \leqq \ell < q^{-1}\pi_a(\mu)$, then there are $N_\ell, A_\ell, B_\ell \in \mathbf{N}$ such that $N_\ell\pi_a(\mu) + \ell q = A_\ell\pi_a(\mu) + B_\ell(n - I)$. Let

$$(2.16)\qquad k \geqq Mq^{-1}\pi_a(\mu) + \max\{q^{-1}B_\ell(n - I) : 1 \leqq \ell < q^{-1}\pi_a(\mu)\},$$

and let

$$\ell(k) = k - [kq\pi_a(\mu)^{-1}]q^{-1}\pi_a(\mu), \tag{2.17}$$
$$M(k) = \pi_a(\mu)^{-1}(kq - B_{\ell(k)}(n - I)).$$

Then a can be $\mu^{(I)}$ connected with itself in kq steps. Indeed, there is a $c \in \mathscr{E}(\mu^{(I)}, I + M(k)\pi_a(\mu))$ such that

$$a = (c_i)_{i=1}^{I} = (c_i)_{i=M(k)\pi_a(\mu)+1}^{I+M(k)\pi_a(\mu)}. \tag{2.18}$$

Hence,

$$c + \sum_{m=1}^{B_{\ell(k)}} (b_i)_{i=I+1}^{n} \in \mathscr{E}(\mu^{(I)}, I + kq). \tag{2.19}$$

LEMMA 2.4. *For $a, b \in \mathscr{E}(\mu^{(I)}, I)$, one has $\pi_a(\mu) = \pi_b(\mu)$.*

PROOF. The ergodicity of μ implies that there are $N, K \in \mathbf{N}$ such that for all $K' \geqq K$, b can be $\mu^{(I)}$ connected with itself in $N + K'\pi_a(\mu)$ steps. Application of Lemma 2.3 concludes the proof.

We shall write $\pi(\mu^{(I)})$ for $\pi_a(\mu)$, $a \in \mathscr{E}(\mu^{(I)}, I)$. It follows from Lemmas 2.3 and 2.4 that the set $\mathscr{E}(\mu^{(I)}, I)$ carries an equivalence relation, where $a, b \in \mathscr{E}(\mu^{(I)}, I)$ are equivalent if they can be $\mu^{(I)}$ connected in $k\pi(\mu^{(I)})$ steps for some $k \in \mathbf{N}$. The classes of this equivalence relation will be called $\mu^{(I)}$ classes.

LEMMA 2.5. *Let $\mathscr{K} \subset \mathscr{E}(\mu^{(I)}, I)$ be a $\mu^{(I)}$ class. Then there exists an $M \in \mathbf{N}$ such that for all $M' \geqq M$ all $a, b \in \mathscr{K}$ can be $\mu^{(I)}$ connected in $M'\pi(\mu^{(I)})$ steps.*

PROOF. For all $c \in \mathscr{K}$, there is a k_c such that for all $k' \geqq k_c$ the sequence c can be $\mu^{(I)}$ connected with itself in $k'\pi(\mu^{(I)})$ steps. For $a, b \in \mathscr{K}$, let $n(a, b) \in \mathbf{N}$ and $\varphi(a, b) \in \Omega^{\ell}$, where $\ell = n(a, b)\pi(\mu^{(I)}) - I$, be such that $a + \varphi(a, b) + b$ is $\mu^{(I)}$ admissible. It follows that for all

$$M' \geqq \max_{a, b \in \mathscr{K}} n(a, b) + \max_{c \in \mathscr{K}} k_c \tag{2.20}$$

and all $b \in \mathscr{K}$, there is a $\psi(b) \in \Omega^m$, where $m = (M' - n(a, b))\pi(\mu^{(I)}) - I$, such that

$$a + \varphi(a, b) + b + \psi(b) + b \in \mathscr{E}(\mu^{(I)}, I + M'\pi(\mu^{(I)})), \qquad a \in \mathscr{K}. \tag{2.21}$$

LEMMA 2.6. *Let $\mathscr{K} \subset \mathscr{E}(\mu^{(I)}, I)$ be a $\mu^{(I)}$ class and let $a \in \mathscr{E}(\mu^{(I)}, n)$ for $n > I$ and $(a_i)_{i=1}^{I}$, $(a_i)_{i=n-I+1}^{n} \in \mathscr{K}$. Then $n - I$ is a multiple of $\pi(\mu^{(I)})$.*

PROOF. There is a $k \in \mathbf{N}$ such that for some $c \in \Omega^m$, where $m = k\pi(\mu^{(I)}) - I$,

$$(a_i)_{i=n-I+1}^{I} + c + (a_i)_{i=1}^{I} \in \mathscr{E}(\mu^{(I)}, I + k\pi(\mu^{(I)})). \tag{2.22}$$

Hence,

$$a + c + (a_i)_{i=1}^{I} \in \mathscr{E}(\mu^{(I)}, n + k\pi(\mu^{(I)})). \tag{2.23}$$

By Lemmas 2.3 and 2.4, $n + k\pi(\mu^{(I)}) - I$ is a multiple of $\pi(\mu^{(I)})$, which concludes the proof.

We shall need the following fact which was already used in the proof of Lemma 3 of [10].

LEMMA 2.7. *For $I \in \mathbf{N}$, $\mu \in \mathscr{W}_I$, $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ and an $\eta > 0$ such that*

$$|\{a \in \Omega^{N'} : |\lambda^{(I)}[a], \mu| < \eta\}| < \exp\{(\tilde{h}(\mu) + \varepsilon)N'\}, \qquad N' \geqq N. \tag{2.24}$$

PROOF. Let $J \geqq I$. Any vector $k = (k_a)$, $a \in \Omega^I$ in $\mathbf{Z}^{\Omega^I}$ such that $k_a \geqq 0$, for all $a \in \Omega^I$, and such that

$$\sum_{a \in \Omega^I} k_a = J - I + 1 \tag{2.25}$$

determines a $\nu_k \in \mathscr{W}_I$

$$\nu_k(a) = (J - I + 1)^{-1} k_a, \qquad a \in \Omega^I. \tag{2.26}$$

Using Stirling's formula, we have, for all such vectors k,

$$\begin{aligned} |\{a \in \Omega^J : \lambda^{(I)}[a] = \nu_k\}| &\leqq |\Omega|^{I-1} \prod_{a \in \Omega^{I-1}} \frac{\left(\sum_{\alpha \in \Omega} k_{(a_1, \cdots, a_{I-1}, \alpha)}\right)!}{\prod_{\alpha \in \Omega} \left(k_{(a_1, \cdots, a_{I-1}, \alpha)}!\right)} \\ &\leqq |\Omega|^{I-1} \prod_{a \in \Omega^I, k_a > 0} \left(\frac{J}{k_a}\right)^{1/2} \exp\{\tilde{h}(\nu_k) J\}. \end{aligned} \tag{2.27}$$

Let $0 < \eta < \rho = \frac{1}{2}\min_{a \in \Omega^I, \mu(a) > 0} \mu(a)$ be such that for $\kappa \in \mathscr{W}_I$, $|\mu, \kappa| < \eta$ implies that $|\tilde{h}(\mu), \tilde{h}(\kappa)| < \frac{1}{2}\varepsilon$, and let $N \in \mathbf{N}$ be such that

$$|\Omega|^{I-1} \rho^{-|\Omega|^I/2} N'^{|\Omega|^I} < \exp\{\tfrac{1}{2}\varepsilon N'\}, \qquad N' \geqq N. \tag{2.28}$$

Then from (2.27),

$$|\{a \in \Omega^{N'} : |\lambda^{(I)}[a], \mu| < \eta\}| < \exp\{(\tilde{h}(\mu) + \varepsilon)N'\}. \qquad N' \geqq N, \tag{2.29}$$

which was to be proved.

