ITERATED LOGARITHM ANALOGUES FOR SAMPLE QUANTILES WHEN $p_n \downarrow 0$

J. KIEFER
CORNELL UNIVERSITY

1. Introduction

This paper is concerned with behavior of the Law of Iterated Logarithm (LIL) type for sample p_n-tiles, $p_n > 0$, when $p_n \downarrow 0$. The results are all stated for uniformly distributed random variables, from which they may easily be translated into results for general laws.

Let $X_1, X_2, \cdots$ be independent identically distributed random variables, uniformly distributed on $[0, 1]$. Let $T_n(x) = \{\text{number of } X_i \leqq x, 1 \leqq i \leqq n\}$, so that $n^{-1}T_n$ is the right continuous *sample distribution function* based on $X_1, X_2, \cdots, X_n$. Define the *sample p_n-tile* $Z_n(p_n)$ as $\min\{z: T_n(z) \geqq np_n\}$. This makes $Z_n(p_n) = np_n$-th order statistic when np_n is a fixed integer. (When $np_n \to \infty$ our results do not depend on the choice of definition of $Z_n(p_n)$ in cases of ambiguity.)

The earliest nontrivial result in this area, due to Baxter [2], is that, for any positive constant c,

$$\limsup_n T_n(c/n) \log\log\log n (\log\log n)^{-1} = 1, \qquad \text{wp } 1. \tag{1.1}$$

On the other hand, it is trivial (and a consequence of Theorem 2 herein, with $k = 1$) that

$$\liminf_n T_n(c/n) = 0, \qquad \text{wp } 1. \tag{1.2}$$

We thus no longer have the symmetry in asymptotic behavior of positive and negative deviations of $T_n(\pi_n) - ET_n(\pi_n)$ that prevails when π_n is constant; indeed, why should we, when $nT_n(c/n)$ is asymptotically Poisson rather than normal?

This difference in behavior means we will have to state results for the two directions of oscillations separately, and (since the analogue of (1.2) will not always be so simple to state) dictates a choice of nomenclature which we had best introduce at the outset: to eliminate possible confusion with reference to the two *directions* of oscillation, we drop the usual "upper or lower class" LIL terms completely, replacing these by "outer or inner class" for sequences $\{f_n\}$ beyond which $T_n(\pi_n)$ moves (in a direction away from $ET_n(\pi_n)$) finitely or infinitely often

Research carried out under ONR contract Nonr 401 (50) and NSF Grant GP 9297.

with probability one. Then, "top or bottom" bounds will refer to the most or least positive oscillations of $T_n(\pi_n)$. Thus, for example, $\{f_n\}$ is a bound of top inner class if $f_n > n\pi_n$ and $T_n(\pi_n) > f_n$ i.o. with probability one. If $\{(1 \pm \varepsilon)f_n\}$ gives top bounds of the two classes, we shall for brevity simply call $\{f_n\}$ a top bound. (If, as when π_n is constant, this yields too gross a result for T_n, we would instead specify a top bound on $T_n(\pi_n) - n\pi_n$.) Thus, log log n/log log log n is a top bound for $T_n(c/n)$ in Baxter's case mentioned above. Bounds for $Z_n(p_n)$ are described similarly. Of course, top bounds for Z_n are related to bottom bounds for T_n and vice versa, by the well known relation

$$T_n(\pi_n) \geqq k_n \Leftrightarrow Z_n(k_n/n) \leqq \pi_n, \tag{1.3}$$

which follows from the definitions.

The proofs of the present paper employ standard techniques and estimates of binomial probabilities, and we have sometimes introduced inessential assumptions to maintain simplicity and brevity. The results are mainly about first order deviations of T_n or Z_n from their expectations; while some "strong form" results are known, with few exceptions (for example, Theorem 1) they entail much longer proofs and I do not presently know the conclusions for the full spectrum of sequences $\{\pi_n\}$ considered herein. Theorems 1 and 2 cover the behavior of $Z_n(k/n)$ with k fixed; Theorem 5 covers the domain of normal limiting behavior and resulting classical LIL form for $T_n(\pi_n)$, and Theorems 3 and 4 cover the behavior in between these two extremes, including Baxter's case; Theorem 6 translates some of the conclusions for T_n into conclusions for Z_n.

We now mention work related to that of the present paper, other than that of Baxter described above. Bahadur [2] used his relation between the Z_n and T_n processes to obtain the LIL for $Z_n(p)$ with p fixed from the classical LIL for binomial $T_n(p)$. The same method can be used to obtain the strong form of inner and outer classes [6]. While the classical techniques of Theorems 5 and 6 herein also yield the LIL for $Z_n(p)$ (the main departure from the usual LIL proof for $T_n(p)$ being that one is now led by (1.3) to the LIL for $T_n(p_n)$ for varying p_n), Bahadur's technique provides a great saving of effort when it comes to the strong form.

Eicker [4] obtained top outer bounds (analogous to (1.1)) for $T_n(\pi_n)$ when $\pi_n \downarrow 0$ and $n\pi_n/\log\log n \to \infty$, but not bottom outer bounds, and also obtained both inner bounds. This is the domain treated in Theorem 5 of the present paper, where the usual binomial-normal LIL form holds. Eicker's inner class proof follows essentially the classical lines which are therefore sketched only in brief outline herein; this proof applies also to certain cases where π_n is bounded away from zero and one. His top outer bound proof uses fine estimates of the probability that the T_n process exceeds certain polygonal bounds; our proof, while much more routine, is considerably shorter, and treats also the bottom outer bound.

Robbins and Siegmund [10] have just announced the use of an interpolating process and the derivation of probability estimates for ever exceeding certain bounds (in the spirit of their earlier work with sums of random variables), in obtaining strong top bounds for the first order statistics $Z_n(1/n)$. Our Theorem 2 with $k = 1$ states only the first order term of this strong form.

In [1], [6], [7], and [8] relations between the $T_n(p)$ and $Z_n(p)$ processes were studied (with respect to varying p as well as n). For example, in the spirit of [1], one thereby obtains in [8] (where some of the results of the present paper were also described) the strong form for $\sup_{0<p<1}|Z_n(p) - p|$ oscillations from the corresponding sample of distribution function results of Chung [3]. Large deviations of either these processes or of their difference, over the domain $0 < p < p_n \downarrow 0$, are related to the present results and will be treated elsewhere; some such considerations have appeared in the work of Chibisov, LeCam, and others. Also related are the numerous papers on weak laws for order statistics, about which we mention only the appearance therein, not surprisingly, of the "fundamental equation" (2.24) which arises below. (See, for example, [9].)

Section 2 contains definitions and relevant binomial tail probability estimates. Statements of the main results are contained in Section 3, along with proofs of the simple first two theorems. The remainder of the paper contains the other proofs.

2. Preliminaries

We shall use i.o., f.o., and a.a.n. in their customary meaning of infinitely often, finitely often, and for almost all (all but finitely many) n. We treat events indexed by the natural numbers $\{n\}$ or a subsequence $\{n_j\}$, and the usual expressions of limiting behavior ($\to$, $\sim$) or of order (such as $O(z(n))$ or $o(g(j))$) refer to behavior as $n \to +\infty$ or $j \to +\infty$. The symbols $\uparrow$, $\downarrow$ are used for monotone, not strict, approach.

We let $\log_1$ denote the natural logarithm and $\log_{j+1} = \log\log_j$. Also, $\log_j^i x = (\log_j x)^i$. In summations and other appearances of such an expression as $\log_j n$, the domain of n is understood to begin where the expression is meaningful.

We shall try to reserve π_n for the argument of T_n and p_n for that of Z_n, with $k_n \geqq 0$ being used for bounds on T_n. We define $T_{n_1,n_2} = T_{n_2} - T_{n_1}$, the observation counter based on $X_{n_1+1}, \cdots, X_{n_2}$. Limiting behavior will often be conveniently described in terms of

$$h_n = n\pi_n/\log_2 n, \qquad H_n = np_n/\log_2 n. \tag{2.1}$$

Either of the Borel-Cantelli lemmas is denoted by BC.

