

# BOUNDS ON INTERVAL PROBABILITIES FOR RESTRICTED FAMILIES OF DISTRIBUTIONS

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## 1. Introduction

A number of improvements of the classical Chebyshev inequalities are known that depend on various restrictions in addition to moment conditions. Most of these results provide bounds on the distribution function  $P\{X \leq t\}$ . In this paper, we consider bounds on  $P\{s < X \leq t\}$ ,  $P\{s < X \leq t | X \leq t\}$  and on  $P\{s < X \leq t | X > s\}$ . Bounds are also obtained on densities and hazard rates. These bounds are obtained under a variety of restrictions, but a unified method is used which yields all results as special cases of a single theorem.

The restrictions we impose yield quite striking improvements over what is obtainable with moment conditions alone. Furthermore, at least some of the conditions arise in practice and can be verified under the proper circumstances by physical considerations. In all cases we assume that  $P\{X \geq 0\} = 1$ .

From a historical viewpoint, a natural condition to consider is that  $1 - F(x) = P\{X > x\}$  is convex on  $[0, \infty)$ . Bounds in this case were obtained by Gauss; a number of extensions and related results have been summarized by Fréchet [7]. Such bounds are often stated as inequalities on  $P\{|Y - m| > x\}$  where  $Y$  is unimodal with mode  $m$ . Of course this implies that  $X = |Y - m|$  satisfies  $P\{X \geq 0\} = 1$  and  $P\{X > x\}$  is convex.

In recent papers (Barlow and Marshall [2], [3]) we considered the condition that the distribution has a monotone hazard rate. If  $F$  has a density  $f$ , the ratio  $q(x) = f(x)/[1 - F(x)]$  is defined for  $F(x) < 1$  and is called the *hazard rate*, or sometimes the failure rate or force of mortality. Whether or not  $F$  has a density,  $F$  is said to have an increasing (decreasing) hazard rate—denoted IHR (DHR)—if  $\log [1 - F(x)]$  is concave where finite (convex on  $[0, \infty)$ ). It is easily seen that in case  $q$  exists, this property is equivalent to  $q$  increasing (decreasing). If  $F$  is a life distribution,  $q(x) dx$  can be interpreted as the conditional probability of death in  $[x, x + dx]$  given that death has not occurred before  $x$ . Because of

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this interpretation, the property of increasing hazard rate has great intuitive appeal as a representation of "wear-out." However, distributions with decreasing hazard rate also arise in reliability, particularly as mixtures of exponential distributions, but also as a reflection of "work-hardening."

We also consider a stronger property than IHR, namely that  $F$  has a density  $f$  which is a Pólya frequency function of order 2 ( $PF_2$ ); that is,  $f$  is logarithmically concave. Such densities are also unimodal.

Another class of distributions for which bounds are obtained is the class of distributions with increasing hazard rate average  $x^{-1} \int_0^x q(z) dz$ . In general  $F$  is said to have an increasing hazard rate average (IHRA) if  $F(0) = 0$  and if  $-\log [1 - F(x)]$  is starshaped where finite. This condition means that

$$(1.1) \quad -x^{-1} \log [1 - F(x)]$$

is increasing in  $x > 0$ , and when  $F$  has a hazard rate  $q$ , it is equivalent to  $x^{-1} \int_0^x q(z) dz$  increasing in  $x > 0$ . This class properly contains the IHR distributions. Its importance in reliability theory has been discussed by Z. W. Birnbaum, J. D. Esary, and A. W. Marshall [6].

Various other restrictions have been imposed to obtain bounds on distribution functions. We mention in particular the results of Mallows [9], [10] who, following Markov and Kreĭn, has obtained inequalities on distributions whose first  $s$  derivatives satisfy certain boundedness and sign change conditions. Such restrictions are not considered here.

We believe that the bounds obtained for interval probabilities may be of more practical interest than bounds obtained only on the distribution function. However, there are actually few such bounds to be found in the literature. Most cases which appear to be examples provide bounds for  $P\{|X - EX| \geq t\}$ , and are more properly regarded as bounds on the distribution function of the positive random variable  $|X - EX|$ . Perhaps the most notable example that cannot be so regarded is the inequality of Selberg [13]. Much more general results can be found, for example, in papers by Hoeffding [8] and Rustagi [12], but these are quite inexplicit.

In reply to a question of Anscombe in the discussion on Mallows' paper [10], Mallows describes a method very similar to ours for obtaining bounds on densities. However, explicit bounds on densities seem not to be known. One reason, perhaps, is that additional restrictions are required to force a density to exist, and to suggest a natural version of it.

## 2. Extremal families

Let  $\mathfrak{F}$  be a class of distributions for which bounds are desired, and suppose that  $F$  in  $\mathfrak{F}$  implies  $F(0-) = 0$ . For example,  $\mathfrak{F}$  may be the class of IHR distributions with first moment  $\mu_1$ . For some  $\mathfrak{F}$ , it is possible to define a class  $\mathfrak{G}$  of "extremal" distributions and show that certain extremums over  $\mathfrak{F}$  are equal to

the corresponding extremums over  $\mathfrak{G}$ . When  $\mathfrak{G}$  is sufficiently simple, the extremums may then be easily obtained.

This method has been used for obtaining Chebyshev-type inequalities, for instance, by Mallows [10]. But it cannot really be called a standard method, and it is not a very well defined one. In fact, a proper definition of "extremal family" seems to depend on the problem at hand, and the definition given below does not coincide with our previous one (Barlow and Marshall [3]).

Since we assume  $F(0-) = 0$ , it is often convenient to consider  $\bar{F}(x) = 1 - F(x)$  in place of  $F(x)$ . Note that  $F(t) - F(s) = \bar{F}(s) - \bar{F}(t)$ , and that  $\int_0^\infty \bar{F}(x) dx = \int_0^\infty x dF(x)$ .

For the sake of definiteness, we assume throughout that distribution functions are right continuous.

In the cases previously considered (Barlow and Marshall [3]), the crossing points of distributions in an extremal family  $\mathfrak{G}$  with fixed  $F$  in  $\mathfrak{F}$  are shown to be continuous in a parameter indexing  $\mathfrak{G}$ . With the help of this fact, it is possible to infer that the crossing points sweep out  $[0, \infty)$ . Thus there exists  $G$  in  $\mathfrak{G}$  such that  $\bar{G}(t-) \geq \bar{F}(t) \geq \bar{G}(t)$ , and consequently,

$$(2.1) \quad \sup_{\mathfrak{G}} \bar{G}(t-) \geq \bar{F}(t) \geq \inf_{\mathfrak{G}} \bar{G}(t).$$

In this paper we consider more closely the intertwining of distributions  $\mathfrak{G}$  with a fixed  $F$  in  $\mathfrak{F}$ . In particular, we require that for  $0 \leq s < t \leq \infty$ , there exists  $G_1$  in  $\mathfrak{G}$  ( $G_2$  in  $\mathfrak{G}$ ) such that  $\bar{G}_1$  ( $\bar{G}_2$ ) crosses  $\bar{F}$  exactly once in  $(s, t]$ , and this crossing is from above (below). With this we have for each  $s < t$  a  $G_1$  and  $G_2$  such that

$$(2.2) \quad \bar{G}_1(s-) \geq \bar{F}(s) \geq \bar{G}_2(s), \quad \bar{G}_2(t-) \geq \bar{F}(t) \geq \bar{G}_1(t).$$