3. A construction of uniquely ergodic sequences

We write $U\mu$ for the Borel measure that is obtained when the Borel measure μ is transported by means of the Borel mapping U.

LEMMA 3.1. *Let $0 < \varepsilon < 1$, $I \in \mathbf{N}$. Let θ be a finite set and let there be given $I(\vartheta)$, $K(\vartheta)$, $M(\vartheta) \in \mathbf{N}$, $\varepsilon_\vartheta > 0$, and $\mu_\vartheta \in \mathscr{W}_{I(\vartheta)}$, $\vartheta \in \theta$. Let there further be given a nonatomic ergodic shift invariant probability measure μ on $\Omega^{\mathbf{Z}}$ such that*

$$\mu\Big(\bigcap_{\vartheta \in \theta} \mathscr{D}_{M(\vartheta), K(\vartheta)}(\mu_\vartheta, \varepsilon_\vartheta)\Big) = 1 \tag{3.1}$$

and

$$L \geqq \max_{\vartheta \in \theta} 6M(\vartheta)\big(K(\vartheta) + 1\big), \tag{3.2}$$

$\tilde{h}(\mu^{(L)}) - h(\mu) > 0$. *Then there exists a shift invariant Borel set $X \subset \Omega^{\mathbf{Z}}$, $\mu(X) = 1$, a one to one Borel mapping $U : X \to \Omega^{\mathbf{Z}}$ that commutes with the shift, and there exist K, M, $L' \in \mathbf{N}$ such that $K > \varepsilon^{-1} I$, $M > \varepsilon^{-1}$, $L' >$* $\max\big(L, 6M(K+1)\big)$,

$$(3.3)\qquad \mu(Z_{(\alpha)}\Delta U^{-1}Z_{(\alpha)}) < \varepsilon, \qquad \alpha \in \Omega,$$

$$(3.4)\qquad U\mu\big(\bigcap_{\vartheta\in\theta} \mathscr{D}_{M(\vartheta),K(\vartheta)}(\mu_\vartheta, \varepsilon_\vartheta)\big) = U\mu\big(\mathscr{D}_{M,K}(\mu^{(I)}, \varepsilon)\big) = 1$$

and

$$(3.5)\qquad \tilde{h}\big((U\mu)^{(L)}\big) > h(\mu).$$

PROOF. Let n be the number of elements in Ω, and set $2^{10}n\delta = \tilde{h}(\mu^{(L)}) - h(\mu)$. The proof will be given in four parts.

We begin the first part of the proof by selecting a $\mu^{(L)}$ class $\mathscr{K} \subset \mathscr{E}(\mu^{(L)}, L)$, $|\mathscr{K}| > 1$. By Lemma 2.5, there exists an $m \in \mathbf{N}$, $m\pi(\mu^{(L)}) > L$, such that all elements of $\mathscr{K}$ can be $\mu^{(L)}$ connected in $m\pi(\mu^{(L)}$ steps. We choose for all a, $a' \in \mathscr{K}$ a $\beta(a, a') \in \Omega^p$, where $p = m\pi(\mu^{(L)}) - L$, such that $a + \beta(a, a') + a$ is $\mu^{(L)}$ admissible. By Lemma 2.3, we can find $a^{(0)}$, $a^{(1)} \in \mathscr{K}$, $t \in \mathbf{N}$, $t\pi(\mu^{(L)}) > 1$, as well as $b^{(0)}, b^{(1)} \in \mathscr{E}\big(\mu^{(L)}, L + t\pi(\mu^{(L)})\big)$, such that

$$(3.6)\qquad a^{(0)} \neq a^{(1)},$$

and

$$(3.7)\qquad \begin{aligned} a^{(0)} &= (b_i^{(0)})_{i=1}^{L} = (b_i^{(1)})_{i=1}^{L} = (b_i^{(0)})_{i=t\pi(\mu^{(L)})+1}^{L+t\pi(\mu^{(L)})}, \\ a^{(1)} &= (b_i)_{i=t\pi(\mu^{(L)})+1}^{L+t\pi(\mu^{(L)})}, \end{aligned}$$

and such that $a^{(0)}$ cannot be $\mu^{(L)}$ connected with itself in less than $t\pi(\mu^{(L)})$ steps. We denote

$$(3.8)\qquad c = (b_i^{(0)})_{i=L+1}^{L+t\pi(\mu^{(L)})}.$$

By the individual ergodic theorem, we can find an $N_1 \in \mathbf{N}$ such that for

$$(3.9)\qquad F_1 = \bigcap_{\substack{k',k''\geqq 0\\ k_1'+k_2''>N_1}} \{x \in \Omega^{\mathbf{Z}} : |\lambda^{(I)}[(x_i)_{i=-k'}^{k''}], \mu^{(I)}| < 2^{-5}\varepsilon\},$$

$$(3.10)\qquad \mu(F_1) > 1 - 2^{-24}\varepsilon^2\delta^2 n^{-1}.$$

By the Shannon-McMillan theorem, we can find a $J_1 \in \mathbf{N}$ such that

$$(3.11)\qquad |\{a \in \mathscr{E}(\mu^{(L)}, J) : (a_i)_{i=1}^{L} \in \mathscr{K}\}| > \exp\{(\tilde{h}(\mu^{(L)}) - \delta)J\}, \qquad J \geqq J_1.$$

(See Lemma 3.1 of [9].) Let $A = (4m + t)\pi(\mu^{(L)}) + 4tm\pi(\mu^{(L)})^2 - L$, and let $P \in \mathbf{N}$ be such that

$$(3.12)\qquad B = Pt\pi(\mu^{(L)}) + L > \max(2^5\varepsilon^{-1}\delta^{-2}nIA, N_1, J_1).$$

We form the sequences $s_0 = a^{(0)} + \Sigma_{k=1}^{P} c$. The set $\{a \in \mathscr{E}(\mu^{(L)}, B) : (a_i)_{i=1}^{L} \in \mathscr{K}\}$, of which s_0 is an element, plays the role of an alphabet in this proof, and we denote it by $\mathscr{A}$. From (3.11) and (3.12), we have

$$(3.13)\qquad |\mathscr{A}| > \exp\{(\tilde{h}(\mu^{(L)}) - 2\delta)(A + B)\}.$$

Next we choose d, $e^{(1)}$, $e^{(2)} \in \mathscr{K}$, $e^{(1)} \neq e^{(2)}$, and we form sequences $\omega_{\rho,\sigma} \in \mathscr{E}(\mu^{(L)}, A)$ by setting

$$
\begin{aligned}
\hat{c} &= \sum_{k=1}^{4m\pi(\mu^{(L)})} c, \\
\omega_{\rho,\sigma} &= \beta(d, a^{(0)}) + b^{(1)} + \beta(a^{(1)}, e^{(\rho)}) + e^{(\rho)} + \beta(e^{(\rho)}, e^{(\sigma)}) + e^{(\sigma)} \\
&\quad + \beta(e^{(\sigma)}, a^{(0)}) + a^{(0)} + (\hat{c}_i)_{i=1}^{4mt\pi(\mu^{(L)})^2 - L}, \qquad \rho, \sigma = 1, 2.
\end{aligned}
\tag{3.14}
$$

Let $h \in \Omega^q$, where $q = (2m - t)\pi(\mu^{(L)}) + 4tm\pi(\mu^{(L)})^2 - L$ be such that $b^{(1)} + h + b^{(1)}$ is $\mu^{(L)}$ admissible. For $f \in \Omega^k$, $f' \in \Omega^{k'}$, $k, k' \geqq L$, such that $(f_i)_{i=k-L+1}^k$, $(f'_i)_{i=1}^L \in \mathscr{K}$, we form sequences $\gamma(f, f') \in \mathscr{E}(\mu^{(L)}, A)$,

$$
\gamma(f, f') = \beta\big((f_i)_{i=k-L+1}^k, a^{(0)}\big) + b^{(1)} + h + b^{(1)} + \beta\big(a^{(1)}, (f'_i)_{i=1}^L\big). \tag{3.15}
$$