We use int $\{x\}$ to denote the largest integer $\leqq x$, and int$^+$ $\{x\}$ for the smallest integer $\geqq x$.

When we consider a subsequence $\{n_j\}$ of the natural numbers, we write

$$I_j = \{n : n_j < n \leqq n_{j+1}\} \quad \text{and} \quad I_{\bar{j}} = \{n : n_j < n < n_{j+1}\}. \tag{2.2}$$

We write $m_j = n_j - n_{j-1}$. The two subsequences $\{n_j\}$ we shall consider, with typical estimates they imply, are

$$\text{(2.3)}\qquad \begin{aligned} n_j &\sim \lambda^j \qquad \text{where} \quad \lambda > 1,\ m_j/n_j \sim (\lambda - 1)/\lambda, \\ \log_2 n_j &\sim \log j, \end{aligned}$$

and, for $\alpha > 0$,

$$\text{(2.4)}\qquad \begin{aligned} n_j &\sim e^{\alpha j/\log j} \qquad\qquad m_j/n_j = 1 - O(j^{-\alpha}), \\ \log_2 n_j &\sim \log j. \end{aligned}$$

Much of our treatment can be carried out in terms of very simple events, essentially as employed in [2] for top bounds on T_n. Suppose $\pi_n \downarrow 0$ and $k_n \uparrow$, and let

$$\text{(2.5)}\qquad A_n = \{T_n(\pi_n) \geqq k_n\}.$$

Let $\{n_j\}$ be *any* increasing sequence of natural numbers, and define

$$\text{(2.6)}\qquad B_j^* = \{T_{n_j, n_{j+1}}(\pi_{n_{j+1}}) \geqq k_{n_{j+1}}\}$$

and

$$\text{(2.7)}\qquad C_j^* = \{T_{n_{j+1}}(\pi_{n_j}) < k_{n_j}\}.$$

Then, since $T_n(\pi)$ is nondecreasing in n and π, and $\pi_n \downarrow$, $k_n \uparrow$,

$$\text{(2.8)}\qquad C_j^* \subset \{T_n(\pi_n) < k_n,\ n \in I_j\},$$

and hence

$$\text{(2.9)}\qquad \{C_j^*,\ \text{a.a.}\ j\} \Rightarrow \{A_n\ \text{f.o.}\}, \qquad \text{wp } 1.$$

In the other direction, obviously

$$\text{(2.10)}\qquad \{B_j^*\ \text{i.o.}\} \Rightarrow \{A_n\ \text{i.o.}\}, \qquad \text{wp } 1,$$

the events of (2.6) of course being useful because they are independent.

We shall see that, in the top outer class proofs of Theorems 1 and 3, we can even avoid the use of subsequences as employed in [2] and in "normal case" proofs such as that of Theorem 5 below, and, as an alternative, work with the events

$$\text{(2.11)}\qquad A_n' = \{X_n \leqq \pi_n,\ T_{n-1}(\pi_n) \geqq k_n - 1\}.$$

As long as $1 \leqq k_n \uparrow$ and $\pi_n \downarrow 0$, it is evident that

$$\text{(2.12)}\qquad \{A_n\ \text{i.o.}\} \Rightarrow \{A_n'\ \text{i.o.}\}, \qquad \text{wp } 1.$$

Similarly, if $\pi_n \downarrow$ and $k_n \uparrow$, for bottom outer bounds on T_n we use

$$\text{(2.13)}\qquad \{T_{n_j}(\pi_{n_j+1}) < k_{n_j+1}\ \text{f.o.}\} \Rightarrow \{T_n(\pi_n) < k_n\ \text{f.o.}\}, \qquad \text{wp } 1.$$

The bottom inner bound treatment is only slightly less simple. If

$$\text{(2.14)}\qquad \begin{aligned} Q_j^* &= \{T_{n_j, n_{j+1}}(\pi_{n_{j+1}}) \leqq k_{n_{j+1}} - \gamma_j\}, \\ R_j^* &= \{T_{n_j}(\pi_{n_{j+1}}) > \gamma_j\}, \end{aligned}$$

where the γ_j are arbitrary nonnegative values, then

$$\{Q_j^* \text{ i.o.}\} \cap \{Q_j^* R_j^* \text{ f.o.}\} \Rightarrow \{T_{n_{j+1}}(\pi_{n_{j+1}}) \leqq k_{n_{j+1}} \text{ i.o.}\}, \qquad \text{wp } 1. \tag{2.15}$$

In fact, in the simple first order proofs of Theorems 2 and 4 it suffices to take $\gamma_j = 0$, and to show $P\{R_j^* \text{ f.o.}\} = 1$ for the second half of the left side of (2.15).

In using the simple devices of (2.5) through (2.15) we require (in addition to $\pi_n \downarrow 0$) $k_n \uparrow$. This is not always too convenient: as described below the statement of Theorem 3, for given $\{\pi_n\}$ the natural formula for a bound k_n in terms of π_n in Theorems 3 and 4 may not yield a monotone k_n. However, in such cases we will be able to replace the nonmonotone $\{k_n\}$ by a sequence $\{k_n^*\}$ such that

$$k_n^* \uparrow, \; k_n^* = [1 + O(1)]k_n, \tag{2.16}$$

and then use the appropriate device of (2.5) through (2.15) on k_n^*; by virtue of our considerations being first order (so that we prove $(1 \pm \varepsilon)k_n^*$ lies in the appropriate class), $\{k_n\}$ is then by definition in the same class as $\{k_n^*\}$.

The study of sequences $\{\pi_n\}$ for which π_n is not monotone, or for which the technique of (2.16) fails, is more complex, requiring in place of (2.5) through (2.15) calculations which are somewhat similar to those stemming from (4.18) and (4.19) in the proof of Theorem 5. We omit such cases. It is obvious that bounds for some $\{\pi_n\}$ for which the departure from monotonicity is slight enough, may be obtained from those of majorizing and minorizing monotone sequences when corresponding bounds for the latter sequences coincide.

To give a little relief from the burden of memory or page turning, we shall reserve further definitions until they are encountered in the proof of Theorem 5.

We now list our binomial estimates. In Theorems 1 and 2 we consider $Z_n(k/n)$ with k fixed, and require only the following simple and familiar estimates ([5], p. 140) for nonnegative integral $\bar{k}$:

$$\begin{aligned} &\limsup_n n\pi_n < \bar{k} \Rightarrow \\ &\qquad \log P\{T_n(\pi_n) \geqq \bar{k}\} - O(1) = \log P\{T_n(\pi_n) = \bar{k}\} \\ &\qquad\qquad = \bar{k} \log (n\pi_n) - n\pi_n - \log (\bar{k}!) + o(1). \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} &\{\pi_n = o(n^{-1/2}), \liminf_n n\pi_n > \bar{k}\} \Rightarrow \\ &\qquad \log P\{T_n(\pi_n) \leqq \bar{k}\} - O(1) = \log P\{T_n(\pi_n) = \bar{k}\} \\ &\qquad\qquad = \bar{k} \log (n\pi_n) - n\pi_n - \log (\bar{k}!) + o(1). \end{aligned} \tag{2.18}$$

In the domain of Theorems 3 and 4, we must take account of the fact that $k_n \to \infty$. Writing

$$\begin{aligned} b(z, N, P) &= \binom{N}{z} P^z(1 - P)^{N-z}, \qquad z \text{ integral}, \\ B(z, N, P) &= \sum_{y \leqq z} b(y, N, P), \\ B^+(z, N, P) &= \sum_{y \geqq z} b(y, N, P), \end{aligned} \tag{2.19}$$

we state the required binomial estimate as

LEMMA 1. *Suppose* $N \to \infty$, $P_N \to 0$, $z_N \to +\infty$. *If*

$$P_N = o(N^{-1/2}), \qquad z_N = o(N^{1/2}), \tag{2.20}$$
$$\limsup_N NP_N/z_N < 1,$$

then

$$\log B^+(z_N, N, P_N) = z_N[\log (NP_N/z_N) - (NP_N/z_N) + 1 + o(1)]. \tag{2.21}$$

Moreover, (2.21) *holds with* B *replacing* B^+, *provided the last condition of* (2.20) *is replaced by* $\liminf_N NP_N/z_N > 1$; *and the right side of* (2.21) *equals* $\log b(\mathrm{int}^+ z_N, N, P_N)$ *or* $\log b(\mathrm{int}\, z_N, N, P)$, *even without any third condition of* (2.20).