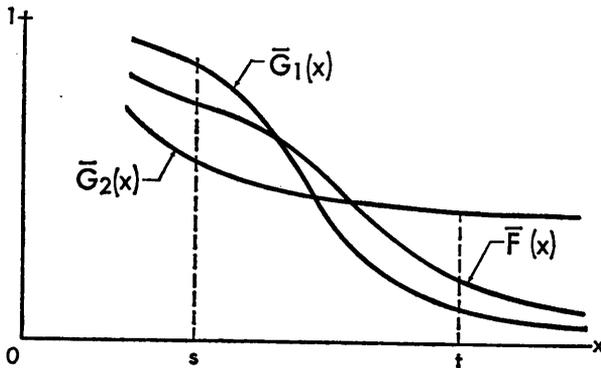


FIGURE 2.1

It follows immediately that

$$(2.3) \quad \bar{G}_2(s) - \bar{G}_2(t-) \leq \bar{F}(s) - \bar{F}(t) \leq \bar{G}_1(s-) - \bar{G}_1(t).$$

Though we guarantee that  $G_1$  and  $G_2$  in  $\mathfrak{G}$  exist, we often cannot be more specific, so that the bounds obtained are

$$(2.4) \quad \inf_{\mathfrak{G}} [\bar{G}(s) - \bar{G}(t-)] \leq \bar{F}(s) - \bar{F}(t) \leq \sup_{\mathfrak{G}} [\bar{G}(s-) - \bar{G}(t)].$$

From the relations between  $G_1$ ,  $G_2$ , and  $F$ , we also have more. Let  $\phi(y, z)$ ,  $0 \leq y, z \leq 1$ , be a function increasing in  $y$  and decreasing in  $z$ . Then

$$(2.5) \quad \phi(\bar{G}_2(s), \bar{G}_2(t-)) \leq \phi(\bar{F}(s), \bar{F}(t)) \leq \phi(\bar{G}_1(s-), \bar{G}_1(t)),$$

and hence,

$$(2.6) \quad \inf_{\mathfrak{G}} \phi(\bar{G}(s), \bar{G}(t-)) \leq \phi(\bar{F}(s), \bar{F}(t)) \leq \sup_{\mathfrak{G}} \phi(\bar{G}(s-), \bar{G}(t)).$$

The functions we consider in this paper are  $\phi_1(y, z) = 1 - z/y$  and  $\phi_2(y, z) = 1 - (1 - y)/(1 - z)$ . We have

$$(2.7) \quad \phi_1(\bar{F}(s), \bar{F}(t)) = [\bar{F}(s) - \bar{F}(t)]/\bar{F}(s) = P\{s < X \leq t | X > s\}$$

and

$$(2.8) \quad \phi_2(\bar{F}(s), \bar{F}(t)) = [F(t) - F(s)]/F(t) = P\{s < X \leq t | X \leq t\}.$$

The definition we give of "extremal family" is motivated partly by the requirement that there exist  $G_1$  and  $G_2$  related properly with  $F$ . The details of the definition are designed to aid in demonstrating that various explicit  $\mathfrak{G}$  that we later define are in fact extremal families. Because these details may otherwise be obscure, we begin by considering as an example the class  $\mathfrak{F}$  of distributions  $F$  satisfying (i)  $F$  has a  $PF_2$  density ( $\log f(x)$  is concave where finite), (ii)  $F(0) = 0$ , and for convenience,  $F(x) < 1$ ,  $x > 0$ , (iii)  $\int_0^\infty \zeta(x)f(x) dx = \nu$  where  $\zeta$  is an increasing function on  $[0, \infty)$  such that  $\zeta(0) \geq 0$ . Let  $w^* = \zeta^{-1}(\nu)$ , and let

$$(2.9) \quad \bar{G}_w(x) = \begin{cases} 1, & x < w, \\ e^{-a(x-w)}, & x \geq w, \end{cases} \quad 0 \leq w \leq w^*,$$

and

$$(2.10) \quad \bar{G}_w(x) = \begin{cases} 1, & x < 0, \\ 1 - (1 - e^{-bx})/(1 - e^{-bw}), & 0 \leq x \leq w, \\ 0, & x > w, \end{cases} \quad w \geq w^*,$$

where  $a$  and  $b$  are determined by the moment condition  $\int_0^\infty \zeta(x) dG_w(x) = \nu$ . Let  $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2$  where  $\mathfrak{G}_1 = \{G_w: 0 \leq w \leq w^*\}$  and  $\mathfrak{G}_2 = \{G_w: w \geq w^*\}$ .

By log concavity of  $f$ , we can show that  $F$  and  $G_w$  in  $\mathfrak{G}_1$  cross at most twice; by the moment condition they cross at least once. Label the crossing of  $\bar{F}$  from above by  $u_w$ , and the crossing of  $\bar{F}$  from below by  $v_w$ ; see figure 2.2.

When  $w = 0$ , there is exactly one crossing; this crossing is from below and so is denoted by  $v_0$ ; see figure 2.3. By the methods of Barlow and Marshall ([3], pp. 1267-1272), it can be shown that  $u_w$  ranges through  $[0, u_{w^*} = w^*]$  and  $v_w$  ranges through  $[v_0, \infty]$  as  $w$  ranges through  $[0, w^*]$ .

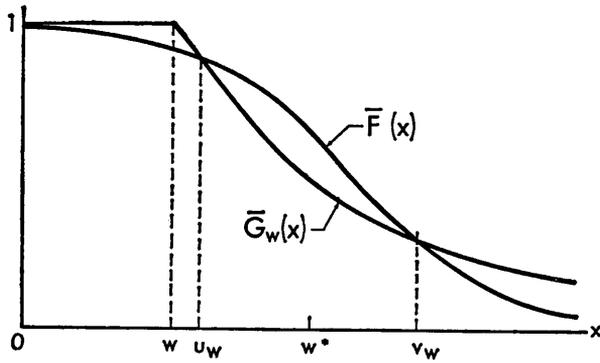


FIGURE 2.2

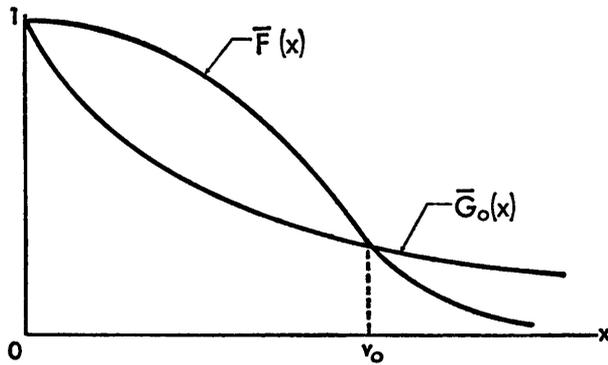


FIGURE 2.3

Also from the log concavity of  $f$ , it follows that  $F$  and  $G_w$  in  $\mathcal{G}_2$  cross at most twice; see figure 2.4.

