A COUNTEREXAMPLE ON MEASURABLE PROCESSES

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In 1947, Doob [7] posed the following question. Suppose $x = \{x_t, 0 \leqq t \leqq 1\}$ is a (jointly) measurable stochastic process with values in a compact space K, for example, the one point compactification $\bar{R}$ of the real line R. Let $\bar{P}_x$ be the distribution of x in the compact function space of all functions from $[0, 1]$ into K, where $\bar{P}_x$ is a regular Borel measure [15]. Then is the evaluation map $E: (t, f) \to f(t)$ necessarily measurable for the product measure $\lambda \times \bar{P}_x$, where λ is Lebesgue measure? I shall give a counterexample, assuming the continuum hypothesis. The counterexample is a Gaussian process. Replacing $([0, 1], \lambda)$ by an equivalent measure space (H, μ), where H is a Hilbert space and μ a suitable Gaussian probability measure, we can take the process x to be the standard Gaussian linear process L on H. Although we shall carry through the details only for this particular process, the method is applicable to various other processes represented by convergent series $\Sigma y_n(t) z_n(\omega)$ with independent terms such that $\Sigma y_n(t)$ and $\Sigma z_n(\omega)$ are not convergent in general. The possibility of weakening the continuum hypothesis assumption will be discussed in an Appendix.

Earlier, M. Mahowald [14] proposed a positive solution to the Kakutani-Doob problem. But the last step in his argument applies the Fubini theorem to sets in a product space which have not been shown to be measurable.

After the counterexample (Proposition 1), we give a few easier facts which also contribute to a broader conclusion that uncountable Cartesian products of compact metric spaces (for example, intervals) are relatively "bad" spaces as regards measurability.

DEFINITION. *Let $(X, \mathscr{B})$ be a measurable space. An X valued stochastic process with parameter set T and probability space $(\Omega, \mathscr{S}, P)$ is a function x from $T \times \Omega$ into X such that for each t in T, $x(t, \cdot)$ is measurable from $(\Omega, \mathscr{S})$ into $(X, \mathscr{B})$.*

Let X^T denote the set of all functions from T into X. Suppose X is a Polish space (complete separable metric space) or a compact Hausdorff space and $\mathscr{B}$ its class of Borel sets. Then for any stochastic process x as in Definition 1, there is a probability measure P_x on X^T such that for any $t_1, \cdots, t_n \in T$ and $B_1, \cdots, B_n \in \mathscr{B}$,

$$(1) \qquad P\{\omega: X(t_j, \omega) \in B_j, j = 1, \cdots, n\} = P_x\{f: f(t_j) \in B_j, j = 1, \cdots, n\},$$

according to a well-known theorem of Kolmogorov.

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Now if X is a compact Hausdorff space K, then K^T is also a compact Hausdorff space with product topology (Tychonov's theorem). If the original process has values in R we can take K to be the one point compactification $\bar{R}$ (or, if desired, a two point or other compactification). The following result is due to Kakutani ([12] and unpublished; see Doob [7] and Nelson [14]).

THEOREM. *Let x be a K valued stochastic process where K is a compact Hausdorff space. Then P_x on K^T has a unique extension to a regular Borel probability measure $\bar{P}_x$.*

Now suppose the parameter set T of a stochastic process x is a measure space $(T, \mathscr{T}, v)$. Then x may or may not be jointly measurable for various σ-algebras in $T \times \Omega$. If we say simply that x is *measurable*, we mean it is measurable for the completion of $v \times P$.

If (X, d) is a metric space and $(T, \mathscr{T}, v)$ a finite measure space $\mathscr{L}^0(T, X)$ denotes the class of all Borel measurable functions from T into X. $L^0(T, X)$ denotes the set of equivalence classes of functions in $\mathscr{L}^0$ for equality v almost everywhere.