Next, let $J^{(0)} \in \mathbf{N}$ be such that

$$
\tilde{h}(\mu^{(J^{(0)})}) < h(\mu) + \delta. \tag{3.16}
$$

By Lemma 2.9, there is a $J_2 \in \mathbf{N}$ and an $\eta > 0$ such that

$$
\begin{aligned}
&\big|\{a \in \Omega^J : |\lambda^{(J^{(0)})}[a], \mu^{(J^{(0)})}| < \eta\}\big| \\
&\quad < \exp\{(\tilde{h}(\mu^{(J^{(0)})}) + \delta)J\} \\
&\quad < \exp\{(h(\mu) + 2\delta)J\}. \qquad J \geqq J_2.
\end{aligned}
\tag{3.17}
$$

We have from the individual ergodic theorem that there is a $Q \in \mathbf{N}$,

$$
Q > \delta^{-1} n J_2, \tag{3.18}
$$

such that for

$$
F_2 = \bigcap_{\substack{k', k'' \geqq 0 \\ k' + k'' > Q(A+B)}} \{x \in \Omega^{\mathbf{Z}} : |\lambda^{(J^{(0)})}[(x_i)_{i=-k'}^{k''}], \mu^{(J^{(0)})}| < \eta\}, \tag{3.19}
$$

$$
\mu(F_2) > 1 - 2^{-24}\varepsilon^2\delta^2 n^{-1}. \tag{3.20}
$$

Let $R \in \mathbf{N}$,

$$
2^5\varepsilon^{-1}Q < R < 2^6\varepsilon^{-1}Q. \tag{3.21}
$$

We set $M = (R + Q)(A + B)$, choose a $K \in \mathbf{N}$, $K > \varepsilon^{-1}I$, and set $L' = 6(R + Q)(A + B)(K + 1)$. Once more appealing to the individual ergodic theorem, we see from (3.11) and (3.20) that there is an $N_2 \in \mathbf{N}$,

$$
N_2 > 2^5\varepsilon^{-1}\delta^{-2} n L', \tag{3.22}
$$

such that for

$$
\begin{aligned}
F_3 = \bigcap_{\substack{k', k'' \geqq 0 \\ k' + k'' \geqq N_2}} \Big\{x \in \Omega^{\mathbf{Z}} : (1 + k' + k'')^{-1} &\sum_{i=-k'}^{k''} \chi_{F_1 \cap F_2}(S^i x) \\
&> 1 - 2^{-12}\varepsilon^2\delta^2 n^{-1}\Big\},
\end{aligned}
\tag{3.23}
$$

$$
\mu(F_3) > 1 - 2^{-5}\varepsilon\delta^2 n^{-1}.
$$

Next, as the second part of the proof, we prepare for the construction of X and U to be given in the third part of the proof. For the actual construction of X and U, we shall need for every $\ell \geqq N_2$ a mapping $a \to q_\ell(a) \in \mathbf{N}$, $(a \in \Omega^\ell)$ and a mapping $\Phi_\ell : \Omega^\ell \to \bigcup_{k=1}^\infty \mathscr{A}^k$, where $\Phi_\ell a \in \mathscr{A}^{q_\ell(a)}$, $a \in \Omega^\ell$. We proceed to describe these mappings. Let s_r, $1 \leqq r \leqq 4$, be different elements of $\mathscr{A}$, all different from s_0, and let $\mathscr{A}_0 = \mathscr{A} - \{s_r : 1 \leqq r \leqq 4\}$. By (3.11) and (3.13),

$$|\mathscr{A}_0| > \exp\{(\tilde{h}(\mu^{(L)}) - 3\delta)(A + B)\}. \tag{3.24}$$

Denote

$$\begin{aligned} C_\ell &= [\ell(A + B)^{-1}] - 1, \\ D_\ell &= [C_\ell(Q + R)^{-1}], \\ K_{\ell,1} &= [\delta Q D_\ell], \qquad & K_{\ell,2} &= [\varepsilon^{-1}\delta^{-1} n Q D_\ell], \\ M_{\ell,1} &= [(1 - 2^5\delta)Q], \qquad & M_{\ell,2} &= [\delta^{-1} n Q], \\ j_\ell &= [\varepsilon\delta^2 n^{-1} C_\ell], \\ \Delta_\ell &= \{k(Q + R) + k' : 0 \leqq k \leqq D_\ell, 1 < k' \leqq Q\}, \\ \Gamma_\ell &= \{1, \cdots, C_\ell\} - \Delta_\ell, \end{aligned} \tag{3.25}$$

and let $\mathscr{P}_j(\Gamma_\ell)$ be the set of subsets of size j of Γ_ℓ, $1 \leqq j \leqq RD_\ell$. Further, for $a \in \Omega^\ell$, let

$$\Gamma(a) = \{k \in \Gamma_\ell : |\lambda^{(I)}[(a_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)}], \mu^{(I)}| \leqq 2^{-5}\varepsilon\}, \tag{3.26}$$

and let $\mathscr{M}_\ell$ be the set of $a \in \Omega^\ell$ such that $|\Gamma(a)| \leqq j_\ell$ and such that

$$|\{0 \leqq j < D_\ell : |\lambda^{(J(0))}[(a_i)_{i=j(Q+R)(A+B)+1}^{j(Q+R)(A+B)+Q(A+B)}], \mu^{(J(0))}| \geqq \eta\}| < \delta^2 n^{-1} D_\ell. \tag{3.27}$$

We see from (3.12), (3.21), (3.22), and (3.24) that there is a one to one mapping

$$\varphi_\ell : \Omega^{\ell - C_\ell B} \to \mathscr{A}_0^{K_{\ell,1}}, \tag{3.28}$$

and we set

$$\xi_{\ell,1} a = \varphi_\ell\left(\sum_{k=0}^{C_\ell - 1} (a_i)_{i=k(A+B)+1}^{k(A+B)+A} + (a_i)_{i=C_\ell(A+B)+1}^{\ell}\right), \qquad a \in \Omega^\ell. \tag{3.29}$$

We compute with the help of (3.12), (3.21), (3.22), and (3.24), that there are one to one mappings

$$\begin{aligned} \xi_{\ell,2,1} &: \bigcup_{j=1}^{j_\ell} (\mathscr{P}_j(\Gamma_\ell) \times \Omega^{jB}) \to \mathscr{A}_0^{K_{\ell,1}}, \\ \xi_{\ell,2,2} &: \Omega^{RD_\ell B} \to \mathscr{A}_0^{K_{\ell,2}}. \end{aligned} \tag{3.30}$$

For $a \in \Omega^\ell$, we set

$$\xi_{\ell,2} a = \begin{cases} \xi_{\ell,2,1}(\Gamma(a), \sum_{k \in \Gamma(a)} (a_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)}) & \text{if} \quad |\Gamma(a)| \leqq j_\ell, \\ \xi_{\ell,2,2}(\sum_{k \in \Gamma_\ell} (a_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)}) & \text{if} \quad |\Gamma(a)| > j_\ell. \end{cases} \tag{3.31}$$

and

$$K_\ell a = \begin{cases} K_{\ell,1} & \text{if } |\Gamma(a)| \leqq j_\ell, \\ K_{\ell,2} & \text{if } |\Gamma(a)| > j_\ell. \end{cases} \tag{3.32}$$