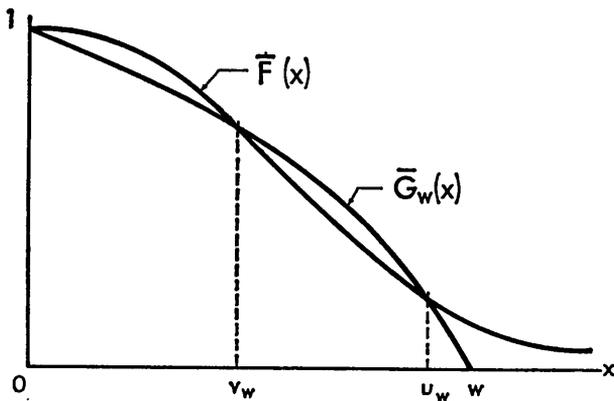


FIGURE 2.4

It can be shown that  $u_w$  ranges through  $[w^*, \infty]$  and  $v_w$  ranges through  $[0, v_0]$  as  $w$  ranges through  $[w^*, \infty]$ .

The above properties of extremal distributions corresponding to a family of distributions with fixed expectation lead to the following definition.

DEFINITION 2.1. A family  $\mathcal{G} = \{G_w: 0 \leq w \leq \infty\}$  is said to be extremal for  $\mathcal{F}$  if (1)  $G \in \mathcal{G}$  and  $F \in \mathcal{F}$  implies  $F$  and  $G$  cross at most twice. For fixed  $F$  in  $\mathcal{F}$ , let  $[m, M]$  be the support of  $F$  (smallest closed interval of  $F$ -probability one). Let  $u_w$  be the crossing of  $\bar{F}$  from above by  $\bar{G}_w$  if such a crossing exists; otherwise, let  $u_w = m$ . Let  $v_w$  be the crossing of  $\bar{F}$  from below by  $\bar{G}_w$  if such a crossing exists; otherwise, let  $v_w = M$ ; (2) there exists  $w^*$  such that:

(a)  $\bar{G}_{w^*}$  crosses  $\bar{F}$  exactly once, and the crossing is from above; see figure 2.5

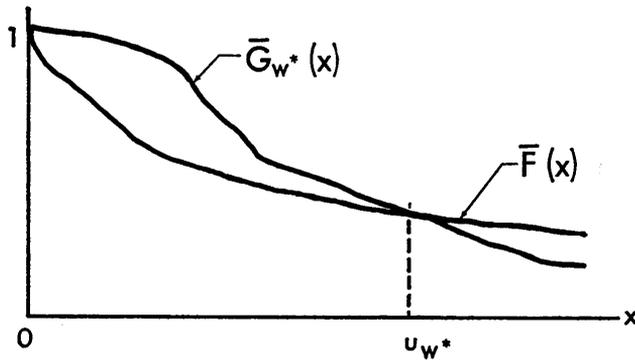


FIGURE 2.5

- (b) As  $w$  decreases from  $w^*$  to 0,
  - (i)  $u$  ranges continuously from  $u_{w^*}$  to  $m$ ,
  - (ii)  $v$  ranges continuously from  $M$  to  $v_0$ ,
  - (iii)  $u < v$ ;

see figure 2.2.

(c) At  $w = 0$ ,  $\bar{G}_w \equiv \bar{G}_0$  crosses  $\bar{F}$  at most once, and the crossing is from below at  $v_0$ . If no crossing exists,  $v = M$ ; see figure 2.3.

- (d) As  $w$  increases from  $w^*$  to  $\infty$ ,
  - (iv)  $u$  ranges continuously from  $u_{w^*}$  to  $M$ ,
  - (v)  $v$  ranges continuously from  $m$  to  $v_0 = v_\infty$ ,
  - (vi)  $u > v$ ;

see figure 2.4.

REMARK. In the above definition we imply that crossings occur at well defined points. However, for certain  $\mathcal{F}$  and  $\mathcal{G}$  it can happen in very special cases that a crossing "point" of  $F$  in  $\mathcal{F}$  and isolated  $G_w$  in  $\mathcal{G}$  is in reality an interval over which  $F$  and  $G_w$  coincide. In such cases, we may want to speak of the crossing as occurring anywhere in the interval of coincidence. The continuity of crossing points is required only to insure that there are no "gaps" where a

crossing from above or below cannot occur. In the case of coincidence over an interval, for example,  $u_{w^*} = [a, b]$ , then it is sufficient that

$$(2.11) \quad \begin{array}{l} \lim_{w \uparrow w^*} u_w \geq a \quad \text{and} \quad \lim_{w \downarrow w^*} u_w \leq b, \\ \text{or} \quad \lim_{w \uparrow w^*} u_w \leq b \quad \text{and} \quad \lim_{w \downarrow w^*} u_w \geq a. \end{array}$$

More precisely, crossing points may be regarded as interval-valued functions, and we require that they be upper semicontinuous (see Berge [5], p. 109).

**THEOREM 2.2.** *If  $\mathcal{G}$  is extremal for  $\mathcal{F}$ ,  $F \in \mathcal{F}$  and  $0 \leq s < t < \infty$ , there exists  $G_1$  and  $G_2$  in  $\mathcal{G}$  such that*

$$(2.12) \quad G_1(s-) \leq F(s) \leq G_2(s),$$

$$(2.13) \quad G_2(t-) \leq F(t) \leq G_1(t).$$

**PROOF.** Consider first the existence of  $G_1$ .

*Case 1* ( $s < t \leq m$ ). By (i), there exists  $w \leq w^*$  such that  $u_w = m$ . Then  $F(t) \leq G_w(t)$ , and by (iii),  $F(s) = G_w(s) = 0$ .

*Case 2* ( $s < m < t \leq u_{w^*}$  or  $m \leq s < t \leq u_{w^*}$ ). By (i), there exists  $w < w^*$  such that  $u_w = t$ . If  $F$  and  $G_w$  are continuous at  $t$ ,  $F(t) = G_w(t)$ , and always,  $F(t) \leq G_w(t+) = G_w(t)$ . By (iii),  $F(s) \geq G_w(s)$ .

*Case 3* ( $s < u_{w^*} < t$ ). Take  $G_1 = G_{w^*}$ .

*Case 4* ( $u_{w^*} \leq s < t$ ). To avoid trivialities, assume  $s < M$ . By (iv), there exists  $w \geq w^*$  such that  $u_w = s$ . If  $G_w$  is continuous at  $s$ , then  $F(s) = G_w(s)$ , and always,  $G_w(s-) \leq F(s)$ . By (vi),  $F(t) \leq G_w(t)$ .

Next, consider the existence of  $G_2$ .

*Case 1* ( $s < t \leq m$ ). Take  $G_2 = G_{w^*}$ .

*Case 2* ( $s < m < t \leq v_0$  or  $m \leq s < t \leq v_0$ ). By (v), there exists  $w \geq w^*$  such that  $v_w = t$ . If  $F$  and  $G_w$  are continuous at  $t$ ,  $F(t) = G_w(t)$  and always,  $G_w(t-) \leq G_w(t) \leq F(t)$ . By (vi),  $F(s) \leq G_w(s)$ .

*Case 3* ( $s < v_0 < t$ ). Take  $G_2 = G_0$ .

*Case 4* ( $v_0 \leq s < t$ ). To avoid trivialities assume  $s < M$ . By (ii), there exists  $w \leq w^*$  such that  $v_w = s$ . If  $F$  and  $G_w$  are continuous at  $s$ , take  $G_2 = G_w$ . Otherwise, by (i) and (ii), there exists  $w_1$  such that  $u_{w_1} < s < v_{w_1} < t$ . Take  $G_2 = G_{w_1}$ .