PROOF. The last condition of (2.20) implies that B^+/b is bounded ([5], p. 140). Putting $z' = \mathrm{int}^+ z_N$ and writing out $\log b(z', N, P_N)$ and using Stirling's approximation, we see that the first condition of (2.20) allows us to neglect $NP_N + \log (1 - P_N)^N$, the second allows us to neglect $\log [N^{z'}/N(N-1) \cdots (N - z' + 1)]$, and the two together imply $z_N P_N = o(1)$ and thus allow us to neglect $z' \log (1 - P_N)$. Since $z_N \sim z'$ and also we can absorb $\log z'$ into the $o(1)z_N$ term, we obtain (2.21). The result for B is obtained in the same way.

We now specialize the parameter values in Lemma 1, as used in the proofs of Theorems 3 and 4. Firstly, as we shall explain after Theorems 1 and 2 where k_n is bounded, we subsequently insure that k_n is unbounded by assuming

$$\liminf_n [\log (n\pi_n)/\log_2 n] \geqq 0 \tag{2.22}$$

in the top bound considerations for $T_n(\pi_n)$ of Theorem 3, and

$$\liminf_n [n\pi_n/\log_2 n] > 1 \tag{2.23}$$

in the bottom bound considerations of Theorem 4.

Secondly, in the domain where (2.22) or (2.23) is satisfied and where also $\pi_n = O(n^{-1} \log_2 n)$, the bounds on $T_n(\pi_n)$ are described in terms of the solutions of a certain transcendental equation.

The fundamental equation. We consider the solutions, for $c > 0$, of the equation

$$\beta(\log \beta - 1) = (1 - c)/c. \tag{2.24}$$

The left side of (2.24) is convex in $\beta > 0$ and attains its minimum value -1 at $\beta = 1$. Hence, (2.24) has a solution $\beta_c' > 1$ if $c > 0$ and a second positive solution $\beta_c'' < 1$ if $c > 1$. For future reference we note that β_c' is decreasing in c while β_c'' is increasing, and that

$$c \to 0 \Leftrightarrow \beta_c' \to \infty \Rightarrow \beta_c' \sim c^{-1}/\log c^{-1} \to +\infty, \; c\beta_c' \sim 1/\log c^{-1} \to 0; \tag{2.25}$$
$$c \to +\infty \Leftrightarrow \beta_c' \to 1 \quad \Rightarrow c\beta_c' \to +\infty;$$

$$c \downarrow 1 \Leftrightarrow \beta_c'' \to 0 \;\Rightarrow \beta_c'' \sim (c-1)/\log (c-1)^{-1}, \qquad c\beta_c'' \to 0;$$

$$c \to +\infty \Leftrightarrow \beta_c'' \to 1, \qquad c\beta_c'' \to +\infty.$$

Finally, on either part $\beta < 1$ or $\beta > 1$ of (2.24), $d(c\beta)/dc = (\beta - 1)/\log \beta > 0$.

We now state the specialization of Lemma 1 used in proving Theorems 3 and 4. Recall the definition (2.1) of h_n.

LEMMA 2. *Suppose* $n \to \infty$, $\pi_n \to 0$, $h_n = O(1)$, *and that* (2.22) *is satisfied. Let* $\{\rho_n\}$ *and* $\{d_n\}$ *be sequences of positive values which are bounded away from* 0 *and* ∞ *and for which* $n\rho_n$ *is integral. Assume*

$$\limsup_n \rho_n/d_n\beta_{h_n}' < 1, \tag{2.26}$$

where β_c' *is defined below* (2.24). *Then*

$$\begin{aligned}\log P\{T_{n\rho_n}(\pi_n) \geqq d_n h_n \beta_{h_n}' \log_2 n\} \\ = \{- d_n + h_n[d_n - \rho_n - d_n\beta_{h_n}' \log (d_n/\rho_n)] + o(1)\} \log_2 n;\end{aligned} \tag{2.27}$$

moreover, $\log P\{T_{n\rho_n}(\pi_n) = \text{int}^+ [d_n h_n \beta_{h_n}' \log_2 n]\}$ *satisfies the same relation.*

If (2.22) *and* (2.26) *are replaced in the above by* (2.23) *and*

$$\liminf_n \rho_n/d_n\beta_{h_n}'' > 1, \tag{2.28}$$

then

$$\begin{aligned}\log P\{T_{n\rho_n}(\pi_n) \leqq d_n h_n \beta_{h_n}^{n} \log_2 n\} \\ = \{- d_n + h_n[d_n - \rho_n - d_n\beta_{h_n}'' \log (d_n/\rho_n)] + o(1)\} \log_2 n.\end{aligned} \tag{2.29}$$

PROOF. We shall demonstrate (2.27); the proof of (2.29) is almost identical. We put $N = n\rho_n$, $P_N = \pi_n$, $z_N = d_n h_n \beta_{h_n}' \log_2 n$ in Lemma 1. Then $z_N \to +\infty$ unless there is a sequence $\{n_j\}$ for which $h_{n_j} \to 0$ and $h_{n_j}\beta_{h_n}' = O(1/\log_2 n_j)$, which by the first line of (2.25) would entail $1/\log h_{n_j}^{-1} = O(1/\log_2 n_j)$; this last is contradicted by the fact that (2.22) (with $h_{n_j} \to 0$) implies $\log h_{n_j}^{-1} = o(\log_2 n_j)$; we conclude that $z_N \to +\infty$. Next, $h_n = O(1)$ implies the first condition of (2.20) as well as $z_N = O(\log_2 n) = O(\log_2 N)$, and this last yields the second condition of (2.20). Also, $NP_N/z_N = \rho_n/d_n\beta_{h_n}'$, so that (2.26) implies the last condition of (2.20). Thus, Lemma 1 applies, and substitution into (2.21) gives

$$\begin{aligned}d_n h_n \beta_{h_n}'\{- \log \beta_{h_n}' + 1 - 1/\beta_{h_n}' + \log (\rho_n/d_n) \\ + [1 - \rho_n/d_n]/\beta_{h_n}' + o(1)\} \log_2 n.\end{aligned} \tag{2.30}$$

Since the first three terms in braces in (2.30) sum to $-1/h_n\beta_{h_n}'$ by (2.24), and since $d_n h_n \beta_{h_n}' o(1) = o(1)$, we obtain (2.27).

When $h_n \to \infty$, the approximations of Lemma 2 are insufficient, and we need the normal approximation instead. This tool is also well known; a careful reading of Feller ([5], pp. 168–173, 178–181) shows that the development there actually applies with only minor and obvious modifications when $\pi_N \to 0$ sufficiently slowly, and we state this result as

LEMMA 3. *If* $x_N = (k_N - N\pi_N)[N\pi_N(1 - \pi_N)]^{-1/2}$, *then*

$$\{N \to \infty, x_N \to +\infty, x_N[N\pi_N(1 - \pi_N)]^{-1/2} \to 0\} \Rightarrow B^+(k_N, N, \pi_N) = (2\pi)^{-1/2}x_N^{-1}\exp\{-x_N^2[1 + o(1)]/2\}. \tag{2.31}$$

(Feller's expression [5], p. 181, (6.11) for the $\frac{1}{2}o(1)x_N^2$ term in (2.31) is also correct under the present conditions; it is conveniently expressed as

$$\tfrac{1}{2}o(1)x_N^2 = \tfrac{1}{6}(2\pi_N - 1)x_N^3[N\pi_N(1 - \pi_N)]^{-1/2} + O(x_N^4[N\pi_N(1 - \pi_N)]^{-1}) \tag{2.32}$$

for use in other limit laws, but this will not be required herein.)