Let I be the unit interval $[0, 1]$. Let x be a measurable process on I with values in a compact metric space K. Since we will show that evaluation E need not be $\lambda \times \bar{P}_x$ measurable, it is natural to consider the extended product measure of Bledsoe and A. P. Morse [5]. Given σ-finite measure spaces $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, v)$, Bledsoe and Morse extend the usual product measure $\mu \times v$ to a measure $\mu \times_M v$) by setting $(\mu \times_M v)(A) = 0$ if and only if both iterated integrals of its indicator function are 0:

$$\int\int \chi_A(x, y)\, d\mu(x)\, dv(y) = \int\int \chi_A(x, y)\, dv(y)\, d\mu(x) = 0. \tag{2}$$

If a real valued function f is $\mu \times_M v$ measurable, then it equals a $\mu \times v$ measurable function almost everywhere for $\mu \times_M v$. Hence $f(x, \cdot)$ is v measurable for μ almost all x, and likewise for $f(\cdot, y)$. It will turn out that for suitable measurable processes x, E is not even $\lambda \times_M \bar{P}_x$ measurable.

Let H be the Hilbert space ℓ_2 of square summable sequences $y = \{y_n\}_{n=1}^{\infty}$. Then the standard Gaussian linear process L on H can be represented by

$$L(y)(\omega) = \sum_{n=1}^{\infty} y_n G_n(\omega), \tag{3}$$

where the G_n are independent normalized Gaussian random variables on a probability space (Ω, P). We shall assume Ω is a countable Cartesian product of real lines and P a product of standard Gaussian measures. For $\omega \in \Omega$ we then have $\omega = \{\omega_n\}_{n=1}^{\infty}$ and $G_n(\omega) \equiv \omega_n$.

Let μ be the Borel probability measure on H for which the coordinate functions y_n are independent Gaussian with

$$Ey_n = 0, \qquad Ey_n^2 = n^{-3/2}. \tag{4}$$

(Since $\Sigma n^{-3/2}$ converges, $\Sigma y_n^2 < \infty$ with probability 1 and μ is indeed a countably additive probability measure on H.) By Lindelöf's and Ulam's theorems, μ is a regular Borel measure on H. For every proper closed linear subspace J of H, $\mu(J) = 0$. There is a one to one measurable function g from I onto H with $\lambda \circ g^{-1} = \mu$.

Added in proof. See P. Halmos and J. von Neumann, "Operator methods in classical mechanics, II," *Ann. of Math.*, Vol. 43 (1942), pp. 332–350, Theorem 2; or, more explicitly, V. A. Rokhlin, "On the fundamental ideas of measure theory," *Mat. Sbornik*, Vol. 25 (1949), pp. 107–150; *Amer. Math. Soc. Translations*, Vol. 71 (1952), pp. 1–54, especially Section 2, No. 7.

Since the problem at hand involves only the measurable structure of the parameter set T of processes (rather than, for example, the topology of T) we can use (I, λ) or (H, μ) interchangeably.

The series (3) defining L converges $(\mu \times P)$ almost surely, and P almost surely for every $y \in H$. Thus L is a measurable process on H. Here is the main result.

PROPOSITION 1. *Assuming the continuum hypothesis, the* $\bar{P}_L$ *inner measure of* $\mathscr{L}^0(H, \bar{R})$ *is* 0, *so* E *is not measurable for* $\mu \times \bar{P}_L$ *nor even* $\mu \times_M \bar{P}_L$.

Most of the proof is in the following lemma.

LEMMA. *Assuming the continuum hypothesis, for every measurable set* $A \subset \Omega$ *with* $P(A) > 0$ *there exists a set* $S \subset H$ *with outer measure* $\mu^*(S) = 1$ *such that for every finite set* $F \subset S$ *and any nonempty open sets* $U_f \subset R$,

$$P\{\omega \in A : L(f)(\omega) \in U_f \text{ for all } f \in F\} > 0. \tag{5}$$

PROOF. (Note that S will necessarily be linearly independent for finite linear combinations and in general will be independent of the standard orthonormal basis of ℓ_2.) By the continuum hypothesis, we take the cardinal c as the set of all countable ordinals. Let $\{C_\alpha : \alpha \in c\}$ be the class of all closed sets C in H with $\mu(C) > 0$ (there are exactly c such sets). We will define $S = \{s_\alpha : \alpha \in c\}$ recursively.