We find from (3.17), (3.18), and (3.24) that there is a one to one mapping

$$\Psi_\ell : \Omega^{Q(A+B)} \to \mathscr{A}_0^{M_{\ell,1}} \cup \mathscr{A}_0^{M_{\ell,2}} \tag{3.33}$$

such that

$$\Psi_\ell a \in \mathscr{A}_0^{M_{\ell,1}} \qquad \text{if } |\lambda^{(J(0))}[a], \mu^{(J(0))}| < \eta. \tag{3.34}$$

Let, for $a \in \Omega^{Q(A+B)}$,

$$M_\ell(a) = \begin{cases} M_{\ell,1} & \text{if } |\lambda^{(J(0))}[a], \mu^{(J(0))}| < \eta, \\ M_{\ell,2} & \text{elsewhere.} \end{cases} \tag{3.35}$$

We define for $a \in \Omega^\ell$,

$$\begin{aligned} \Phi_\ell a = {} & (s_0) + \xi_{\ell,1} a + (s_1) + \xi_{\ell,2} a + (s_2) \\ & + \sum_{j=0}^{D_\ell - 1} \left(\Psi_\ell\left((a_i)_{i=j(Q+R)(A+B)+1}^{j(Q+R)(A+B)+Q} \right) + (s_3) \right) + (s_4), \end{aligned} \tag{3.36}$$

$$q_\ell(a) = 2^2 + D_\ell + K_{\ell,1} + K_\ell a + \sum_{j=0}^{D_\ell - 1} M_\ell\left((a_i)_{i=j(Q+R)(A+B)+1}^{j(Q+R)(A+B)+Q} \right). \tag{3.37}$$

We have from (3.18),

$$q_\ell(a) \leqq 2^2 \varepsilon^{-1} \delta^{-1} n Q D_\ell, \qquad a \in \Omega^\ell, \tag{3.38}$$

and we have

$$q_\ell(a) \leqq (1 - 4\delta) Q D_\ell, \qquad a \in \mathscr{M}_\ell. \tag{3.39}$$

Observe that we have constructed the Φ_ℓ in such a way that $\Gamma(a)$ is uniquely determined by $\Phi_\ell a$ and that a is uniquely determined by $\Phi_\ell a$ and the $(a_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)}$, $k \in \Gamma_\ell - \Gamma(a)$.

For the third part of the proof, let

$$E \subset \bigcup_{a \in \mathscr{X}} S^L Z_a \tag{3.40}$$

be a Borel set of positive μ measure such that

$$E \cap S^{-i} E = \phi, \qquad 1 \leqq i \leqq N_2, \tag{3.41}$$

and such that for all $x \in E$ the sets $\{i \in \mathbf{N} : S^i x \in E\}$ and $\{i \in \mathbf{N} : S^{-i} x \in E\}$ are infinite. Let

$$E = \bigcup_{r=1}^{2n^{L'}} E_r \tag{3.42}$$

be a partition of E,

$$2\mu(E_r) = n^{-L'}\mu(E), \qquad 1 \leqq r \leqq 2n^{L'}. \tag{3.43}$$

We denote for $x \in \bigcup_{i=-\infty}^{\infty} S^i E$,

$$\begin{aligned} i^-(x) &= \min\{i \geqq 0 : S^{-i}x \in E\}, \\ i^+(x) &= \min\{i \in \mathbf{N} : S^{i+1}x \in E\}, \\ \ell(x) &= i^-(x) + i^+(x). \end{aligned} \tag{3.44}$$

We shall construct a shift invariant Borel set

$$X \subset \bigcup_{i=-\infty}^{\infty} S^i E \tag{3.45}$$

and one to one Borel mappings

$$U_r : X \to \Omega^{\mathbf{Z}}, \qquad 1 \leqq r \leqq 2n^{L'}, \tag{3.46}$$

that commute with the shift. The construction of these U_r will be achieved by assigning to every $x \in E \cap X$

$$u_x(k) \in \mathscr{A}, \qquad 1 \leqq k \leqq C_{\ell(x)}. \tag{3.47}$$

and

$$u_x(C_{\ell(x)} + 1) \in \mathscr{E}\big(\mu^{(L)}, \ell(x) - C_{\ell(x)}(A + B) - A\big). \tag{3.48}$$

We shall have

$$u_x(1) = s_0, \qquad x \in E \cap X. \tag{3.49}$$

Moreover, the $u_x(C_{\ell(x)} + 1)$ will have the property that their first L elements and hence also their last L elements form sequences that are in $\mathscr{K}$. Hence, according to Lemma 2.6, it will be possible to define the U_r by the requirement that they commute with the shift, and by setting $U_r x$, $x \in E \cap X$, equal to y, where

$$(y_i)_{i=1}^{\ell(x)} = \omega_{\rho(x),\sigma(x)} + s_0 + \sum_{k=1}^{C_{\ell(x)}} \big(\gamma\big(u_x(k), u_x(k+1)\big) + u_x(k+1)\big), \tag{3.50}$$

$$\begin{aligned} \rho(x) &= \begin{cases} 1 & \text{if } x \in E_r, \\ 2 & \text{if } x \notin E_r, \end{cases} \\ \sigma(x) &= \begin{cases} 1 & \text{if } S^{-\ell(S^{-1}x)}x \in E_r, \\ 2 & \text{if } S^{-\ell(S^{-1}x)}x \notin E_r. \end{cases} \end{aligned} \tag{351}$$

In order to define the $u_x(C_{\ell(x)} + 1)$, for all $\ell \geqq N_2$, let $d^{(\ell)}$ be an element of $\mathscr{E}\big(\mu^{(L)}, \ell - C_\ell(A + B) - A\big)$ that ends in d and does not contain $b^{(0)}$ as a subsequence. Then set

$$u_x(C_{\ell(x)} + 1) = d^{(\ell(x))}. \qquad x \in E. \tag{3.52}$$

In order to produce the $u_x(k)$, $1 \leqq k \leqq C_\ell(x)$, we shall first construct certain $v_x(j) \in \mathscr{A}$, $1 \leqq j \leqq QD_{\ell(x)}$. For this we need the mappings q_ℓ and Φ_ℓ as defined in the second part of the proof. We set

$$q(x) = q_{\ell(x)}\big((x_i)_{i=-i^-(x)+1}^{i^+(x)}\big), \qquad x \in \bigcup_{i=-\infty}^{\infty} S^iE. \tag{3.53}$$

We have from (3.21) and (3.23) that

$$\mu\big(\{x \in \Omega^{\mathbf{Z}} : (x_i)_{i=-i^-(x)+1}^{i^+(x)} \in \mathscr{M}_{\ell(x)}\}\big) > 1 - 2^{-5}\varepsilon\delta^2 n^{-1}. \tag{3.54}$$

From this and from (3.38) and (3.39), we compute that

$$\int_{\Omega^{\mathbf{Z}}} \ell(x)^{-1}\big(q(x) - QD_{\ell(x)}\big)\, d\mu(x) < -2^{-7}\delta\varepsilon(A + B)^{-1}. \tag{3.55}$$

Hence, by the individual ergodic theorem, the set

$$X = \left\{x \in \bigcup_{i=-\infty}^{\infty} S^iE : \sum_{k=1}^{\infty} \big(q(S^kx) - QD_{\ell(S^kx)}\big)\chi_E(S^kx) = -\infty\right\} \tag{3.56}$$

has μ measure one. We set for $x \in X$

$$\begin{aligned} \Xi_x &= \{(i,j) \in \mathbf{Z} \times \mathbf{N} : S^ix \in E \cap X,\ 1 \leqq j \leqq q(S^ix)\}, \\ \hat{\Xi}_x &= \{(i,j) \in \mathbf{Z} \times \mathbf{N} : S^ix \in E \cap X,\ 1 \leqq j \leqq QD_{\ell(S^ix)}\}. \end{aligned} \tag{3.57}$$