Finally, we shall also use the elementary fact that

$$\inf_{N,\pi} B^+(N\pi - 1, N, \pi) > 0; \tag{2.33}$$

the "bad case" where the $N\pi - 1$ is required rather than $N\pi$ is of course $N \to \infty$, $N\pi \downarrow 0$. Similarly,

$$\inf_{N,\pi} B(N\pi + 1, N, \pi) > 0. \tag{2.34}$$

In each of these expressions we include zero in the domain of N; both probabilities are then one, corresponding to the interpretation $b(0, 0, \pi) = 1$ which is appropriate in the application.

It seems essential that the proofs, as carried out in the present paper, be divided into the several cases as treated. For, the estimates of Lemma 2 are useless in the "normal" case of Theorem 5, just as those of Lemma 3 are useless for proving Theorems 3 and 4. Again, the geometric $\{n_j\}$ of (2.3) is inadequate in the inner class bottom bound proofs of Theorems 2 and 4, where (2.4) is used, but the latter cannot be used in the corresponding outer class proofs; in Theorem 5 we again use geometric $\{n_j\}$ where, also, it is impossible to avoid using subsequences by using (2.11) and (2.12) as in parts of Theorems 1 and 3. (For certain strong form results, other sequences $\{n_j\}$ must of course be considered.)

3. Main results

The short proofs of Theorems 1 and 2 will be given in this section, but proofs of the other theorems stated in this section will be deferred in favor of discussion here. For *bounded* top bounds on $T_n(\pi_n)$ or corresponding bottom bounds on the kth order statistic $Z_n(k/n)$ (k fixed), the situation is completely known [8], and is elementary to verify. We forego the artificial generality of bounded but varying k_n, or oscillatory $n\pi_n$, and state the result simply as

THEOREM 1. *If k is a positive integer and $n\pi_n \downarrow 0$, then*

$$P\{T_n(\pi_n) \geqq k \text{ i.o.}\} = P\{Z_n(k/n) \leqq \pi_n \text{ i.o.}\} = \begin{Bmatrix} 0 \Leftrightarrow \infty > \\ 1 \Leftrightarrow \infty = \end{Bmatrix} \sum_n n^{k-1}\pi_n^k \tag{3.1}$$

(*so that* $\log [nZ_n(k/n)]$ *has*

$$-k^{-1}\{\log_2 n + \log_3 n + \cdots + (1+\varepsilon)\log_j n\} \tag{3.2}$$

as bottom outer or inner bound, depending on whether $\varepsilon > 0$ *or* $\varepsilon \leqq 0$).

PROOF. *Outer class.* By (1.3) we are concerned with the events A_n of (2.5), with $k_n = k$. The geometric sequence $\{n_j\}$ of (2.3) can be used with (2.7) and (2.9) to give a proof, in standard manner. However, we emphasize the simplicity of the present case by giving a proof without a subsequence, using the events $\{A'_n\}$ of (2.11) with $k_n = k$ and the relation (2.12). If the series of (3.1) converges, then BC and the estimate (2.17) with $\bar{k} = k - 1$ imply $P\{A'_n \text{ i.o.}\} = 1$.

Inner class. Let $n_j = 2^j$. Since $n\pi_n \downarrow 0$, divergence of the series of (3.1) implies divergence of $\Sigma_j (n_j \pi_{n_j})^k$. The estimate (2.17) for $P\{B_j^*\}$, BC, and (2.10) complete the proof of Theorem 1.

In view of Theorem 1, our further concern with top bounds k_n on $T_n(\pi_n)$ is with the case $k_n \to +\infty$. Thus, π_n should be such that the series of (3.1) diverges for each fixed k. This is obviously the case if (2.22) holds, but not if $\limsup_n [\log(n\pi_n)/\log_2 n] < 0$. Hence, ignoring oscillatory behavior where neither of these holds, we hereafter assume π_n satisfies (2.22) in discussing these top bounds. This condition is used in order to apply Lemma 2 in the proof of Theorem 3.

Even the first order lower bounds on $Z_n(k/n)$ in Theorem 1 depend on k, and thereby exhibit quite a different behavior from the upper bounds, to which we now turn.

THEOREM 2. *If k is a positive integer, then*

$$P\{nZ_n(k/n) > (1+\varepsilon)\log_2 n \text{ i.o.}\} = \begin{cases} 0 & \textit{if} \quad \varepsilon > 0, \\ 1 & \textit{if} \quad \varepsilon \leqq 0. \end{cases} \tag{3.3}$$

PROOF. *Outer class.* By BC, (1.3), and (2.13) with $k_n = k$, it suffices to show that, for $d > 1$, there is a $\lambda > 1$ for which $P\{T_{n_j}(dn_{j+1}^{-1}\log_2 n_{j+1}) \leqq k - 1\}$ is summable, where $n_j = \text{int } \lambda^j$. But, by (2.3) and (2.18) this probability is

$$\exp\{(k-1)\log[d\lambda^{-1}\log j] - d\lambda^{-1}\log j + O(1)\}, \tag{3.4}$$

which is summable if $1 < \lambda < d$.

Inner class. We now put $n_j = \text{int }\{e^{\alpha j \log j}\}$ with $\alpha > 0$. Then, by (2.4) and (2.18),

$$\begin{aligned} P\{&T_{n_j, n_{j+1}}(n_{j+1}^{-1}\log_2 n_{j+1}) \leqq k - 1\} \\ &= \exp\{(k-1)\log_3 n_{j+1} - [1 - O(j^{-\alpha})]\log_2 n_{j+1} + O(1)\}, \end{aligned} \tag{3.5}$$

whose sum diverges. By (2.14) and (2.15) with $\gamma_j = 0$ and $k_n = k - 1$, the proof is completed upon computing, again from (2.4) and (2.18) (now with $\bar{k} = 0$),

$$\begin{aligned} P\{&T_{n_j}(n_{j+1}^{-1}\log_2 n_{j+1}) \geqq 1\} \\ &= 1 - \exp\{-O(1)j^{-\alpha}\log_2 n_{j+1} + O(1) = O(j^{-\alpha}\log j). \end{aligned} \tag{3.6}$$

REMARKS. In (2.15), $\{Q_j^* \text{ i.o. wp } 1\}$ is essentially automatic in this case, and it is the event $\{Q_j^* R_j^* \text{ f.o. wp } 1\}$ for which geometric $n_j \sim \lambda^j$ is inadequate. With

n_j as chosen in the proof and $\alpha > 1$, not merely $\{Q_j^* R_j^* \text{ f.o. wp } 1\}$, but even $\{R_j^* \text{ f.o. wp } 1\}$ is satisfied; however, in strong form analogues one cannot always be so cavalier.

In view of Theorem 2, our further investigation of bottom bounds k_n on $T_n(\pi_n)$, with $k_n \to +\infty$, will be made under the assumption (analogous to (2.22)) that (2.23) holds. This will be discussed further, just after the statement of Theorem 4.

We now turn to the domain of behavior between that of the kth order statistic (fixed k) and that of "normal" LIL. This includes Baxter's case. We recall the definition (2.1) of h_n.