Let α be a countable ordinal and suppose given $S_\alpha = \{s_\beta : \beta < \alpha\}$, such that all finite sets $F \subset S_\alpha$ have the property stated in the lemma (the empty set does have that property, in case $\alpha = 0$).

There exists an increasing sequence $\mathscr{A}_n$ of σ-algebras of subsets of Ω such that the union of the $\mathscr{A}_n$ generates the σ-algebra of Borel sets in Ω, and such that each $\mathscr{A}_n$ is generated by a finite class $\mathscr{F}_n$ of disjoint sets such that whenever ω and $\zeta \in A \in \mathscr{F}_n$ and

$$\max\{|\omega_j|, |\zeta_j| : 1 \leqq j \leqq n\} \leqq n, \tag{6}$$

then $|\omega_j - \zeta_j| < 1/n$ for $j = 1, \cdots, n$.

For any measurable set $B \subset \Omega$, the conditional expectation $E(\chi_B | \mathscr{A}_n)$ is just $P(B \cap A)/P(A)$ on each set $A \in \mathscr{F}_n$. (For our P we may and do choose $\mathscr{F}_n$ so that $P(A) > 0$ for all $A \in \mathscr{F}_n$.) By the martingale convergence theorem, $E(\chi_B | \mathscr{A}_n)$ converges to 1 as $n \to \infty$ for P almost all points of B. These points will be called *density points* of B (for $\{\mathscr{A}_n\}$). For any $\zeta \in \Omega$, let $E^n(\zeta)$ be the unique set in $\mathscr{F}_n$ to which ζ belongs. Then ζ is a density point of B iff $\zeta \in B$ and

$$\lim_{n \to \infty} \frac{P\{E^n(\zeta) \cap B\}}{P\{E^n(\zeta)\}} = 1. \tag{7}$$

For any measurable set $A \subset \Omega$, finite set $F \subset H$, and collection $\{U_f: f \in F\}$ of nonempty open subsets of R, let

$$A_{F,U} = \{\omega \in A: L(f)(\omega) \in U_f \text{ for all } f \in F\}. \tag{8}$$

Now a sequence $\{r_n\}_{n=1}^{\infty}$ of real numbers will be called *recurrent* iff for every nonempty open $U \subset R$, $r_n \in U$ for some (and hence infinitely many) n. Note that this property depends only on the tail of the sequence, not on $r_1, \cdots, r_N$ for any fixed finite N.

If $y_n = (-1^k \omega_n / n^{3/4}$ for $n = 2^k, \cdots, 2^{k+1} - 1; k = 0, 1, \cdots,$ then for P almost all ω, the partial sums $\Sigma_{j=1}^n y_j \omega_j$ form a recurrent sequence. To prove this let

$$Z_k \equiv \sum_{2^k \leqq j < 2^{k+1}} \frac{(-1)^k \omega_j^2}{j^{3/4}}. \tag{9}$$

Then

$$EZ_k = (-1)^k \sum_{2^k \leqq j < 2^{k+1}} j^{-3/4} \tag{10}$$

which is asymptotic to $(-1)^k 2^{(k+8)/4}(2^{1/4} - 1)$;

$$\sigma^2(Z_k) = \sum_{2^k \leqq j < 2^{k+1}} 2j^{-3/2}, \tag{11}$$

which goes to 0 like $2^{-k/2}$ as $k \to \infty$. Thus with probability 1, $|Z_k - EZ_k| < 1$ for all large enough k. Hence $\Sigma^n y_j \omega_j$ walks back and forth with the length of individual steps converging to 0 with probability 1, while oscillating through larger and larger total distances between $n = 2^k$ and $n = 2^{k+1}$ as $k \to \infty$.