Exploiting the defining property of X, we obtain for all $x \in X$ a mapping $\tau_x : \Xi_x \to \hat{\Xi}_x$ by setting for $(i,j) \in \Xi_x$,

(3.58)

$$\hat{\imath}_x(i,j) = \min\Big\{i' > i : j - QD_{\ell(S^ix)} + \sum_{i<k\leqq i'} \big(q(S^kx) - QD_{\ell(S^kx)}\big)\chi_E(S^kx) \leqq 0\Big\},$$

$$\hat{\jmath}(i,j) = j - QD_{\ell(S^ix)} + \sum_{i<k<\hat{\imath}(i,j)} \big(q(S^kx) - QD_{\ell(S^kx)}\big)\chi_E(S^kx) + q(S^{\hat{\imath}(i,j)}x),$$

and by setting

$$\tau_x(i,j) = \big(\hat{\imath}_x(i,j), \hat{\jmath}_x(i,j)\big). \tag{3.59}$$

The τ_x are one to one (see the second part of the proof of Theorem (2.1) of [9]). Denote now

$$\big(w_x(j)\big)_{j=1}^{q(x)} = \Phi_{\ell(x)}\big((x_i)_{i=1}^{\ell(x)}\big), \qquad x \in E \cap X, \tag{3.60}$$

and set for $x \in E \cap X$,

$$v_x(j) = \begin{cases} w_{S^ix}(j') & \text{if } (1,j) \in \tau_x(\Xi_x) \text{ and if } \tau_x(i,j') = (1,j), \\ s_3 & \text{if } (1,j) \notin \tau_x(\Xi_x). \end{cases} \tag{3.61}$$

Finally, we choose a $g \in \mathscr{A}$,

$$|\lambda^{(l)}[g], \mu^{(l)}| < 2^{-5}\varepsilon, \tag{3.62}$$

and we let ζ_ℓ stand for a one to one mapping of Δ_ℓ onto $\{1, \cdots, QD_\ell\}$. We define then for $x \in E \cap X$, $1 \leqq k \leqq C_{\ell(x)}$,

$$(3.63)\quad u_x(k) = \begin{cases} (x_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)} & \text{if } k \in \Gamma_{\ell(x)} - \Gamma\big((x_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)}\big), \\ g & \text{if } k \in \Gamma\big((x_i)_{i=(k-1)(A+B)+A+1}^{k(A+B)}\big), \\ v_x(\zeta_{\ell(x)}k) & \text{if } k \in \Delta_{\ell(x)}. \end{cases}$$

This construction is such that (3.49) holds.

Let Λ be the mapping that assigns to an $x \in X$ the

$$(3.64)\qquad z \in \left(\{\phi\} \cup \bigcup_{k=1}^{\infty} \mathscr{A}^k\right)^{\mathbf{Z}}$$

that is given by

$$(3.65)\qquad z_i = \begin{cases} \Phi_{\ell(x)}\big((x_i)_{i=1}^{\ell(x)}\big) & \text{if } S^i x \in E, \\ \phi & \text{if } S^i x \notin E. \end{cases}$$

The mapping that carries $a = \Lambda x$, $x \in X$, into the

$$(3.66)\qquad \hat{z} \in \left(\{\phi\} \cup \bigcup_{k=1}^{\infty} \mathscr{A}^k\right)^{\mathbf{Z}}$$

that is given by

$$(3.67)\qquad \hat{z}_i = \begin{cases} (v_x(k))_{k=1}^{QD_{\ell(x)}} & \text{if } S^i x \in E, \\ \phi & \text{if } S^i x \notin E, \end{cases}$$

is one to one and commutes with the shift as can be shown by an argument that is similar to the one that was given in the second part of the proof of Theorem (2.1) of [9]. Let

$$(3.68)\qquad \begin{aligned} \hat{s} &= a^{(0)} + \sum_{i=1}^{\hat{B}-L} c, \\ \hat{B} &= 4mt\pi(\mu^{(L)})^2 + P. \end{aligned}$$

The $\omega_{\rho,\sigma}$, $\gamma(f, f')$, and the $d^{(\ell)}$ were structured such that $x \in E \cap X$ if and only if

$$(3.69)\qquad (U_r x)_{i=A+B-\hat{B}+1}^{A+B} = \hat{s}, \qquad 1 \leqq r \leqq 2n^{L'}.$$

These two facts and the observation that was made at the end of the second part of the proof allow us to conclude that the U_r are one to one.

For the fourth and final part of the proof of Lemma 3.1, we proceed to show that all of the U_r have the properties (3.3) and (3.4) and that at least one of them has also property (3.5).

We have from (3.12), (3.21), and (3.22) that

$$(3.70)\quad \mu(\{x \in X : |\{i \in \mathbf{Z}: -i^-(x) < i \leqq i^+(x), (U_r x)_i \neq x_i\}| < \tfrac{1}{2}\varepsilon\ell(x)\}) > 1 - \tfrac{1}{2}\varepsilon \qquad 1 \leqq r \leqq 2n^{L'}.$$

Hence, the individual ergodic theorem shows that the U_r have property (3.3). The U_r were also constructed such that we can infer from (3.11), (3.21), and (3.62) that for all $x \in E \cap X$,

$$(3.71)\qquad \left|\lambda^{(I)}\left[\left((U_r x)_i\right)_{i=j(Q+R)(A+B)+1}^{(j+1)(Q+B)(A+B)}\right], \mu^{(I)}\right| < \varepsilon, \qquad \begin{array}{l} 0 \leqq j < D_{\ell(x)}, \\ 1 \leqq r \leqq 2n^{L'}. \end{array}$$

From this and from (3.1) and (3.2) together with Lemma 2.1, we see that the U_r have property (3.4).

Last but not least there is an r_0 and an $a \in \Omega^{L'}$ such that

$$(3.72)\qquad (a_i)_{i=1}^{\hat{B}} = \hat{s},$$

and

$$(3.73)\qquad \mu(E_{r_0}) > \mu(U_{r_0}^{-1} Z_a \cap E_{r_0}) > 0.$$

With

$$(3.74)\qquad \begin{aligned} G_- &= \{x \in Z_a : (x_i)_{i=\hat{B}-A-B+1}^{\hat{B}} \in \{\omega_{1,1}, \omega_{1,2}\}\}, \\ G_+ &= \{x \in Z_a : (x_i)_{i=k(x)+1}^{k(x)+A} \in \{\omega_{1,2}, \omega_{2,2}\}\}, \\ k(x) &= \min\{k > L' : (x_i)_{i=k+1}^{k+A+B} \in \{\omega_{\rho,\sigma} + s_0 : \rho, \sigma = 1, 2\}\}, \end{aligned}$$

we have from (3.43) and (3.73),

$$(3.75)\qquad \begin{aligned} 0 &< U_{r_0}\mu(Z_a \cap G_-) < U_{r_0}\mu(Z_a), \\ 0 &< U_{r_0}\mu(Z_a \cap G_+) < U_{r_0}\mu(Z_a), \end{aligned}$$

and

$$(3.76)\qquad U_{r_0}\mu(Z_a \cap G_- \cap G_+) = 0.$$

Hence, $U = U_{r_0}$ has property (3.4).

LEMMA 3.2. *Let $0 < \varepsilon < 1$, $I \in \mathbf{N}$, and let μ be a nonatomic ergodic shift invariant measure on $\Omega^{\mathbf{Z}}$ such that $h(\mu) < \log|\Omega|$. Then there exist a shift invariant Borel set $X \subset \Omega^{\mathbf{Z}}$, $\mu(X) = 1$, and a one to one Borel mapping $U: X \to \Omega^{\mathbf{Z}}$ that commutes with the shift and K, L, $M \in \mathbf{N}$, such that*

$$(3.77)\qquad K > \varepsilon^{-1} I, \qquad M > \varepsilon^{-1}, \qquad L > 6M(K+1),$$

and such that

$$(3.78)\qquad U\mu(\mathscr{D}_{M,K}(\mu^{(I)}, \varepsilon)) = 1$$

and

$$(3.79)\qquad \tilde{h}((U\mu)^{(L)}) > h(\mu).$$

The proof of this lemma is similar to the proof of Lemma 3.1.