THEOREM 3. *Suppose* $\pi_n \downarrow 0$, (2.22) *is satisfied*, $h_n = O(1)$, *and that there is a* k_n^* *satisfying* (2.16) *for* $k_n = h_n \beta'_{h_n} \log_2 n$. *Then*

$$\limsup_n T_n(\pi_n)/h_n \beta'_{h_n} \log_2 n = 1 \qquad \text{wp } 1. \tag{3.7}$$

In particular, if $\pi_n \downarrow 0$, *a top bound* k_n^* *on* $T_n(\pi_n)$ *is given in various ranges of* π_n *by*:

$$h_n \to c > 0 \Rightarrow k_n^* = c\beta'_c \log_2 n; \tag{3.8}$$

$$h_n \to 0 \quad \text{and} \quad \log_2 n/\log h_n^{-1} \sim g_n \uparrow +\infty \Rightarrow k_n^* = g_n; \tag{3.9}$$

in particular,

$$\begin{aligned} k_n^* &= \log_2 n/\log_3 n \quad \text{if} \quad \log(n\pi_n) = o(\log_3 n), \\ k_n^* &= \log_2 n/(B+1)\log_3 n \quad \text{if} \quad \pi_n \sim An^{-1}\log_2^{-B} n, \quad B > -1, \\ k_n^* &= \log_2 n/\log(n\pi_n)^{-1} \quad \text{if} \quad \begin{cases} \log_2 n/\log(n\pi_n)^{-1} \uparrow +\infty, \\ \log_3 n/\log(n\pi_n)^{-1} \to 0. \end{cases} \end{aligned} \tag{3.10}$$

The use of (2.16) here and in Theorem 4 is not as unnatural as it may first appear. For example, with the positive constant c near zero in Theorem 3 or near one in Theorem 4, and $h_n = c + (-1)^n/3n \log n \log_2 n$, we have $\pi_n \downarrow$, $h_n \to c$ (and, in the case of Theorem 4, even $n\pi_n \uparrow$); but the "natural" k_n given in the denominator of (3.7) or (3.11) is not monotone, which of course $k_n^* = c\beta_c \log_2 n$ is, and the conclusion of each theorem is still valid. The particular cases of (3.8), (3.9), (3.10) have been stated in terms of a simple increasing k_n^* rather than $h_n \beta'_{h_n} \log_2 n \sim \log_2 n/\log h_n^{-1}$.

We also note that, in both Theorem 3 and Theorem 4, h_n of exactly order one does not imply that $\lim h_n$ exists. One can obtain $1 < \liminf_n h_n < \limsup_n h_n < +\infty$ while $\pi_n \downarrow$, $n\pi_n \uparrow$, and even the "natural" $k_n \uparrow$ (discussed in the previous paragraph), by letting $h_{n+1} - h_n$ take successive blocks of positive and negative steps of size $\varepsilon/n \log n \log_2 n$ with ε sufficiently small.

THEOREM 4. *Suppose* $\pi_n \downarrow 0$, (2.23) *is satisfied*, $h_n = O(1)$, *and that there is a* k_n^* *satisfying* (2.16) *for* $k_n = h_n \beta''_{h_n} \log_2 n$. *Then*

$$\liminf_n T_n(\pi_n)/h_n \beta''_{h_n} \log_2 \mathrm{n} = 1 \qquad \text{wp } 1. \tag{3.11}$$

In particular, if $\pi_n \downarrow 0$,

$$(3.12) \qquad h_n \to c > 1 \Rightarrow k_n^* = c\beta_c'' \log_2 n$$

is a bottom bound on $T_n(\pi_n)$.

By putting $h_n = 1 + \delta$ with $\delta > 0$ in (3.12), and then letting $\delta \to 0$ and using the third line of (2.25), we obtain that (in a case where (2.23) is not satisfied)

$$(3.13) \qquad \limsup_n h_n \leqq 1 \Rightarrow \liminf_n T_n(\pi_n)/\log_2 n = 0, \qquad \text{wp } 1.$$

In fact, when $h_n \equiv 1$ we can use (3.4) or (1.3) and (3.3) with $k = 1$ to obtain

$$(3.14) \qquad \pi_n = n^{-1} \log_2 n \Rightarrow \liminf_n T_n(\pi_n) = 0. \qquad \text{wp } 1.$$

Since we shall see that the behavior of $c\beta_c'$ as $c \to 0$ in (2.25) yields the top bounds (3.8), (3.9), (3.10) as $h_n \to 0$, it is tempting to try to use the third line of (2.25) to obtain seemingly analogous *bottom* bounds in terms of $h_n - 1 \downarrow 0$. However, the latter are less accurately viewed as first order results in $h_n - 1$ than as second order results in h_n, which cannot be obtained from the behavior of $c\beta_c''$ as $c \downarrow 1$ without more effort, because of the failure of (2.23). We do not have complete results in this domain, and shall not discuss it further except to mention here, as an example of what is involved in the subdomain of smallest values of $h_n - 1$ of interest, that the determination of which sequences $h_n - 1$ of the particular form $L \log_3 n/\log_2 n$ (L constant) continue to imply the conclusion of (3.14) when $h_n \downarrow 1$, requires a more delicate argument than that used in proving Theorems 2 and 4. (Of course, the strong top bounds on $Z_n(1/n)$ [10] imply this second order consequence, but yield nothing about sequences $h_n - 1$ which vanish more slowly.)

We now turn to sequences $\{\pi_n\}$ which vanish slowly enough that "normal" LIL behavior prevails. Of course, we may write $\pi_n(1 - \pi_n) \sim \pi_n$.

THEOREM 5. *If* $\pi_n \downarrow 0$, $h_n \to +\infty$ *and* $n\pi_n \uparrow$, *then, for either choice of sign,*

$$(3.15) \qquad \limsup_n \pm [T_n(\pi_n) - n\pi_n][2n\pi_n \log_2 n]^{-1/2} = 1, \qquad \text{wp } 1.$$

REMARKS. The assumption that $n\pi_n$ is nondecreasing, although natural enough, is not essential, but is used to simplify the outer class proofs. For example, if $n^L\pi_n$ is increasing for any positive value L, it is only necessary to put $n_j \sim (1 + \varepsilon/3)^{2j/(L+1)}$ to use the same proof. However, if $n\pi_n$ oscillates too much, the right side of (4.14) need not be bounded away from zero, and a longer proof is needed. It will be evident that only minor changes in the proof are required to cover various other cases, for example, $\pi_n \downarrow \pi_0 > 0$, in which case the coefficient of $\log_2 n$ in (3.15) must of course be altered to $2n\pi_0(1 - \pi_0)$. It will be seen that the inner class proof does not use monotonicity of π_n or of $n\pi_n$. Eicker points out that his inner class results apply to more general sequences of sets than $[0, \pi_n]$; here, if J_n is a subset of $[0, 1]$ of Lebesgue measure π_n, we consider $T_n(J_n) = \{\text{number of } X_i \text{ in } J_n, 1 \leqq i \leqq n\}$. One must note, however, that such inner bounds on $T_n(J_n)$ may not be sharp if J_n moves too rapidly, and may even give the wrong order. One need only cite the familiar example of J_n chosen so that the random variables $T_n(J_n)$ are independent, in which case $\pm [T_n(J_n) - n\pi_n]$ has $[2n\pi_n \log n]^{1/2}$ as top bound under our assumptions on π_n.

In (3.1) equivalent T_n and Z_n bounds were treated, and (3.14) gives a consequence of Theorem 2 for T_n. It remains to translate the results of Theorems 3 to 5 into conclusions for the sample quantiles $Z_n(p_n)$ beyond the domain of Theorems 1 and 2. For each positive value v, we define c'_v to be the positive value satisfying the pair of equations

$$\beta'_{c'_v}(\log \beta'_{c'_v} - 1) = (1 - c'_v)/c'_v, \qquad c'_v \beta'_{c'_v} = v, \tag{3.16}$$

with $\beta' > 1$. We define c''_v by the analogue of (3.16) obtained by replacing β' by $\beta'' < 1$. The existence and uniqueness of c'_v and c''_v follows from (2.25) and the sentence following it. We recall the definition (2.1) of H_n.

THEOREM 6. *Suppose $p_n \downarrow 0$. If $H_n \uparrow +\infty$, then, for either choice of sign,*

$$\limsup_n \pm [Z_n(p_n) - p_n][2p_n n^{-1} \log_2 n]^{-1/2} = 1, \qquad \text{wp } 1. \tag{3.17}$$

If $0 < v < \infty$ and $H_n \to v$, then

$$\begin{aligned} c''_v &= \limsup_n nZ_n(p_n)/\log_2 n, \qquad \text{wp } 1 \\ c'_v &= \liminf_n nZ_n(p_n)/\log_2 n, \qquad \text{wp } 1 \end{aligned} \tag{3.18}$$

If $H_n \to 0$ (and $np_n \geqq 1$ to avoid trivialities), then

$$\limsup_n nZ_n(p_n)/\log_2 n = 1, \qquad \text{wp } 1; \tag{3.19}$$

while if $h_n \downarrow 0$ and $np_n \uparrow +\infty$, then

$$\liminf_n H_n \log [nZ_n(p_n)/\log_2 n] = -1, \qquad \text{wp } 1. \tag{3.20}$$

REMARKS. As in the case of Theorem 5, the assumption $H_n \uparrow$ is stronger than needed, but it is made to simplify the proof. Also, if $p_n \downarrow p_0 > 0$, essentially the same proof yields (3.17) with $2p_n$ replaced by $2p_0(1 - p_0)$. (Compare the remarks below Theorem 5.) In particular, when $p_n = p_0$ we obtain the LIL for sample p_0-tiles, but not the strong form [6], which would require considerably more effort using the present route. Part of (3.17) was stated in [8] under unnecessary restrictions. More satisfactory forms than (3.19) and (3.20) are obviously related to strong forms of Theorems 3 and 4. As they stand, (3.19) and (3.20) are also correct for the case $p_n = k/n$ of Theorems 2 and 1, and the grossness of (3.20) as a description for the latter is evident.