Let $\mathscr{R}$ be the set of all ω such that the sums just discussed are recurrent, so that $\mu(\mathscr{R}) = 1$. Let

$$\begin{aligned} T_{F,U} = \{y \in H&: \text{for some density point } \omega \text{ of } A_{F,U} \cap \mathscr{R}, \\ y_n &= (-1)^k \omega_n n^{-3/4} \text{ for } 2^k \leqq n < 2^{k+1} \\ &\qquad\qquad \text{for all large enough } k \text{ and } n\}. \end{aligned} \tag{12}$$

Then $\mu(T_{F,U}) = 1$ whenever $P(A_{F,U}) > 0$ because

$$\omega \to \{(-1)^k \omega_n n^{-3/4}, 2^k \leqq n < 2^{k+1}, k = 0, 1, \cdots\}, \tag{13}$$

is a measure preserving map of (Ω, P) onto (H, μ), so $\mu(T_{F,U}) \geqq P(A_{F,U} \cap \mathscr{R}) > 0$. But $\mu(T_{F,U}) = 0$ or 1, so $\mu(T_{F,U}) = 1$.

Now given $S_\alpha = \{s_\beta: \beta < \alpha\}$ such that $P(A_{F,U}) > 0$ whenever F is a finite subset of S_α, let $T_\alpha = \bigcap_{F,U} T_{F,U}$ where F runs over all finite subsets of S_α and U over all collections $\{U_f, f \in F\}$ of nonempty intervals in R with rational endpoints. Then $\mu(T_\alpha) = 1$ since S_α is countable. Thus we can and do choose $s_\alpha \in (T_\alpha \cap C_\alpha)$. This completes the recursive definition of S. Since S intersects each C_α and μ is regular, $\mu^*(S) = 1$.

Now we prove that S has the other property stated in the lemma. It suffices to prove inductively that S_α has that property for each countable ordinal α, assuming that S_β does for all $\beta < \alpha$. As noted, there is no problem if $\alpha = 0$. Also if α is a limit ordinal, the result is immediate. So we need only prove it for $S_{\alpha+1}$, assuming it for S_α.

Let G be a finite subset of $S_{\alpha+1} = \{s_\beta : \beta \leqq \alpha\}$. We may assume $G = F \cup \{s_\alpha\}$, $F \subset S_\alpha$, and for each U, $P(A_{F,U}) > 0$. We must prove $P(A_{G,U}) > 0$ for any nonempty open set $U_{s_\alpha} \subset R$.

Let ζ be a density point of $A_{F,U}$ such that for some M_0, we have $(s_\alpha)_j = (-1)^k \zeta_j j^{-3/4}$ whenever $2^k \leqq j < 2^{k+1}$ and $j \geqq M_0$. Given any $\varepsilon > 0$, there is an $M > M_0$ such that

$$P\{\omega : |\sum_{j \geqq M} (s_\alpha)_j \omega_j| < \varepsilon\} > \tfrac{1}{2}, \tag{14}$$

since $\Sigma_{j \geqq M} (s_\alpha)_j \omega_j$ is a Gaussian random variable with variance $\Sigma_{j \geqq M} (s_\alpha)_j^2 \to 0$ as $M \to \infty$.