THEOREM 3.1. *Let $0 < \delta < 1$, $I \in \mathbf{N}$, and let μ be a nonatomic ergodic shift invariant probability measure on $\Omega^{\mathbf{Z}}$,*

$$(3.80)\qquad h(\mu) < \log|\Omega|.$$

Then there exists a uniquely ergodic sequence $x \in \Omega^{\mathbf{Z}}$ such that the systems $(\Omega^{\mathbf{Z}}, \mu, S)$ and $(\mathscr{O}_x, \mu_x, S)$ are isomorphic, and such that

$$(3.81)\qquad |\mu(Z_a) - \mu_x(Z_a)| < \delta, \qquad a \in \Omega^I.$$

PROOF. We are going to construct inductively a decreasing sequence $X_j \subset \Omega^{\mathbf{Z}}$ of shift invariant Borel sets, $\mu(X_j) = 1$, and a sequence $V_j \colon X_j \to \Omega^{\mathbf{Z}}$ of one to one Borel mappings that commute with the shift. At the same time, we shall obtain inductively also $A(j)$, $K(j)$, $M(j) \in \mathbf{N}$, $\mu_j \in \mathscr{W}_j$, and sets

$$\mathscr{C}(\alpha, j) \subset \prod_{-A(j) < i \leqq A(j)} \Omega, \qquad \alpha \in \Omega, \tag{3.82}$$

such that for all $j \in \mathbf{N}$ with $n = |\Omega|$

$$M(j) > jn^{2j}, \tag{3.83}$$

$$K(j) > n^{2j}, \tag{3.84}$$

$$L(j) \geqq 6 \max_{1 \leqq k \leqq j} M(k)\big(K(k) + 1\big), \tag{3.85}$$

$$\tilde{h}\big((V_j\mu)^{(L(j))}\big) > h(\mu), \tag{3.86}$$

$$V_jX_j \subset \bigcap_{k=1}^{j} \mathscr{D}_{M(k),K(k)}(\mu_k, n^{-2k}), \tag{3.87}$$

$$\mu\Big(\bigcup_{\alpha\in\Omega} (V_{j-1}^{-1} Z_{(\alpha)} \Delta V_j^{-1} Z_{(\alpha)})\Big) < \delta I^{-1} 2^{-j} n^{-3A'(j-1)}, \qquad A'(j) = \max_{1 \leqq k \leqq j} A(k), \tag{3.88}$$

$$\big(A'(0) = 0,\ V_0 = 1\big),$$

$$\mu\Big(Z_{(\alpha)} \Delta V_j^{-1}\Big(\bigcup_{a \in \mathscr{C}(\alpha, j)} Z_a\Big)\Big) < 2^{-j}. \tag{3.89}$$

We set $A(1) = 1$ and we use (3.80) and Lemma 3.2 to obtain a set X_1 and a mapping V_1 as well as K_1, L_1, M_1 with the desired properties, setting

$$\mathscr{C}(\alpha, 1) = \{(\alpha)\}, \qquad \alpha \in \Omega. \tag{3.90}$$

Assume now that we have already carried out the construction up to index j. Because of (3.85), (3.86), and (3.87), we can apply Lemma 3.1 to the measure $V_j\mu$ with $\theta = \{1, \cdots, j\}$ and

$$\varepsilon_k = n^{-2k}, \qquad 1 \leqq k \leqq j, \tag{3.91}$$

and produce a shift invariant Borel set $Y \subset V_jX_j$, $\mu(V_j^{-1}Y) = 1$, a one to one Borel mapping $U \colon Y \to \Omega^{\mathbf{Z}}$ that commutes with the shift, and $K(j + 1)$, $L(j + 1)$, $M(j + 1)$, such that

$$\begin{aligned} &M(j + 1) > (j + 1)n^{2j+2}, \\ &K(j + 1) > n^{2j+2}, \\ &L(j + 1) \geqq 6 \max_{1 \leqq k \leqq j+1} M(k)\big(K(k) + 1\big), \end{aligned} \tag{3.92}$$

such that

$$V_j\mu\Big(\bigcup_{\alpha\in\Omega} (Z_{(\alpha)} \Delta U^{-1} Z_{(\alpha)})\Big) < \delta I^{-1} 2^{-j-1} n^{-3A'(j)} \tag{3.93}$$

such that

$$\tilde{h}\big((UV_j\mu)^{(L(j+1))}\big) > h(\mu), \tag{3.94}$$

and such that with

$$\mu_{j+1} = (UV_j\mu)^{(j+1)},$$

$$UY \subset \bigcap_{1\leqq k \leqq j+1} \mathscr{D}_{M(k),K(k)}(\mu_k, n^{-2k}). \tag{3.95}$$

We set then

$$X_{j+1} = V_j^{-1}Y, \tag{3.96}$$

$$V_{j+1}x = UV_jx, \qquad x \in X_{j+1}.$$

We find from (3.93) that

$$\mu\Big(\bigcup_{\alpha\in\Omega} (V_j^{-1}Z_{(\alpha)}\Delta V_{j+1}^{-1}Z_{(\alpha)})\Big) < \delta I^{-1}2^{-j-1}n^{-3A'(j)}. \tag{3.97}$$

Since V_{j+1} is one to one, we can find an $A(j+1)$ and

$$\mathscr{C}(\alpha, n+1) \subset \prod_{A(j+1)<i\leqq A(j+1)} \Omega, \qquad \alpha \in \Omega, \tag{3.98}$$

such that

$$\mu\Big(Z_{(\alpha)}\Delta V_{j+1}^{-1}\Big(\bigcup_{a\in\mathscr{C}(\alpha,j+1)} Z_a\Big)\Big) < 2^{-j-1}. \tag{3.99}$$

This concludes the induction.

It follows from (3.88) that there is a Borel mapping $W: \Omega^{\mathbf{Z}} \to \Omega^{\mathbf{Z}}$ that commutes with the shift such that for μ-a.a. $x \in \Omega^{\mathbf{Z}}$,

$$(Wx)_i = \lim_{n\to\infty} (V_jx)_i, \qquad i \in \mathbf{Z}, \tag{3.100}$$

and

$$\mu(Z_{(\alpha)}\Delta W^{-1}Z_{(\alpha)}) < \delta I^{-1}. \qquad \alpha \in \Omega. \tag{3.101}$$

We infer from (3.88) that for all $j \in \mathbf{N}$,

$$\mu(V_j^{-1}Z_{(\alpha)}\Delta W^{-1}Z_{(\alpha)}) \leqq \sum_{k\geqq j} \mu(V_k^{-1}Z_{(\alpha)}\Delta V_{k+1}^{-1}Z_{(\alpha)}) < 2^{-j}n^{-3A'(j)}. \tag{3.102}$$

Hence,

$$\mu(V_j^{-1}Z_a\Delta W^{-1}Z_a) < 2^{-j}n^{-2A(j)}, \qquad a \in \prod_{A(j)<i\leqq A(j)} \Omega. \tag{3.103}$$

Hence, also

$$\mu\Big(V_j^{-1}\Big(\bigcup_{a\in\mathscr{C}(\alpha,j)} Z_a\Big)\Delta W^{-1}\Big(\bigcup_{a\in\mathscr{C}(\alpha,j)} Z_a\Big)\Big) < 2^{-j}, \qquad \alpha \in \Omega, \tag{3.104}$$

and from (3.89),

$$\mu\Big(Z_{(\alpha)}\Delta W^{-1}\Big(\bigcup_{a\in\mathscr{C}(\alpha,j)} Z_a\Big)\Big) < 2^{-j+1}, \qquad \alpha \in \Omega. \tag{3.105}$$