Proofs of Theorems 3 through 6

PROOF OF THEOREM 3. In each particular case of (3.8), (3.9), (3.10), the assumptions preceding (3.7) as well as the correctness of the stated k_n^* follow easily from the first line of (2.25) and from the validity of (3.7), and we turn to the proof of the latter.

Inner class. Let $\varepsilon > 0$ be specified, $\varepsilon < 1$. Let $n_j \sim \varepsilon^{-j}$. Then $m_j \sim (1-\varepsilon)n_j$ by (2.3). Hence, we can compute

$$\log P\{T_{n_{j-1},n_j}(\pi_{n_j}) \geqq d_{n_j}h_{n_j}\beta'_{h_{n-j}} \log_2 n_j\} \tag{4.1}$$

by using (2.27) with $\rho_n \sim d_n \sim 1-\varepsilon$ and with n replaced by n_j in (2.27); here d_n is chosen so that (2.16) is satisfied, where $k_n^* = (1-\varepsilon)^{-1} d_n h_n \beta'_{h_{n_j}} \log_2 n$. Note that (2.26) is satisfied because (2.24) and (2.25) and $h_n = O(1)$ imply

$$\liminf_n \beta'_{h_n} > 1. \tag{4.2}$$

Also, $h_n\beta'_{h_n}$ is bounded, by (2.25). Hence, from (2.27), the expression (4.1) equals $-(1-\varepsilon)[1+o(1)] \log_2 n_j$, so (2.10) and BC yield the desired result. (The last clause of Lemma 1 indicates that we do not need to verify (2.26) for the above half of the proof, but (4.1) is needed below, anyway.)

Outer class. As in the case of Theorem 1, there is a proof using (2.7) to (2.9) with the n_j of (2.3), which we omit in order to demonstrate the simplicity of the situation by working directly with (2.12) and the A'_n of (2.11) with k_n replaced there by

$$k'_n = d_n h_n \beta'_{h_n} \log_2 n, \tag{4.3}$$

and $d_n - 1 \to \varepsilon$, small and positive, with d_n chosen so that $k'_n \uparrow$ and k'_n is integral. By (4.2), for small enough ε and with $\rho_n = 1$, (2.26) is satisfied. Because of (2.26) (as used in (2.20)) we have for the event of (2.11) with k'_n for k_n,

$$\begin{aligned} \log P\{A'_n\} &= \log [\pi_n B^+(k'_n - 1, n-1, \pi_n)] \\ &\sim \log [\pi_n b(k'_n - 1, n-1, \pi_n)] \\ &= \log [k'_n n^{-1} b(k'_n, n, \pi_n)]. \end{aligned} \tag{4.4}$$

From the previously obtained boundedness of $h_n\beta'_{h_n}$, we have $\log k'_n = o(\log_2 n)$. Consequently, from (4.4) and (2.27),

$$\begin{aligned} \log P\{A'_n\} = -\log n + \{&-(1+\varepsilon) \\ &+ h_n[\varepsilon - (1+\varepsilon)\beta'_{h_n} \log(1+\varepsilon)] + o(1)\} \log_2 n. \end{aligned} \tag{4.5}$$

By (4.2), the quantity in square brackets in (4.5) is negative for ε sufficiently small (and positive) and for all large n. This, (2.12), and BC complete the proof.

PROOF OF THEOREM 4. Equation (3.12) follows from (3.11).

Outer class. Given ε, small and positive, let d_n be chosen so that $d_n h_n \beta'_{h_n} \log_2 n \uparrow$ and $d_n \to 1-\varepsilon$. Put $n_j \sim \lambda^j$. Because of (2.13), we compute, using (2.29) with n replaced by n_{j+1} and $\rho_{n_{j+1}} = n_j/n_{j+1} \sim \lambda^{-1}$,

$$\begin{aligned} &\log P\{T_{n_j}(\pi_{n_{j+1}}) < d_{n_{j+1}} h_{n_{j+1}} \beta''_{h_{n_{j+1}}} \log_2 n_{j+1}\} \\ &\quad = \{-(1-\varepsilon) + h_{n_{j+1}}[1 - \varepsilon - \lambda^{-1} - (1-\varepsilon)\beta''_{h_n} \\ &\qquad \log(\lambda(1-\varepsilon))] + o(1)\} \log_2 n_{j+1}, \end{aligned} \tag{4.6}$$

provided (2.28) is satisfied. Write $\liminf_n h_n = \bar{h}$. Since $\bar{h} > 1$ by (2.23), the structure (2.24)—(2.25) of the functional equation implies that $1 < 1 - \bar{h}\beta_{\bar{h}}'' \log \beta_{\bar{h}}'' = \bar{h}(1 - \beta_{\bar{h}}'')$. Moreover, by the comment following (2.25) and the fact that $(\beta - 1)/\log \beta < 1$ for $0 < \beta < 1$, we conclude that $h - h\beta_h''$ increases in h. Hence, $h_n(1 - \beta_{h_n}'') > 1 + 2\delta$ for some $\delta > 0$ and all large n. Now let $\lambda > 1$ be chosen so close to one that $\lambda d\beta_h < 1$ (which yields (2.28)) and such that $1 - \lambda^{-1} < \varepsilon\delta/(1 + 2\delta)$. Regarding the expression in braces on the right side of (4.6), other than the $o(1)$ term, as a function of $\varepsilon > 0$, upon expanding it in powers of ε and noting that $\log \lambda = 1 - \lambda^{-1} + O(\varepsilon^2)$ we obtain

$$
\begin{aligned}
(4.7) \qquad & -(1 - \varepsilon) + h_{h_{j+1}}[1 - \varepsilon - \lambda^{-1} \\
&\qquad - (1 - \varepsilon)\beta_{h_{n_{j+1}}}''(-\varepsilon + 1 + \lambda^{-1} + O(\varepsilon^2)] \\
&= -(1 - \varepsilon) + h_{n_{j+1}}(1 - \beta_{h_{n_{j+1}}}'')(1 - \lambda^{-1} - \varepsilon) + O(\varepsilon^2) \\
&< -(1 - \varepsilon) + (1 + 2\delta)\big(-\varepsilon + \varepsilon\delta/(1 + 2\delta)\big) + O(\varepsilon^2) \\
&< -1 - \delta\varepsilon + O(\varepsilon^2).
\end{aligned}
$$

Thus, for ε sufficiently small and positive, the probability of (4.6) is summable.