Thus by independence, for all $N \geqq M$,

$$P(E^N(\zeta) \cap \{\omega : |\sum_{j > N} (s_\alpha)_j \omega_j| < \varepsilon\}) > \tfrac{1}{2} P(E^N(\zeta)). \tag{15}$$

On the other hand, since ζ is a density point of $A_{F,U}$,

$$P(A_{F,U} \cap E^N(\zeta)) > \tfrac{1}{2} P(E^N(\zeta)) \tag{16}$$

for all large enough N, and then

$$P(A_{F,U} \cap E^N(\zeta) \cap \{\omega : |\sum_{j > N} (s_\alpha)_j \omega_j| < \varepsilon\}) > 0. \tag{17}$$

For ω in the set just shown to have positive probability,

$$|L(s_\alpha)(\omega) - \sum_{j=1}^{N} (s_\alpha)_j \zeta_j| < \varepsilon. \tag{18}$$

But since $\Sigma_{j \leqq N} (s_\alpha)_j \zeta_j$ is a recurrent sequence of real numbers and N can be chosen arbitrarily large, this implies $P(A_{G,U}) > 0$, proving the lemma.

Now suppose C is a compact subset of $\bar{R}^H$, $\bar{P}_L(C) > 0$, and $C \subset \mathscr{L}^0(H, \bar{R})$. Let $\mathcal{O} = \{e_n\}$ be the standard orthonormal basis in H ($y \equiv \Sigma y_n e_n$); C is included in a compact Baire set C_1 with $\bar{P}_L(C) = \bar{P}_L(C_1) = P_L(C_1)$ (this follows easily from the theory in [11], Chapter 10). Whether $f \in C_1$ depends only on $f(y)$ for y in a countable set $Y \subset H$. Let η be the map from $R^{\mathcal{O}}$ into R^Y defined by

$$\eta(\omega)(y) = \Sigma y_n \omega_n. \tag{19}$$

This map is defined for P almost all $\omega \in R^{\mathcal{O}}$ and is P measurable. We have $C_1 = \pi^{-1}(C_2)$ where C_2 is a Borel set in $\bar{R}^Y$ and π is the natural projection of $\bar{R}^H$ onto $\bar{R}^Y$. By definition of L we have:

$$P(\eta^{-1}(C_2)) = P_L(C_1). \tag{20}$$

Thus we can apply the lemma to $\eta^{-1}(C_2)$ and get $S \subset H$ with $\mu^*(S) = 1$ such that $P(\eta^{-1}(C_2)_{F,U}) > 0$ for every finite $F \subset S$ and collection $\{U_f : f \in F\}$ of nonempty open sets in R. Equivalently,

(21) $$P_L\{\varphi \in C_1 : \varphi(f) \in U_f \text{ for all } f \in F\} > 0.$$

Since $C \subset C_1$ and $\bar{P}_L(C) = \bar{P}_L(C_1) = P_L(C_1)$, we then have:

(22) $$\bar{P}_L\{\varphi \in C : \varphi(f) \in U_f \text{ for all } f \in F\} > 0.$$

Since C is compact for pointwise convergence, this implies that all functions from S into $\bar{R}$ are restrictions of functions in C.

For any μ measurable set $D \subset H$ let $v(D \cap S) = \mu^*(D \cap S)$. Then v is a countably additive probability measure ([11], p. 75). For each single point $p \in S$. $v(\{p\}) = 0$. Hence, again applying the continuum hypothesis, v is not defined on all subsets of S ([14], Théorème 1). Thus not every function from S into $\bar{R}$ extends to a μ measurable function from H into $\bar{R}$. This contradicts $C \subset \mathscr{L}^0(H, \bar{R})$. Thus the $\bar{P}_L$ inner measure of $\mathscr{L}^0(H, \bar{R})$ is 0.

If E were $(\mu \times_M \bar{P}_L)$ measurable, then by a Fubini theorem ([5], Theorem 5.3), $\bar{P}_L$ almost all functions would be μ measurable. This contradiction shows that E is not measurable for $\mu \times_M \bar{P}_L$, nor *a fortiori* for $\mu \times \bar{P}_L$, *Q.E.D.*

It is notable that if $S(n) = \Sigma_{j=1}^n y_j \omega_j$ in the above proof, E is $\mu \times \bar{P}_{S(n)}$ measurable for every n but not $\mu \times \bar{P}_L$ measurable, although $S(n) \to L$ almost surely. The following discussion of related, simpler phenomena may help the reader's intuition (as it did the author's) as to what can go wrong. Let Z_2 denote the group with two elements $\{0, 1\}$.