If  $\mathcal{G}$  is an extremal family, we use the notation  $\mathcal{G}_1 = \{G_w: 0 \leq w \leq w^*\}$  and  $\mathcal{G}_2 = \{G_w: w \geq w^*\}$ .

**THEOREM 2.3.** *Let  $\mathcal{G}$  be extremal for  $\mathcal{F}$ . If  $F \in \mathcal{F}$ ,  $0 \leq s < t \leq \infty$  and if  $\phi(y, z)$  is increasing in  $y$  and decreasing in  $z$ , then*

$$(2.14) \quad \phi(\bar{F}(s), \bar{F}(t)) \geq \begin{cases} \inf_{G \in \mathcal{G}_2} \phi(\bar{G}(s), \bar{G}(t-)), & s < t \leq v_0, \\ \phi(\bar{G}_0(s), \bar{G}_0(t-)), & s < v_0 < t, \\ \inf_{G \in \mathcal{G}_1} \phi(\bar{G}(s), \bar{G}(t-)), & v_0 \leq s < t; \end{cases}$$

$$(2.15) \quad \phi(\bar{F}(s), \bar{F}(t)) \leq \begin{cases} \sup_{G \in \mathcal{G}_1} \phi(\bar{G}(s-), \bar{G}(t)), & s < t \leq u_{w^*}, \\ \phi(\bar{G}_{w^*}(s-), \bar{G}_{w^*}(t)), & s < u_{w^*} < t, \\ \sup_{G \in \mathcal{G}_2} \phi(\bar{G}(s-), \bar{G}(t)), & u_{w^*} \leq s < t. \end{cases}$$

That  $\inf_{\mathfrak{G}} \phi(\bar{G}(s), \bar{G}(t)) \leq \phi(\bar{F}(s), \bar{F}(t)) \leq \sup_{\mathfrak{G}} \phi(\bar{G}(s), \bar{G}(t))$  follows directly from theorem 2.2. The more detailed results of theorem 2.3 are easily obtained from the proof of theorem 2.2. These detailed results are useful in case  $u_{w^*}$  or  $v_0$  are known; they are also useful even when only bounds on  $u_{w^*}$  or  $v_0$  are known. Otherwise, explicit results can be obtained only by computing the extremum over the whole class  $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2$ .

**THEOREM 2.4.** *If  $\mathfrak{G}$  is extremal for  $\mathfrak{F}$  and if  $G \in \mathfrak{G}$  implies that there exists a sequence  $\{F_n\}_{n=0}^{\infty}$ ,  $F_n \in \mathfrak{F}$ , such that  $F_n \rightarrow G$  in distribution, then the inequalities of theorem 2.3 are sharp.*

Of course the conditions of this obvious theorem are met if  $\mathfrak{G} \subset \mathfrak{F}$ . However, in the special cases considered later where  $\mathfrak{F}$  is the class of DHR distributions or the class of distributions with decreasing densities with a fixed moment  $\int_{0-}^{\infty} \zeta(x) dF(x)$ ,  $\mathfrak{G} \not\subset \mathfrak{F}$  because the moment condition may be violated. In these cases, theorem 2.4 applies to yield sharpness of the inequalities.

### 3. Bounds on probabilities of intervals

To apply theorem 2.3 in a special case, it is of course necessary first to obtain the extremal family. Such families satisfying definition 2.1 do not always exist; in fact, the requirement that  $G$  in  $\mathfrak{G}$  crosses  $F$  in  $\mathfrak{F}$  at most twice is geared to families  $\mathfrak{F}$  of distributions satisfying only a single moment condition. We offer no guide for determining whether an extremal family exists, and no guide for finding it when it does exist.

Before discussing more interesting examples, we mention the case that  $\mathfrak{F}$  is restricted by  $F(0-) = 0$  and a moment condition  $\int_{0-}^{\infty} \zeta(x) dF(x) = \nu$ ,  $\zeta$  strictly monotone and nonnegative. In this case,  $G_w$  in  $\mathfrak{G}_1$  is degenerate at  $w \leq \zeta^{-1}(\nu)$ , and  $G_w$  in  $\mathfrak{G}_2$  places mass  $[\nu - \zeta(0)]/[\zeta(w) - \zeta(0)] = p$  at  $w \geq \zeta^{-1}(\nu)$  and mass  $1 - p$  at the origin. Here, most bounds of theorem 2.3 are trivially 0 or 1. With  $\zeta(x) = x$ ,  $\nu = \mu < s$ , one also obtains that  $P\{s \leq X \leq t | X \leq t\} \leq \mu/s$  and  $P\{s \leq X \leq t\} \leq \mu/s$ . Both of these bounds are immediate from the original results of Chebyshev.

The remainder of this section is devoted to examples that are not quite as trivial as this classical case.

In order to give explicit results at points of discontinuity of a bound, we assume in the remainder of this paper that  $F$  is right continuous.

**3.1. Decreasing densities.** Let  $\mathfrak{F}$  be the class of distributions  $F$  such that  $F(0-) = 0$ ,  $\bar{F}(x)$  is convex on  $[0, \infty)$  and  $\int_{0-}^{\infty} \zeta(x) dF(x) = \nu < \infty$  where  $\zeta$  is a nonnegative strictly monotone function on  $[0, \infty)$ . In this case,  $w^*$  is defined by

$$(3.1) \quad \int_{0-}^{w^*} \zeta(x) dx = w^* \nu,$$

$$(3.2) \quad \mathfrak{G}_1 = \{G_w: 0 \leq w \leq w^*\},$$

where

$$(3.3) \quad \bar{G}_w(x) = \begin{cases} 1, & x < 0, \\ 1 - x/w, & 0 \leq x \leq w, \\ 0, & x > w, \end{cases}$$

and  $\mathfrak{G}_2 = \{G_w: w \geq w^*\}$ , where

$$(3.4) \quad \bar{G}_w(x) = \begin{cases} 1, & x < 0, \\ \alpha(1 - x/w), & 0 \leq x \leq w, \\ 0, & x > w. \end{cases}$$

The constant  $\alpha$  is determined by the moment condition  $\int_{0-}^{\infty} \zeta(x) dG_w(x) = \nu$ . In this case,  $u_{w^*}$  depends on  $F$  and  $\nu_0 = M$ .