Exploiting shift invariance, we conclude from this that there exists a shift invariant Borel set $X \subset \bigcap_{n=1}^{\infty} X_j$, $\mu(x) = 1$, such that W if restricted to X is one to one. We claim that, for all $x \in X$, Wx is a uniquely ergodic sequence such that

$$\mu_{Wx} = W\mu. \tag{3.106}$$

To establish this, we prove first that

$$WX \subset \bigcap_{j=1}^{\infty} \mathscr{D}_{M(j),K(j)}(\mu_j, n^{-2j}). \tag{3.107}$$

Indeed, for $j \in \mathbf{N}$, $i' \in \mathbf{Z}$ and $x \in X$ there is a $j' \geqq j$ such that

$$(Wx)_i = (V_{j'}x)_i, \qquad i' \leqq i < i' + 3M(j)(K(j) + 1), \tag{3.108}$$

and by (3.87), $V_{j'}x \in \mathscr{D}_{M(j),K(j)}(\mu_j, n^{-2j})$ and (3.107) follows. By Lemma 2.2, for

$$N(j) = n^{2j}M(j)(K(j) + 1), \tag{3.109}$$

$WX \subset \mathscr{U}_{N(j)}(\mu_j, 2^5 n^{-2j}), j \in \mathbf{N}$.

From the individual ergodic theorem, therefore, for all $j \in \mathbf{N}$,

$$|\mu_j, (W\mu)^{(j)}| < 2^5 n^{-2j}. \tag{3.110}$$

Hence,

$$WX \subset \bigcap_{j=1}^{\infty} \mathscr{U}_{N(j)}((W\mu)^{(j)}, 2^6 n^{-2j}). \tag{3.111}$$

This implies that every element in WX is uniquely ergodic and that (3.106) holds. Equation (3.81) follows from (3.101).

COROLLARY 3.1. *For every ergodic invertible p preserving transformation T of the Lebesgue measure space $(E, \mathscr{B}, p)$ with finite entropy $h(T)$ there exists a uniquely ergodic sequence $x \in \Omega^{\mathbf{Z}}$, $e^{h(T)} < |\Omega| \leqq e^{h(T)} + 1$, such that the systems (E, p, T) and $(\mathscr{O}_x, \mu_x, S)$ are isomorphic.*

PROOF. We know from the finite generator theorem (see [9] and [10]) that there is a shift invariant probability measure μ on $\Omega^{\mathbf{Z}}$, $e^{h(T)} < |\Omega| \leqq e^{h(T)} + 1$, such that the systems (E, p, T) and $(\Omega^{\mathbf{Z}}, \mu, S)$ are isomorphic. Hence, the corollary is a consequence of Theorem 3.3.

4. Infinite entropy

We define for a mapping $\xi: \theta \to \theta'$:

$$\hat{\xi}x = (\xi x_i)_{i=-\infty}^{\infty}, \qquad x \in \theta^{\mathbf{Z}}. \tag{4.1}$$

For the case of infinite entropy, we need yet another version of Lemma 3.1.

LEMMA 4.1. *Let $0 < \varepsilon < 1$, $I \in \mathbf{N}$. Let θ be a finite set and let there be given $I(\vartheta)$, $K(\vartheta)$, $M(\vartheta) \in \mathbf{N}$, $\varepsilon > 0$, finite sets Γ_ϑ, mappings $\varphi_\vartheta: \Omega \to \Gamma_\vartheta$, and probability measures μ_ϑ on $\Gamma_\vartheta^{I(\vartheta)}$, $\vartheta \in \theta$. Let there further be given a nonatomic ergodic* *shift invariant probability measure μ on $\Omega^{\mathbf{Z}}$ such that $h(\mu) < \log |\Omega|$ and*

$$\mu\Big(\bigcap_{\vartheta\in\theta} \hat{\varphi}_{\varphi}^{-1}\mathscr{D}_{M(\vartheta),K(\vartheta)}(\mu_\vartheta, \varepsilon_\vartheta)\Big) = 1. \tag{4.2}$$

Then there exists a shift invariant Borel set $X \subset \Omega^{\mathbf{Z}}$, $\mu(X) = 1$, a one to one Borel mapping $U: X \to \Omega^{\mathbf{Z}}$ that commutes with the shift such that

$$\mu(Z_{(\alpha)}\Delta U^{-1}Z_{(\alpha)}) < \varepsilon, \qquad \alpha \in \Omega, \tag{4.3}$$

and also $M, K \in \mathbf{N}$, such that

$$M > \varepsilon^{-1}I, \qquad K > \varepsilon^{-1}, \tag{4.4}$$

and such that

$$U\mu\Big(\bigcap_{\vartheta\in\theta} \hat{\varphi}_{\vartheta}^{-1}\mathscr{D}_{M(\vartheta),K(\vartheta)}(\mu_\vartheta, \varepsilon_\vartheta)\Big) = U\mu\big(\mathscr{D}_{M,K}(\mu^{(I)}, \varepsilon)\big) = 1. \tag{4.5}$$

In the proof of our last theorem, we shall use the shift space $(\{0, 1, 2\}^{\mathbf{N}})^{\mathbf{Z}}$. Again, we say that an $x \in (\{0, 1, 2\}^{\mathbf{N}})^{\mathbf{Z}}$ is uniquely ergodic if the shift is a uniquely ergodic homeomorphism of $\mathcal{O}_x = \overline{\{S^i x : i \in \mathbf{Z}\}}$ and we denote the corresponding invariant probability measure again by μ_x.

THEOREM 4.1. *For every ergodic invertible p preserving transformation T of a Lebesgue measure space $(E, \mathscr{B}, p)$ there exists a uniquely ergodic point x in $(\{0, 1, 2\}^{\mathbf{N}})^{\mathbf{Z}}$ such that the systems (E, p, T) and $(\mathcal{O}_x, \mu_x, S)$ are isomorphic.*

PROOF. Let $(E, \mathscr{B}, p)$ be the unit interval together with Lebesgue measure and let $\{F(j): j \in \mathbf{N}\}$ be a collection of Borel subsets of E that generates $\mathscr{B}$. We set

$$\varphi_{j,k}a = (a_i)_{j=1}^{k}, \qquad a \in \{0, 1, 2\}^j, j \geqq k. \tag{4.6}$$

We are going to construct inductively a decreasing sequence $X_j \subset E$ of T invariant Borel sets, $p(X_j) = 1$, and a sequence $V_j: X_j \to (\{0, 1, 2\}^j)^{\mathbf{Z}}$ of Borel mappings such that the diagrams

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "V_j"] & E \arrow[d, "V_j"] \\ (\{0,1,2\}^j)^{\mathbf{Z}} \arrow[r, "S"] & (\{0,1,2\}^j)^{\mathbf{Z}} \end{tikzcd} \tag{4.7}$$

are commutative. At the same time we shall obtain inductively also $K(j)$, $M(j) \in \mathbf{N}$, probability measures μ_j on $(\{0, 1, 2\}^j)^j$, such that for all $j \in \mathbf{N}$,

$$M(j) < j3^{2j^2}, \tag{4.8}$$

$$K(j) > 3^{2j^2}, \tag{4.9}$$

$$V_jX_j \subset \bigcap_{k=1}^{j} \hat{\varphi}_{j,k}^{-1}\mathscr{D}_{M(k),K(k)}(\mu_k, 3^{-2k^2}), \tag{4.10}$$

and $A(j) \in \mathbf{N}$ and sets

$$\mathscr{C}(j, k) \subset \prod_{A(j)<i\leqq A(j)} \{0, 1, 2\}^j, \qquad 1 \leqq k \leqq j, \tag{4.11}$$

such that

$$p\big(F(k)\Delta V_j^{-1}\big(\bigcup_{a\in\mathscr{C}(j,k)} Z_a\big)\big) < 2^{-j}, \qquad 1 \leqq k \leqq j. \tag{4.12}$$