Given finite measures P_n on a topological space S, we say that $P_n \to P_0$ (weak star) as $n \to \infty$ iff for every bounded continuous real valued function f on S, $\int f dP_n \to \int f dP_0$. (We assume the integrals are defined; if S is metrizable, this means the P_n must be defined on all Borel sets.)

If x_n are processes with the same probability space such that $x_n \to x_0$ almost surely, then by the bounded convergence theorem, $\bar{P}_{x_n} \to \bar{P}_{x_0}$ (weak star).

Conversely, Skorokhod [17] proved that if S is a complete separable metric space and P_n are probabilities on S with $P_n \to P_0$ (weak star), then there exist S valued random variables x_n over some probability space (Ω, P) with distributions P_n (that is, $P \circ x_n^{-1} = P_n$ for all n) such that $x_n \to x_0$ P almost surely. We shall see that this theorem does not hold for the compact Hausdorff space Z_2^I in place of S. Thus the hypothesis of metrizability cannot simply be removed.

As a compact Abelian group, Z_2^I has a Haar measure Q which equals $\bar{P}_v$, where v is a Z_2 valued process on I with independent values at different points and for each t, $P\{v(t) = 0\} = \frac{1}{2}$.

PROPOSITION 2. *There exist processes x_n such that $\bar{P}_{x_n} \to Q$ (weak star) with each $\bar{P}_{x_n}$ concentrated in a finite set of functions. Hence E is $(\lambda \times \bar{P}_{x_n})$ measurable for all n, but it is not $(\lambda \times Q)$ measurable.*

PROOF. Let $x_n(t, \omega) = v(j/n, \omega)$, $j/n \leqq t < (j + 1)/n$, $j = 0, \cdots, n + 1$, and $x_n(1, \omega) \equiv v(1, \omega)$. By the Stone-Weierstrass theorem, any continuous real function on Z_2^I can be uniformly approximated by one which depends on only finitely many coordinates. If F is a finite subset of I, then for n large enough, $\{x_n(t, \omega) : t \in F\}$ have the same joint distribution as $\{v(t, \omega) : t \in F\}$, namely

Haar measure on Z_2^F. Hence $\bar{P}_{x_n} \to Q$ (weak star). Clearly there are only finitely many functions $x_n(\cdot, \omega)$, so E is $(\lambda \times \bar{P}_{x_n})$ measurable. It is not $(\lambda \times Q)$ measurable, and in fact there is no measurable process y with $\bar{P}_y = Q$ (see Doob [6], Theorems 2.3 and 2.4 or Ambrose [1], Theorem 3 in the light of [2], Theorem 6), *Q.E.D.*

We conclude also that there cannot be measurable processes y_n with $P_{y_n} \equiv P_{x_n}$ and $y_n \to y_0$ almost surely, for then $\bar{P}_{y_0} = Q$ and y_0 is measurable. Hence Skorohod's theorem does not extend to Z_2^I.

The completion of P_v is defined on all Borel sets, as Kakutani showed ([13], Theorem 3). In other words, the Haar measure Q is "completion regular" ([11], p. 230 and p. 288, Theorem I). But P_v is defined only on Baire sets in Z_2^I, that is, measurable sets depending on only countably many coordinates. Now the only Baire set included in $\mathscr{L}^0(I, Z_2)$ is the empty set. Also the only Baire set disjoint from $\mathscr{L}^0(I, Z_2)$ is empty. Hence