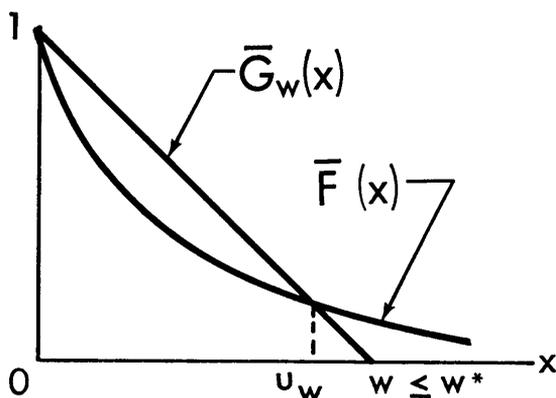


FIGURE 3.1.1

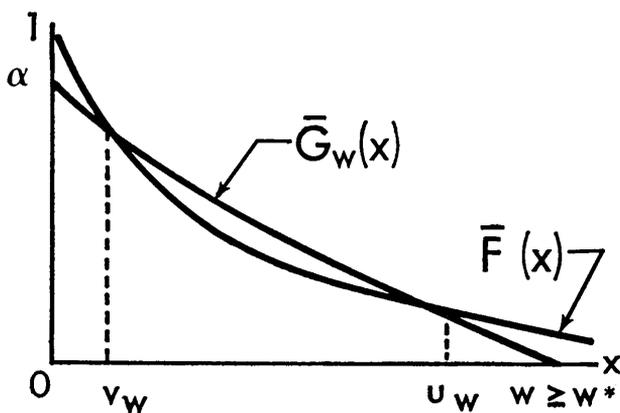


FIGURE 3.1.2

Using theorem 2.3, it is straightforward to obtain explicit results when  $\zeta(x) = x^r$ . In this case we denote  $\nu = \mu_r$  and obtain the following theorem.

THEOREM 3.1.1. *If  $F(0-) = 0$ ,  $\bar{F}(x)$  is convex on  $[0, \infty)$  and  $\int_0^\infty x^r dF(x) = \mu_r$ , then for all  $0 \leq s < t$ ,*

$$(3.5) \quad F(t) - F(s) \leq \begin{cases} [r/(r+1)s]^r \mu_r, & w^* = [(r+1)\mu_r]^{1/r} \leq (r+1)s/r \leq t \\ (r+1)\mu_r(t-s)/t^{r+1}, & w^* \leq t \leq (r+1)s/r \\ 1 - s[(r+1)\mu_r]^{-1/r}, & (r+1)s/r \leq w^* \leq t \\ 1 - s/t, & t \leq w^*. \end{cases}$$

The special case obtained from this theorem by letting  $t \rightarrow \infty$  has been given by Fréchet [7]; for further comments, see example 2.2 of Barlow and Marshall [2]. Other bounds obtainable from theorem 2.3 are trivial (0 or 1) with the exception of the upper bound for  $[F(t) - F(s)]/F(t)$ . The upper bound for this conditional probability coincides with the upper bound for  $F(t) - F(s)$  given in the above theorem.

3.2. *Increasing hazard rates.* Consider now the class  $\mathcal{F}$  of distributions  $F$  such that  $F(0-) = 0$ ,  $F$  is IHR, and  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$ , where  $\zeta$  is a non-negative strictly monotone function on  $[0, \infty)$ . In this case,  $w^* = \zeta^{-1}(\nu)$ ,  $\mathcal{G}_1 = \{G_w: 0 \leq w \leq w^*\}$  where  $G_w$  is given by (2.9), and  $\mathcal{G}_2 = \{G_w: w \geq w^*\}$  where

$$(3.6) \quad \bar{G}_w(x) = \begin{cases} e^{-bx}, & 0 \leq x < w, \\ 0, & x \geq w, \end{cases}$$

and  $b$  is determined by the moment condition  $\int_0^\infty \zeta(x) dG_w(x) = \nu$ . It is not difficult to see that for all  $w \geq w^*$ , there exists  $b$  satisfying this condition.

The continuity of crossing points  $u_w$  and  $v_w$  can be checked using arguments similar to those of Barlow and Marshall ([3], p. 1269).

Since  $G_{w^*}$  is degenerate at  $w^*$ , it follows that  $u_{w^*} = w^* = \zeta^{-1}(\nu)$  and the details of theorem 2.3 are useful in computing upper bounds.

It is clear from the definition of  $G_w$  that  $w < u_w < v_w$  when  $w < w^*$  and  $v_w < w = u_w$  when  $w > w^*$ . Using these facts, one can examine the proof of theorem 2.2 to obtain the following refinement of theorem 2.3: if  $\phi(y, z)$  is increasing in  $y$  and decreasing in  $z$ , then

$$(3.7) \quad \phi(\bar{F}(s), \bar{F}(t)) \leq \begin{cases} \sup_{w \leq t} \phi(\bar{G}_w(s), \bar{G}_w(t)), & s < t \leq u_{w^*} = \zeta^{-1}(\nu) \\ \phi(\bar{G}_{w^*}(s), \bar{G}_{w^*}(t)) = \phi(1, 0), & s < \zeta^{-1}(\nu) < t \\ \phi(\bar{G}_s(s-), 0), & \zeta^{-1}(\nu) \leq s < t. \end{cases}$$

Although  $v_0$  depends on  $F$ , it is known (Barlow and Marshall [2], lemmas 3.1 and 3.2) that in case  $\zeta(x)$  is increasing and convex, then  $v_0 \geq \nu$ . This is useful in computing lower bounds, since  $t \leq \nu$  implies  $t \leq v_0$ .

Because of their importance in reliability applications, we give a number of inequalities for IHR distributions. Since our main interest is in  $\zeta(x) = x^r$ , we state the results for this case unless there is no loss of simplicity in stating more general results.

THEOREM 3.2.1. *If  $F(0) = 0$ ,  $F$  is IHR and  $\int_0^\infty x^r dF(x) = \mu_r$  ( $r \geq 1$ ), then*

$$(3.8) \quad F(t) - F(s) \geq \begin{cases} 0, & s < t < \mu_r^{1/r} \text{ or } \mu_r^{1/r} \leq s < t \\ \min [e^{-s/\lambda_r^{1/r}} - e^{-t/\lambda_r^{1/r}}, e^{-bs} - e^{-bt}], & s < \mu_r^{1/r} \leq t \end{cases}$$

where  $b$  satisfies  $r \int_0^t x^{r-1} e^{-bx} dx = \mu_r$  and  $\lambda_r = \mu_r/\Gamma(r+1)$ .

PROOF. The lower bound of 0 is attained by the degenerate distribution, which by assumption is right continuous. Suppose that  $s < \mu_r^{1/r} \leq t$ , and let  $w = t$  so that  $G_w \in \mathcal{G}_2$ . If  $\bar{F}(s) \geq \bar{G}_t(s)$ ,

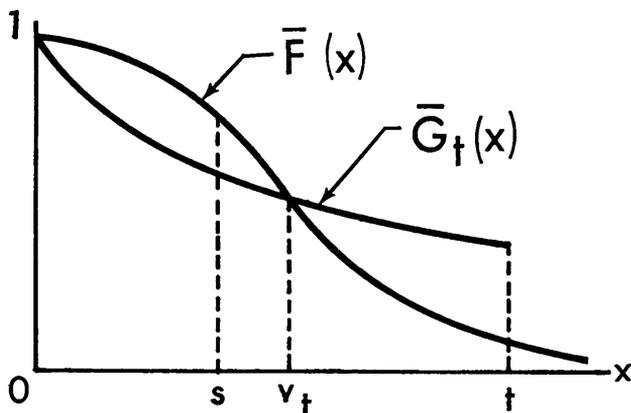


FIGURE 3.2.1

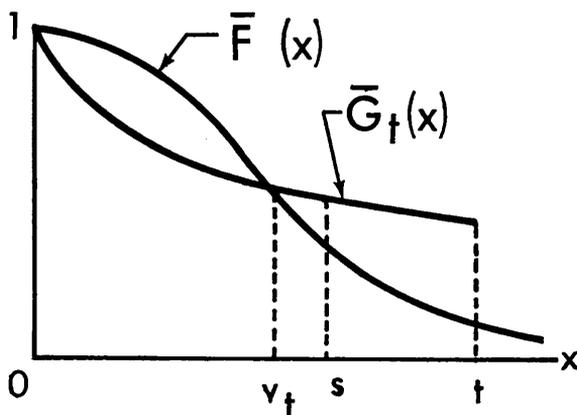


FIGURE 3.2.2

then since  $\bar{G}_t(t-) \geq \bar{F}(t)$  (figure 3.2.1),

$$(3.9) \quad F(t) - F(s) \geq G_t(t-) - G_t(s).$$

Next suppose  $\bar{F}(s) < \bar{G}_t(s)$  (figure 3.2.2). Since  $r \geq 1$ ,  $x^r$  is convex, and it follows that  $v_0 \geq \mu_r^{1/r}$ ; but  $v$  ranges monotonically through  $[0, v_0]$  as  $w$  ranges through  $[\mu_r^{1/r}, \infty]$ . Hence there exists  $w \geq t$  such that  $v_w = s$ , and we conclude that