Inner class. As in the proof of Theorem 2, we now use the $n_j = \text{int}\,\{e^{\alpha j \log j}\}$ with $\alpha > 0$, of (2.4). Hence, choosing $d_n - 1 \to \varepsilon$ small and positive and such that $d_n h_n \beta_{h_n}'' \log_2 n \uparrow$, we obtain from (2.29) with n_{j+1} for n and $\rho_{n_j} = m_j/n_j = 1 - O(j^{-\alpha})$,

$$
\begin{aligned}
(4.8) \qquad & \log P\{T_{n_j, n_{j+1}}(\pi_{n_{j+1}}) \leqq d_{n_{j+1}} h_{n_{j+1}} \beta_{h_{n_{j+1}}}'' \log_2 n_{j+1}\} \\
&= \{-(1 + \varepsilon) + h_{n_{j+1}}[\varepsilon - (1 + \varepsilon)\beta_{h_{n_{j+1}}}'' \log(1 + \varepsilon)] \\
&\qquad + o(1)\} \log_2 n_{j+1}.
\end{aligned}
$$

(While the last comment in the statement of Lemma 1 implies that this half of the proof does not require (2.28), the latter is in fact satisfied provided $(1 + \varepsilon)^{-1} > \limsup_n \beta_{h_n}''$, the last quantity being less than one by (2.24) and (2.25) since $h_n = O(1)$.) As in the outer class proof, we again have $h_n(1 - \beta_{h_n}'') > 1 + 2\delta$ for some $\delta > 0$ and all large n, and consequently the expression in braces on the right side of (4.8) is greater than $-1 + \delta\varepsilon$ for ε sufficiently small and positive and for all large n. Hence, the probabilities of (4.8) have divergent sum. In view of (2.14) and (2.15), it remains to compute

$$
\begin{aligned}
(4.9) \qquad P\{T_{n_j}(\pi_{n_{j+1}}) \geqq 1\} &= 1 - (1 - \pi_{n_{j+1}})^{n_j} \\
&\sim (n_j/n_{j+1}) h_{n_{j+1}} \log_2 n_{j+1} \\
&= O(j^{-\alpha} \log j),
\end{aligned}
$$

which is summable if $\alpha > 1$. This completes the proof of Theorem 4.

PROOF OF THEOREM 5. *Outer class.* Given $\varepsilon > 0$, put $n_j \sim (1 + \varepsilon/3)^j$ and modify the previous notation by writing

$$
\begin{aligned}
& k_n(\varepsilon) = n\pi_n + (1 + \varepsilon)[2n\pi_n \log_2 n]^{1/2}, \\
(4.10) \qquad & D_n(\varepsilon) = \{T_n(\pi_n) \geqq k_n(\varepsilon)\}, \qquad D_j^*(\varepsilon) = \bigcup_{n \in I_j} D_n(\varepsilon).
\end{aligned}
$$

The desired outer top bound result is

$$P\{D_n(\varepsilon)\text{ i.o.}\} = 0. \tag{4.11}$$

The first step is common in such proofs. If we show that

$$\liminf_j P\{D_{n_{j+1}}(\varepsilon/3) | D_j^*(\varepsilon)\} > 0, \tag{4.12}$$

then $P\{D_n(\varepsilon)\text{ i.o.}\} = P\{D_j^*(\varepsilon)\text{ i.o.}\} = 0$, by BC if $P\{D_{n_{j+1}}(\varepsilon/3)\}$ is summable, which it is by Lemma 3.

For brevity, we hereafter write k_n for $k_n(\varepsilon)$ (never for $k_n(\varepsilon/3)$). For $\nu \in I_j$, we define

$$G_\nu = \{T_\nu(\pi_{n_{j+1}}) \geqq k_\nu \pi_{n_{j+1}}/\pi_\nu - 1\}. \tag{4.13}$$

If the event $D_j^*(\varepsilon)$ of (4.10) occurs, define the random variable N_j^* by $N_j^* = \min\{n: D_n(\varepsilon)\text{ occurs}, n \in I_j\}$. Clearly,

$$\begin{aligned} P\{D_{n_{j+1}}(\varepsilon/3) \,|\, D_j^*(\varepsilon)\} &= EP\{D_{n_{j+1}}(\varepsilon/3) \,|\, D_j^*(\varepsilon); N_j^*; T_{N_j^*}(\pi_{N_j^*})\} \\ &\geqq \inf_{z \geqq k, \nu \in I_j^-} P\{D_{n_{j+1}}(\varepsilon/3) \,|\, T_\nu(\pi_\nu) = z\}. \end{aligned} \tag{4.14}$$

Since

$$\begin{aligned} P\{D_{n_{j+1}}(\varepsilon/3) | T_\nu(\pi_\nu) = z\} &\geqq P\{G_\nu D_{n_{j+1}}(\varepsilon/3) | T_\nu(\pi_\nu) = z\} \\ &= P\{D_{n_{j+1}}(\varepsilon/3) \,|\, G_\nu; T_\nu(\pi_\nu) = z\} \\ &\qquad \cdot P\{G_\nu \,|\, T_\nu(\pi_\nu) = z\}, \end{aligned} \tag{4.15}$$

we obtain (4.12) from (4.14) if there is a $\delta > 0$ such that, for all large j and $\nu \in I_j^-$,

$$\inf_{z \geqq k_\nu} P\{D_{n_{j+1}}(\varepsilon/3) | G_\nu; T_\nu(\pi_\nu) = z\} > \delta \tag{4.16}$$

and

$$\inf_{z \geqq k_\nu} P\{G_\nu | T_\nu(\pi_\nu) = z\} > \delta. \tag{4.17}$$

The conditional probability of (4.17) is clearly

$$B^+(k_\nu \pi_{n_{j+1}}/\pi_\nu - 1, z, \pi_{n_{j+1}}/\pi_\nu), \tag{4.18}$$

which is a minimum for $z = \text{int}^+\{k_\nu\}$. This and (2.33) yield (4.17).

Since $\pi_{n_{j+1}} \leqq \pi_\nu$, the probability of $D_{n_{j+1}}(\varepsilon/3)$ conditioned on values of $T_\nu(\pi_\nu)$ and $T_\nu(\pi_{n_{j+1}})$ is the same as that conditioned only on the last. Hence abbreviating $\text{int}^+\{k_\nu \pi_{n_{j+1}}/\pi_\nu - 1\}$ by μ, we see that the left side of (4.16) is at least

$$\begin{aligned} &\inf_{z \geqq k_\nu, z \geqq y \geqq \mu} P\{D_{n_{j+1}}(\varepsilon/3) \,|\, T_\nu(\pi_{n_{j+1}}) = y; T_\nu(\pi_\nu) = z\} \\ &\quad = \inf_{y \geqq \mu} P\{D_{n_{j+1}}(\varepsilon/3) | T_\nu(\pi_{n_{j+1}}) = y\} \\ &\quad = \inf_{y \geqq \mu} P\{T_{\nu, n_{j+1}}(\pi_{n_{j+1}}) \geqq k_{n_{j+1}}(\varepsilon/3) - y\} \\ &\quad = B^+(k_{n_{j+1}}(\varepsilon/3) - \text{int}^+\{k_\nu \pi_{n_{j+1}}/\pi_\nu - 1\}, n_{j+1} - \nu, \pi_{n_{j+1}}). \end{aligned} \tag{4.19}$$

By (2.33) and the fact that $\text{int}^+ \{x\} \geqq x$, we will thus establish (4.16) if we show that

$$k_{n_{j+1}}(\varepsilon/3) - [k_\nu(\varepsilon)\pi_{n_{j+1}}/\pi_\nu - 1] \leqq (n_{j+1} - \nu)\pi_{n_{j+1}} - 1 \tag{4.20}$$

for all large j and $\nu \in I_j^-$. Dividing both sides of (4.20) by $\pi_{n_{j+1}}$ and using (4.10), we obtain that (4.20) is equivalent to

$$\begin{aligned}(1 + \varepsilon/3)n_{j+1}[2(\log_2 n_{j+1})/n_{j+1}\pi_{n_{j+1}}]^{1/2} + 2\pi_{n_{j+1}}^{-1}&\\ \leqq (1 + \varepsilon)\nu[2(\log_2 \nu)/\nu\pi_\nu]^{1/2}&.\end{aligned} \tag{4.21}$$

The ratio of $2\pi_{n_{j+1}}^{-1}$ to the term preceding it approaches zero. Also, $n_{j+1}\pi_{n_{j+1}} > \nu\pi_\nu$ and $n_{j+1}(\log_2^{1/2} n_{j+1})/\nu \log_2^{1/2} \nu < n_{j+1}(\log_2^{1/2} n_{j+1})/n_j \log_2^{1/2} n_j \sim 1 + \varepsilon/3$. We conclude that (4.21) is satisfied for all large j and $\nu \in I_j^-$, completing the proof of (4.11).