$$Q^*(\mathscr{T}) = Q^*(Z_2^I \sim \mathscr{T}) = I. \tag{23}$$

where $\mathscr{T} = \mathscr{L}^0(I, Z_2)$. Likewise (23) remains true when $\mathscr{T}$ is taken to be the set of all Borel measurable functions from I into Z_2. Or, $\mathscr{T}$ may be the still smaller set of all functions f from I into Z_2 such that $\{t: f(t) = 1\}$ is a countable intersection of open sets. Thus these sets $\mathscr{T}$ of measurable functions are non-measurable subsets of Z_2^I; they are not Borel sets and are not measurable for the completion of Q, in particular. The best positive result known in this direction seems to be that the set of functions with only jump discontinuities is a Borel set in Z_2^I (Nelson [15]).

If L is the Gaussian process in the proof of Proposition 1 above, I do not know $\bar{P}_L^*(\mathscr{T})$ for any of the three classes of $\mathscr{T}$ of measurable functions mentioned above. Thus it seems conceivable that we could have $\bar{P}_L^*(\mathscr{L}^0(I, R)) = 0$ even though the process L is measurable.

Here is another simple example affecting uncountable product spaces. Let I^I be the space of all functions from I into itself with the product topology.

PROPOSITION 3. *There exist continuous functions f_n from I into I^I with $f_n(x) \to f(x)$ in I^I as $n \to \infty$ for all $x \in I$, yet f is not Borel measurable.*

PROOF. Let $f_n(x)(y) = \max\{1 - n|x - y|, 0\}$. Then $f_n(x) \to f(x)$, where $f(x)(y) = 1$ if $x = y$ and 0 if $x \neq y$. Then f is one to one and its range has discrete relative topology. So for every subset A of I there is an open set $U \subset I^I$ with $f^{-1}(U) = A$, so f is not measurable. *Q.E.D.*

Note that I^I is homeomorphic to a subset of a compact, Hausdorff, Abelian topological group K, or of a topological vector space S with various good properties. So I^I could be replaced by such a K or S in the statement of Proposition 3.

But as our first three propositions show, apparently "good" topological algebraic structures may have pathological measure theoretic properties. The following classes of topological spaces are more pleasant for measure theory.

DEFINITIONS. *A* Polish *space is a separable, metrizable topological space which is complete for some metrization.*

A Hausdorff topological space $(X, \mathcal{T})$ *is called a* Souslin *space if there is a continuous map f from a Polish space onto X. If f can be chosen one to one, then X is called a* Lusin *space.*

A topological space $(X, \mathcal{T})$ *is* regular *iff for every* $p \in X$ *and closed set* $F \subset X$ *with* $p \notin F$, *there exist disjoint open U, V with* $p \in U$ *and* $F \subset V$.

The space $(X, \mathcal{T})$ *is* normal *iff whenever E and F are disjoint and closed, there are disjoint open U and V with* $E \subset U$ *and* $F \subset V$.

A normal space is called perfectly normal *iff every open set is a countable union of closed sets.*

Fernique ([9], Proposition I.6.1, p. 19) shows that every regular Lusin space is paracompact and perfectly normal. His proof shows also that a regular Souslin space is paracompact and hence normal. An open subset of a Souslin space is Souslin because an open subset of a Polish space is Polish. Hence Fernique's proof shows that every regular Souslin space is perfectly normal.

Fernique and L. Schwartz [16], [17] have observed that regular Souslin and Lusin spaces have other properties convenient for measure theory. The classes of such spaces are stable for a great many operations and are broad enough to include the separable spaces of the theory of distributions.

Thus the following proposition has reasonable applications to nonmetrizable spaces. (Note, in correction to [9], Théorème I.4.2(a), that the hypothesis "perfectly normal" cannot be weakened to "normal", in view of Proposition 3 above.)