$$(3.10) \quad F(t) - F(s) \geq \inf_{w \geq t} (e^{-bs} - e^{-bt})$$

where  $b$  satisfies  $r \int_0^\infty x^{r-1} e^{-bx} dx = \mu_r$ . Since extremal distributions satisfy the

moment condition, they must cross at least once, and we conclude that  $b$  is a monotone increasing function of  $w$ . Differentiating with respect to  $b$ , we see that  $e^{-bs} - e^{-bt}$  is increasing for  $b \leq (t - s)^{-1} \log t/s$  and decreasing otherwise. Hence, the infimum is attained for  $w = t$  or  $w = \infty$ .||

The above theorem can be stated with the more general condition that  $\int_0^\infty \zeta(x) dF(x) = \nu$  where  $\zeta(x) \geq 0$  is convex and strictly increasing. The crucial fact used in the proof is that  $v_0 \geq \zeta^{-1}(\nu)$ .

**THEOREM 3.2.2.** *If  $F(0) = 0$ ,  $F$  is IHR and  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$  where  $\zeta$  is a strictly monotone nonnegative function on  $[0, \infty)$ , then*

$$(3.11a) \quad F(t) - F(s) \leq \max \left\{ \sup_{0 \leq w \leq s} [e^{-a(s-w)} - e^{-a(t-w)}], \sup_{s \leq w \leq t} [1 - e^{-a(t-w)}] \right\}$$

if  $s < t < \zeta^{-1}(\nu)$ ,

and

$$(3.11b) \quad F(t) - F(s) \leq \begin{cases} 1, & s < \zeta^{-1}(\nu) \leq t, \\ e^{-bs}, & \zeta^{-1}(\nu) \leq s < t, \end{cases}$$

where  $a$  satisfies  $\int_w^\infty \zeta(x) a e^{-a(x-w)} dx = \nu$  and  $b$  satisfies  $\int_0^s \zeta(x) b e^{-bx} dx + \zeta(s) e^{-bs} = \nu$ .

If  $\zeta(x) = x$  so that  $\nu = \mu_1$ , then we have more explicitly that

$$(3.12) \quad F(t) - F(s) \leq 1 - e^{-(t-s)/(\mu_1-s)}, \quad s < t < \mu_1.$$

**PROOF.** Suppose  $s < t < \zeta^{-1}(\nu)$ . Then there exists  $w$  such that  $w \leq t = u_w < \zeta^{-1}(\nu)$  (see figure 3.2.3), since  $u_w > w$  ranges continuously through

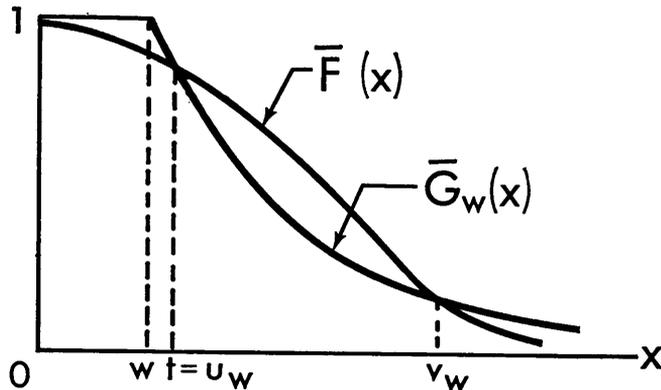


FIGURE 3.2.3

$[0, \zeta^{-1}(\nu)]$  as  $w$  ranges over the same interval. Hence,

$$(3.13) \quad F(t) - F(s) \leq \sup_{0 \leq w \leq t} [G_w(t) - G_w(s)],$$

and we have the first bound.

In case  $\zeta(x) = x$  and  $\nu = \mu_1 = 1$ , we have  $a = 1/(1 - w)$ . If  $w < s$ , then

$$(3.14) \quad (1 - w)^2 \frac{d}{dw} [G_w(t) - G_w(s)] = (1 - s)e^{-(s-w)/(1-w)} - (1 - t)e^{-(t-w)/(1-w)} \geq 0,$$

since  $s < t$  and the supremum over  $0 < w < s$  is achieved with  $w = s$ . If  $s < w < t$ , then

$$(3.15) \quad (1 - w)^2 \frac{d}{dw} [1 - e^{-(t-w)/(1-w)}] = (t - w)e^{-(t-w)/(1-w)} \leq 0$$

and again the supremum is attained with  $w = s$ .

The bound for  $s < \zeta^{-1}(\nu) \leq t$  is attained by the distribution degenerate at  $\zeta^{-1}(\nu)$ .

If  $\zeta^{-1}(\nu) \leq s < t$ , then since  $\bar{G}_s(s-) = e^{-bs} \geq \bar{F}(s)$  and  $\bar{G}_s(s+) = 0$ , the last bound is immediate.||

**THEOREM 3.2.3.** *If  $F(0) = 0$ ,  $F$  is IHR and  $\int_0^\infty x^r dF(x) = \mu_r$  ( $r \geq 1$ ), then*

$$(3.16) \quad \frac{F(t) - F(s)}{F(t)} \geq \begin{cases} 0, & \mu_r^{1/r} \leq s < t, \\ [e^{-s/\lambda_r^{1/r}} - e^{-t/\lambda_r^{1/r}}]/[1 - e^{-t/\lambda_r^{1/r}}], & s < \mu_r^{1/r}, \end{cases}$$

where  $\lambda_r = \mu_r/\Gamma(r + 1)$ .