Also, we shall have with $A'(j) = \max_{1\leqq k\leqq j} A(k)$,

$$p\left(V_{j-1}^{-1}Z_{(a)}\Delta V_j^{-1}\left(\bigcup_{\alpha=0}^{2} Z_{((a_1,\cdots,a_{j-1},\alpha))}\right)\right) < 2^{-j}3^{-3A'(j-1)}, \qquad a \in \{0, 1, 2\}^{j-1}, j > 1. \tag{4.13}$$

Let

$$V_0x = \big(\chi_{F(1)}(S^ix)\big)_{i=-\infty}^{\infty} \in \{0, 1, 2\}^{\mathbf{Z}}, \qquad x \in E. \tag{4.14}$$

We can apply Lemma 3.2 to obtain $M(1)$, $K(1) > 9$, a shift invariant Borel set $Y_1 \subset \{0, 1, 2\}^{\mathbf{Z}}$, $p(V_0^{-1}Y_1) = 1$, and a one to one Borel mapping

$$U_1: Y_1 \to \{0, 1, 2\}^{\mathbf{Z}} \tag{4.15}$$

that commutes with the shift such that

$$U_1Y \subset \mathscr{D}_{M(1),K(1)}\big((U_1V_0p)^{(1)}, 3^{-2}\big). \tag{4.16}$$

We can set

$$\begin{aligned} X_1 &= V_0^{-1}Y_1, \\ V_1x &= U_1V_0x, \qquad x \in X. \end{aligned} \tag{4.17}$$

Since V_1 is one to one there exist $A(1)$ and $\mathscr{C}(1, 1)$ with the required properties.

Assume that the construction has already been carried out up to index $j \in \mathbf{N}$. We define then a Borel mapping $Q: X_j \to \big(\{0, 1, 2\}^{j+1}\big)^{\mathbf{Z}}$ by setting

$$Qx = \big((V_jx)_i + \big(\chi_{F(j+1)}(S^ix)\big)\big)_{i=-\infty}^{\infty}, \qquad x \in X_j. \tag{4.18}$$

We apply Lemma 4.1 to the measure Qp and to $\Gamma_k = \{0, 1, 2\}^k$, $\varphi_{j,k}$, and $\varepsilon_k = 3^{-2k^2}$, $1 \leqq k \leqq j$, to obtain

$$M(j + 1) > (j + 1)3^{2(j+1)^2}, \qquad K(j + 1) > 3^{2(j+1)^2}, \tag{4.19}$$

a shift invariant Borel set $Y_{j+1} \subset \big(\{0, 1, 2\}^{j+1}\big)^{\mathbf{Z}}$, $p(Q^{-1}Y_{j+1}) = 1$, and a one to one Borel mapping

$$U_{j+1}: Y_{j+1} \to \big(\{0, 1, 2\}^{j+1}\big)^{\mathbf{Z}} \tag{4.20}$$

such that with

$$\begin{aligned} \mu_{j+1} &= (U_{j+1}V_jp)^{(j+1)}, \\ U_{j+1}Y_{j+1} &\subset \bigcap_{k=1}^{j+1} \hat{\varphi}_{j,k}^{-1}\mathscr{D}_{M(k),K(k)}(\mu_k, 3^{-2k^2}), \end{aligned} \tag{4.21}$$

and such that

$$p\big(Q^{-1}(Z_{(a)}\Delta U_{j+1}^{-1}Z_{(a)})\big) < 2^{-j-1}3^{-3A'(j)}, \qquad a \in \{0, 1, 2\}^{j+1}. \tag{4.22}$$

We set

$$X_{j+1} = Q^{-1}Y_{j+1}, \qquad V_{j+1}x = U_{j+1}Qx. \qquad x \in X_{j+1}. \tag{4.23}$$

Since U_{j+1} is one to one, we can infer the existence of suitable $A(j + 1, k)$ and $\mathscr{C}(j + 1, k)$.

It follows from (4.13) that there exists a Borel mapping $W: E \to (\{0, 1, 2\}^{\mathbf{N}})^{\mathbf{Z}}$ such that for p-a.a. $x \in E$,

$$(Wx)_{\ell,i} = \lim_{j=\infty} (V_j x)_{\ell,i}, \qquad \ell \in \mathbf{N},\ i \in \mathbf{Z}. \tag{4.24}$$

Since $\mathscr{B}$ is generated by $\{F(j): j \in \mathbf{N}\}$, we can infer from (4.12) and (4.13) that there exists a T invariant Borel set $X \subset \bigcap_{j=1}^{\infty} X_j$, $p(X) = 1$, such that W if restricted to X is one to one. Set

$$\psi_j y = (y_k)_{k=1}^{j}, \qquad y \in \{0, 1, 2\}^{\mathbf{N}},\ j \in \mathbf{N}. \tag{4.25}$$

To establish that for all $x \in X$, Wx is uniquely ergodic such that

$$\mu_{Wx} = Wp, \tag{4.26}$$

it is, by (4.8), (4.9), Lemma 2.2, and by the individual ergodic theorem, enough to show that for all $j \in \mathbf{N}$,

$$\hat{\psi}_j Wx \in \mathscr{D}_{M(j),K(j)}(\mu_j, 3^{-2j^2}). \tag{4.27}$$

That this is indeed so follows from (4.10) and (4.24).

REFERENCES

[1] CH. GRILLENBERGER, "Zwei kombinatorische Konstruktionen für strikt ergodische Folgen," thesis, Universität Erlangen-Nürnberg, 1970.

[2] F. HAHN and Y. KATZNELSON, "On the entropy of uniquely ergodic transformations," *Trans. Amer. Math. Soc.*, Vol. 126 (1967), pp. 335–360.

[3] K. JACOBS, "Systèmes dynamiques Riemaniens," *Czechoslovak Math. J.*, Vol. 20 (1970), pp. 628–631.

[4] ———, "Lipschitz functions and the prevalence of strict ergodicity for continuous time flows," Contributions to Ergodic Theory and Probability, *Lecture Notes in Mathematics*, Berlin–Heidelberg–New York, Springer, 1970, Vol. 160, pp. 87–124.

[5] K. JACOBS and M. KEANE, "0-1 sequences of Toeplitz type," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 13 (1969), pp. 123–131.

[6] R. I. JEWETT, "The prevalence of uniquely ergodic systems," *J. Math. Mech.*, Vol. 19 (1970), pp. 717–729.

[7] S. KAKUTANI, "Ergodic theory of shift transformations," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1967, Vol. 2, pp. 405–414.

[8] M. KEANE, "Generalized Morse sequences," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 10 (1968), pp. 335–353.

[9] W. KRIEGER, "On entropy and generators of measure-preserving transformations," *Trans. Amer. Math. Soc.*, Vol. 149 (1970), pp. 453–464.

[10] ———, "On generators in exhaustive σ-algebras of ergodic measure-preserving transformations," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, Vol. 20 (1971), pp. 75–82.
[11] J. C. OXTOBY, "Ergodic sets," *Bull. Amer. Math. Soc.*, Vol. 58 (1952), pp. 116–136.
[12] W. A. VEECH, "Strict ergodicity in zero-dimensional dynamical systems and the Kronecker-Weyl theorem mod. 2," *Trans. Amer. Math. Soc.*, Vol. 140 (1969), pp. 1–33.