The proof of the outer bottom result is very similar, so we shall merely list the changes. In (4.10) we replace $(1 + \varepsilon)$ by $-(1 + \varepsilon)$ in the definition of $k_n(\varepsilon)$, and $\geqq$ by $\leqq$ in the definition of $D_n(\varepsilon)$, as well as in the domain of z in (4.14), (4.16), and (4.17). The event G_ν of (4.13) is replaced by

$$\{T_\nu(\pi_{n_{j+1}}) \leqq k_\nu\pi_{n_{j+1}}/\pi_\nu + 1\}; \tag{4.22}$$

also, int^+ is replaced everywhere by int. The probability (4.18) is replaced by

$$B(k_\nu\pi_{n_{j+1}}/\pi_\nu + 1, z, \pi_{n_{j+1}}/\pi_\nu), \tag{4.23}$$

whose minimum subject to $z \leqq k_\nu$ is at $z = \text{int}\{k_\nu\}$; this minimum is bounded away from zero, by (2.34). Finally, in (4.19) μ becomes $\text{int}\{k_\nu\pi_{n_{j+1}}/\pi_\nu + 1\}$, and the domain of the first infimum is $\{y \leqq z \leqq k_\nu, y \leqq \mu\}$; when we replace the resulting domain $\{y \leqq \min(k_\nu, \mu)\}$ by $\{y \leqq \mu\}$, we cannot increase the infimum, and the analogues of last two expressions of (4.19) give, for a lower bound on the analogue of (4.16),

$$\begin{aligned}\inf_{y \leqq \mu} P\{T_{\nu, n_{j+1}}(\pi_{n_{j+1}}) \leqq k_{n_{j+1}}(\varepsilon/3) - y\}&\\ = B(k_{n_{j+1}}(\varepsilon/3) - \text{int}\{k_\nu\pi_{n_{j+1}}/\pi_\nu + 1\}, n_{j+1} - \nu, \pi_{n_{j+1}})&.\end{aligned} \tag{4.24}$$

By (2.34) and the fact that $\text{int}\{x\} \leqq x$, we obtain as the analogue of (4.20),

$$k_{n_{j+1}}(\varepsilon/3) - [k_\nu\pi_{n_{j+1}}/\pi_\nu + 1] \geqq (n_{j+1} - \nu)\pi_{n_{j+1}} + 1. \tag{4.25}$$

Recalling that $(1 + \varepsilon)$ has been replaced by $-(1 + \varepsilon)$ in the definition (4.10) of $k_n(\varepsilon)$, we see that (4.25) is again equivalent to (4.21).

Inner class. The proof follows usual LIL lines and has been given by Eicker [4] in essentially this form, so we only sketch it. For the top inner bound, one shows, with $n_j \sim \lambda^j$, that

$$P\{T_{n_j}(\pi_{n_j}) > n_j\pi_{n_j} + (1 - \varepsilon)[2h_{n_j}]^{1/2} \log_2 n_j \text{ i.o.}\} = 1, \tag{4.26}$$

by showing that

(4.27) $$P\{T_{n_{j-1},n_j}(\pi_{n_j}) > [n_j - (1 - [4\lambda/h_{n_j}]^{1/2})n_{j-1}]\pi_{n_j} + (1-\varepsilon)[2h_{n_j}]^{1/2}\log_2 n_j \text{ i.o.}\} = 1,$$

and

(4.28) $$P\{T_{n_{j-1}}(\pi_{n_j}) > (1 - [4\lambda/h_{n_j}]^{1/2})n_{j-1}\pi_{n_j}, \text{ a.a. } j\} = 1.$$

Of these, (4.27) is a consequence of (2.31) and BC provided λ is chosen so large that $[\lambda/(\lambda-1)]^{1/2}[(4/\lambda)^{1/2} + (1-\varepsilon)2^{1/2}] < 2^{1/2}$. The probability complementary to (4.28) is proved summable by using the standard Markov-Cramér inequality with abbreviations $T = T_{n_{j-1}}(\pi_{n_j})$, $\delta = [4\lambda/h_{n_j}]^{1/2}$, to obtain

(4.29) $$\begin{aligned} P\{T \leqq (1-\delta)ET\} &= P\{e^{-\delta T} \geqq e^{-(1-\delta)\delta ET}\} \\ &\leqq e^{(1-\delta)\delta ET}\, Ee^{-\delta T} \\ &= \exp\{n_{j-1}[\pi_{n_j}(1-\delta)\delta \\ &\qquad + \log(1 - \pi_{n_j} + \pi_{n_j}e^{-\delta})]\} \\ &< \exp\{-\delta^2 n_{j-1}\pi_{n_j}/3\}, \end{aligned}$$

the last inequality for δ and π_{n_j} sufficiently small. For the bottom inner bound, replace $>$ by $<$, $(1-\varepsilon)$ by $-(1-\varepsilon)$, and $-[4\lambda/h_{n_j}]^{1/2}$ by $[4\lambda/h_{n_j}]^{1/2}$ in (4.26), (4.27), (4.28); and replace (4.29) by

(4.30) $$P\{T \geqq (1+\delta)ET\} \leqq e^{-(1+\delta)\delta ET} Ee^{\delta T},$$

with the same final estimate as in (4.29).

REMARK. The coefficient δ of T and ET in the second expression of (4.29) and (4.30) is not the usual minimizing value for exponential binomial bounds, but is less cumbersome and is close enough to yield the desired conclusions.

PROOF OF THEOREM 6. If $H_n \uparrow +\infty$ and $h_n = H_n \pm \lambda(2H_n)^{1/2}$ where $\lambda = 1+\varepsilon$ or $1-\varepsilon$, then $h_n \uparrow$ for large n. Applying Theorem 5 for these four possible choices of h_n, and noting that $h_n \mp \lambda(2h_n)^{1/2} = H_n + O(1)$, yields (3.17). Similarly, (3.18) follows from Theorems 3 and 4. If $H_n \to 0$, we obtain (3.19) from Theorem 2 and from (3.18) for v small and positive; by (2.24) and (2.25), $\lim_{v\downarrow 0} c_v'' = 1$. Finally, (3.20) follows from (1.3) with $k_n = np_n$ and $h_n = \exp\{-(1\pm\varepsilon)H_n^{-1}\}$ upon invoking Theorem 3, the condition $np_n \to +\infty$ of the latter then being equivalent to (2.22).

REFERENCES

[1] R. R. BAHADUR, "A note on quantiles in large samples," *Ann. Math. Statist.*, Vol. 37 (1966), pp. 577–580.

[2] G. BAXTER, "An analogue of the law of the iterated logarithm," *Proc. Amer. Math. Soc.*, Vol. 6 (1955), pp. 177–181.

[3] K. L. CHUNG, "An estimate concerning the Kolmogoroff limit distribution," *Trans. Amer. Math. Soc.*, Vol. 67 (1949), pp. 36–50.

[4] F. EICKER, "A log log law for double sequences," to appear.

[5] W. FELLER, *An Introduction to Probability Theory and its Applications*, Vol. 1, New York, Wiley, 1957 (2nd ed.).

[6] J. KIEFER, "On Bahadur's representation of sample quantiles," *Ann. Math. Statist.*, Vol. 38 (1967), pp. 1323–1342.
[7] ———, "Deviations between the sample quantile process and the sample df," *Proceedings of the First International Conference on Nonparametric Inference*, Cambridge, Cambridge University Press, 1969, pp. 299–319.
[8] ———, "Old and new methods for studying order statistics and sample quantiles," *Proceedings of the First International Conference on Nonparametric Inference*, Cambridge, Cambridge University Press, 1969, pp. 349–357.
[9] V. F. KOLCHIN, "On the limiting behavior of extreme order statistics in a polynomial scheme," *Theor. Probability Appl.*, Vol. 14 (1969), pp. 458–469.
[10] H. ROBBINS and D. SIEGMUND, "An iterated logarithm law for maxima and minima" (abstract), *Ann. Math. Statist.*, Vol. 41 (1970), p. 757.