**PROOF.** For  $\mu_r^{1/r} < s < t$ , the bound is attained with  $G_w \in \mathcal{G}_2$  and  $w = s$ .

Suppose  $s \leq \mu_r^{1/r}$ . Since  $r \geq 1$ , we know that  $v_0 \geq \mu_r^{1/r}$ . Hence, by theorem 2.3 we need consider only  $\mathcal{G}_2$ . It is easily seen that  $[G_w(t) - G_w(s)]/G_w(t)$  is decreasing in  $w$ ,  $\mu_r^{1/r} < w < t$ , and hence

$$(3.17) \quad [F(t) - F(s)]/F(t) \geq \inf_{w \geq t} [G_w(t) - G_w(s)]/G_w(t);$$

that is, we want  $w \geq t$  to maximize  $G_w(s)/G_w(t) = (1 - e^{-bs})/(1 - e^{-bt})$ , where  $b$  satisfies  $r \int_0^w x^{r-1} e^{-bx} dx = \mu_r$ . Since  $b$  is an increasing function of  $w$ , we maximize with respect to  $b \geq 0$ . Now  $d/(G_w(s)/G_w(t)) db \geq 0$  if and only if  $te^{bs} - se^{bt} \leq t - s$ . Since  $t > s$ ,  $d/(te^{bs} - se^{bt}) db \leq 0$ , and we have that  $te^{bs} - se^{bt} \leq te^{bs} - se^{bt}|_{b=0} = t - s$ . Letting  $w \rightarrow \infty$ , we obtain  $b = 1/\lambda_r^{1/r}$ , and hence the second bound.

As in the case of theorem 3.2.1, we could obtain similar bounds if  $\zeta \geq 0$  is convex and increasing, by using the fact that  $v_0 \geq \zeta^{-1}(\nu)$ .

**THEOREM 3.2.4.** *If  $F(0) = 0$ ,  $F$  is IHR and  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$  where  $\zeta \geq 0$  is a strictly monotone function on  $[0, \infty)$ , then*

$$(3.18) \quad \frac{F(t) - F(s)}{F(t)} \leq \begin{cases} 1, & s < \zeta^{-1}(\nu), \\ 1 - G_s(s) = e^{-bs}, & s \geq \zeta^{-1}(\nu), \end{cases}$$

where  $b$  satisfies  $\int_0^s \zeta(x) b e^{-bx} dx + \zeta(s) e^{-bs} = \nu$ .

**PROOF.** The bound for  $s < \zeta^{-1}(\nu)$  is attained by  $G_w$  in  $\mathcal{G}_1$ ,  $s < w < t$ . For  $s \geq \zeta^{-1}(\nu)$  we need consider only  $\mathcal{G}_2$ , and by the monotonicity obtained in the proof of theorem 3.2.3, the result follows.||

**THEOREM 3.2.5.** *If  $F(0) = 0$ ,  $F$  is IHR and  $\int_0^\infty x^r dF(x) = \mu_r$ , then*

$$(3.19) \quad \frac{F(t) - F(s)}{1 - F(s)} \geq \begin{cases} 0, & s < t < \mu_r^{1/r}, \\ 1 - e^{-b(t-s)}, & t \geq \mu_r^{1/r}, \end{cases}$$

where  $b$  satisfies  $r \int_0^t x^{r-1} e^{-bx} dx = \mu_r$ .

**PROOF.** The proof parallels the proof of theorem 3.2.1 to a certain extent.

Clearly, if  $t < \mu_r^{1/r}$ , the bound is attained with  $G_t \in \mathcal{G}_1$ . Suppose  $t \geq \mu_r^{1/r}$  and  $s < v_0$ . If  $\bar{F}(s) \geq \bar{G}_t(s)$  (see figure 3.2.1), then

$$(3.20) \quad \frac{F(t) - F(s)}{\bar{F}(s)} = 1 - \frac{\bar{F}(t)}{\bar{F}(s)} \geq 1 - \frac{\bar{G}_t(t-)}{\bar{G}_t(s)} = \frac{G_t(t-) - G_t(s)}{\bar{G}_t(s)}.$$

If  $\bar{F}(s) < \bar{G}_t(s)$  (see figure 3.2.2) and  $s < v_0$ , choose  $w \geq t$  so that  $v_w = s$ . In this case

$$(3.21) \quad \frac{F(t) - F(s)}{\bar{F}(s)} \geq \frac{G_w(t) - G_w(s)}{\bar{G}_w(s)} = 1 - e^{-b(t-s)}$$

where  $b$  satisfies  $r \int_w^0 x^{r-1} e^{-bx} dx = \mu_r$ . Since  $b$  increases with  $w$  and  $1 - e^{-b(t-s)}$  increases with  $b$ , the minimum is attained with  $w = t$  as before.

Now suppose  $s > v_0$ . Choose  $G_w \in \mathcal{G}_1$  ( $0 \leq w \leq \mu_r^{1/r}$ ) so that  $\bar{G}_w(s) = \bar{F}(s)$ , that is,  $v_w = s$  (see figure 3.2.4).

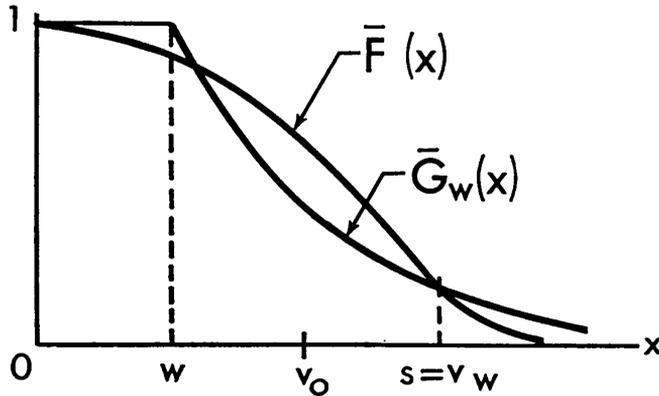


FIGURE 3.2.4

Clearly,

$$(3.22) \quad \frac{F(t) - F(s)}{\bar{F}(s)} \geq \frac{G_w(t) - G_w(s)}{\bar{G}_w(s)} = 1 - e^{-a(t-s)}.$$

But  $e^{-a(t-s)}$  is maximized for  $w = 0$ , since  $a$  is increasing with  $w$ . Now  $G_0 = G_w$ , so we have already found that

$$(3.23) \quad 1 - e^{-b(t-s)} \leq 1 - e^{-a(t-s)}$$

where  $a$  and  $b$  satisfy

$$(3.24) \quad r \int_0^t x^{r-1} e^{-bx} dx = \mu_r = \int_0^\infty x^r a e^{-ax} dx.$$

**THEOREM 3.2.6.** *If  $F(0) = 0$ ,  $F$  is IHR and  $\int_0^\infty \xi(x) dF(x) = \nu < \infty$  where  $\xi \geq 0$  is a strictly increasing function on  $[0, \infty)$ , then*

$$(3.25) \quad \frac{F(t) - F(s)}{1 - F(s)} \leq \begin{cases} \sup_{s \leq w \leq t} 1 - e^{-a(t-w)}, & s < t < \xi^{-1}(\nu), \\ 1, & t \geq \xi^{-1}(\nu), \end{cases}$$

where  $a$  is determined by

$$(3.26) \quad \int_w^\infty \zeta(x)ae^{-a(x-w)} dx = \nu.$$

If  $\zeta(x) = x$  so that  $\nu = \mu_1$ , then we have more explicitly that

$$(3.27) \quad \frac{F(t) - F(s)}{1 - F(s)} \leq 1 - e^{-(t-s)/(\mu_1-s)}, \quad s < t < \mu_1.$$

PROOF. First suppose  $s < t < \zeta^{-1}(\nu)$ . Choose  $w < t$  such that  $u_w = t$  (see figure 3.2.3). Then  $\bar{F}(t) = \bar{G}_w(t)$  and  $\bar{F}(s) \leq \bar{G}_w(s)$ . Hence,

$$(3.28) \quad \frac{F(t) - F(s)}{1 - F(s)} = 1 - \frac{\bar{F}(t)}{\bar{F}(s)} \leq 1 - \frac{\bar{G}_w(t)}{\bar{G}_w(s)} = \begin{cases} 1 - e^{-a(t-s)}, & w \leq s \\ 1 - e^{-a(t-w)}, & w > s \end{cases}$$

where  $a$  is determined by

$$(3.29) \quad \int_w^\infty \zeta(x)ae^{-a(x-w)} dx = \int_0^\infty \zeta(x+w)ae^{-ax} dx = \nu.$$

Since  $\zeta$  is increasing,  $a$  is increasing with  $w$ ; furthermore,  $1 - e^{-a(t-s)}$  is increasing in  $a$ , and we conclude that  $\max_{w \leq s} 1 - e^{-a(t-s)}$  occurs at  $w = s$ .