PROPOSITION 4. *Let* f_n *be measurable functions from a measurable space* $(X, \mathcal{S})$ *into a perfectly normal space* $(S, \mathcal{T})$ *such that* $f_n(x) \to f(x)$ *for all* $x \in X$. *Then f is also measurable.*

PROOF. It suffices to show that for every open $U \subset S, f^{-1}(U) \in \mathcal{S}$. We have $U = \bigcup_{n=1}^{\infty} F_n$ where the F_n are closed. Since $(X, \mathcal{T})$ is normal we can assume F_n is included in the interior of F_{n+1} for all n. Then

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} f_m^{-1}(F_n). \tag{24}$$

Q.E.D.

Yet, unfortunately, regular Lusin spaces do not have all possible good properties for measures. In particular, we shall see that Skorohod's theorem does not extend to Lusin spaces although it holds for all separable metric spaces, complete or not ([8] Theorem 3, p. 1569).

Let H be a separable, infinite dimensional Hilbert space with orthogonal basis $\{\varphi_m\}_{m=1}^{\infty}$. Let $\mathfrak{w}$ be the weak topology on H. Then $(H, \mathfrak{w})$ is clearly a Lusin space. As a Hausdorff topological group, it is (completely) regular ([12], Theorem 8.4, p. 70). Thus $(H, \mathfrak{w})$ is perfectly normal. The following example is suggested by an example of Fernique (see Badrikian [3], exposé 8, No. 6).

PROPOSITION 5. *Let* μ_n *be the probability measure on H which gives mass* $(\frac{1}{2})^n$ *to* $n\varphi_m$ *for* $m = 1, \cdots, 2^n$. *Then* μ_n *converges (weak star) on* $(H, \mathfrak{w})$ *to the unit* *mass at* 0. *But there are no random variables* X_n *with distributions* μ_n *which converge to* 0 *with positive probability*.

PROOF. Let f be a bounded continuous real function on $(H, \mathfrak{w})$. Given $\varepsilon > 0$ there exist $\psi_1, \cdots, \psi_r$ in H such that if $|(\varphi, \psi_j)| < 1$ for $j = 1, \cdots, r$, then $|f(\varphi) - f(0)| < \varepsilon$. Thus it suffices to show that

$$\int |(\varphi, \psi)|^2 \, d\mu_n(\varphi) \to 0 \tag{25}$$

for each $\psi \in H$. Now $\psi = \Sigma a_m \varphi_m$ where $\Sigma |a_m|^2 < \infty$, and

$$\int |(\varphi, \psi)|^2 \, d\mu_n(\varphi) = \frac{1}{2^n} \sum_{m=1}^{2^n} n^2 |a_m|^2, \tag{26}$$

which goes to 0 as $n \to \infty$ by the dominated convergence theorem.

Now if X_n are H valued random variables with distributions μ_n, then $\|X_n\| \to \infty$ with probability 1, and then by the Banach-Steinhaus theorem, X_n cannot converge to 0 in $(H, \mathfrak{w})$. *Q.E.D.*

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APPENDIX

Here is some discussion on weakening the continuum hypothesis assumption in the proof of Proposition 1. The continuum hypothesis was used via the following two assertions, which we may and do state in terms of Lebesgue measure λ rather than μ.

ASSERTION A.1. *For any collection of fewer than c sets of* λ *measure* 0, *their union also has* λ *measure*. 0.

ASSERTION A.2. *If* $\lambda^*(S) > 0$, *then not every subset of* S *is of the form* $A \cap S$ *where* A *is* λ *measurable*.

Now A.1 implies A.2 according to K. Kunen (Doctoral dissertation, Stanford University, 1968, "Inaccessibility properties of cardinals," Theorem 14.7 ii). Also it appears that A.1 may be strictly weaker than the continuum hypothesis, although at this writing I can only rèfer the interested reader to R. Solovay and J. Silver, to whom I am grateful for several conversations on these matters.

Note added in proof. Assertion A.2 has been proved, assuming only the usual set theory with the axiom of choice, in a letter to me from David Fremlin of Cambridge, England.

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