Clearly, the bound for  $t > \zeta^{-1}(\nu)$  is attained by any  $G_w$  with  $\zeta^{-1}(\nu) < w < t$ .||

Note that using theorems 3.2.5 and 3.2.6, we have also obtained bounds on  $P\{X \geq t|X \geq s\} = [1 - F(t)]/[1 - F(s)]$ , since

$$(3.30) \quad \frac{1 - F(t)}{1 - F(s)} = 1 - \frac{F(t) - F(s)}{1 - F(s)}.$$

3.3. *PF<sub>2</sub> densities.* Let  $\mathfrak{F}$  denote the class of distributions  $F(t) = \int_0^t f(x) dx$  such that  $F(0) = 0$ ,  $f$  is  $PF_2$  on  $[0, \infty)$ , and  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$  where  $\zeta \geq 0$  is a strictly monotone function on  $[0, \infty)$ . The extremals for this case have been introduced in (2.9) and (2.10).

Using theorem 2.3 together with information on the extremals for  $PF_2$  densities given by Barlow and Marshall ([3], pp. 1268-1269), we obtain

THEOREM 3.3.1. *If  $F(0) = 0$ ,  $f$  is  $PF_2$  on  $[0, \infty)$  and  $\int_0^\infty \zeta(x)f(x) dx = \nu < \infty$  where  $\zeta \geq 0$  is strictly monotone on  $[0, \infty)$ , then*

$$(3.31) \quad F(t) - F(s) \geq \begin{cases} 0, & 0 \leq s < t < \zeta^{-1}(\nu) = w^* \quad \text{or} \quad \zeta^{-1}(\nu) \leq s < t, \\ \inf_{w \geq t} \int_s^t be^{-bx} dx/[1 - e^{-bw}], & s < \zeta^{-1}(\nu) \leq t, \end{cases}$$

(3.32a)

$$F(t) - F(s) \leq \max \left\{ \sup_{0 \leq w \leq s} [e^{-a(s-w)} - e^{-a(t-w)}], \sup_{s \leq w \leq t} [1 - e^{-a(t-w)}], \right.$$

if  $s < t < \zeta^{-1}(\nu)$

and

$$(3.32b) \quad F(t) - F(s) \leq \begin{cases} 1, & s < \zeta^{-1}(\nu) \leq t, \\ \sup_{w \geq s} \int_s^t be^{-bx} dx/[1 - e^{-bw}], & \zeta^{-1}(\nu) \leq s < t, \end{cases}$$

where  $a$  and  $b$  are chosen to satisfy

$$(3.33) \quad \int_0^w \zeta(x)be^{-bx} dx/[1 - e^{-bw}] = \int_w^\infty \zeta(x)ae^{-a(x-w)} dx = \nu.$$

In section 4, we obtain explicit bounds in special cases utilizing bounds on the density.

PROOF. To show the lower bound, suppose first  $t > v_0$  and  $s < w^* = \zeta^{-1}(\nu)$ . Clearly,  $F(t) - F(s) \geq G_0(t) - G_0(s) = e^{-bs} - e^{-bt}$  where  $\int_0^\infty \zeta(x)be^{-bx} dx = \nu$ . If  $s < w^* < t < v_0$ , choose  $G_w$  in  $\mathfrak{G}_2$  such that  $v_w = t$  (see figure 3.3.1). This is

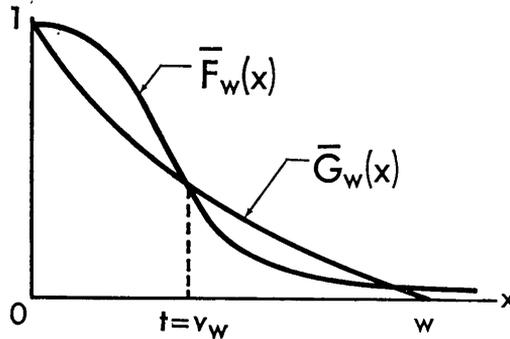


FIGURE 3.3.1

possible, since  $v_w$  ranges through  $[0, v_0]$  as  $w$  ranges through  $[w^*, \infty]$ . Clearly  $F(t) - F(s) \geq G_w(t) - F_w(s)$ . The remaining lower bound is attained by the degenerate distribution.

The upper bounds in case  $s < \zeta^{-1}(\nu)$  are given in theorem 3.2.2. Suppose that  $\zeta^{-1}(\nu) \leq s < t$ . There exists a unique crossing of  $f$  from below by the density  $g_w$  of  $G_w$ ,  $w > w^*$  (see Barlow and Marshall [3], p. 1269); denote this crossing by  $x_w^*$ . If  $s > x_w^*$ , the bound is clear (see figure 3.3.2). If  $s < x_w^*$ , there exists  $w$  such that  $x_w^* = s$  (see figure 3.3.3). Barlow and Marshall [3] show that for this  $w$ ,

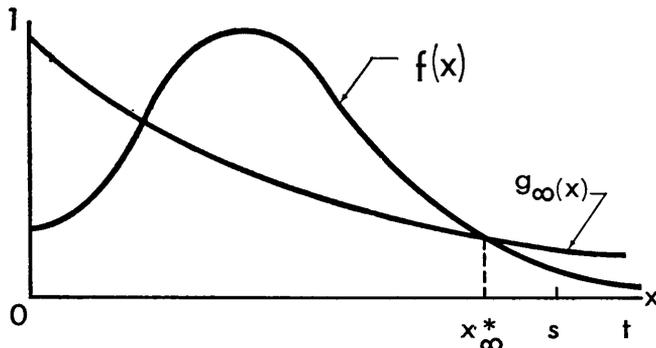


FIGURE 3.3.2

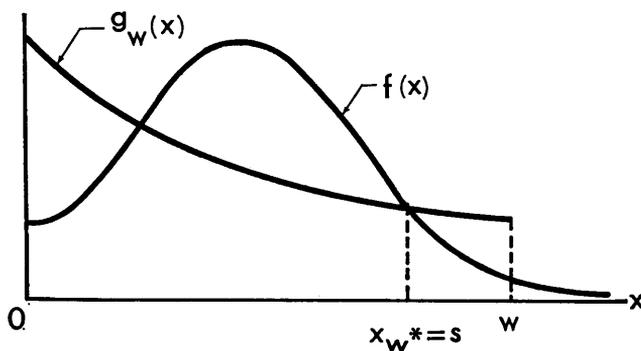


FIGURE 3.3.3

$\int_s^\infty f(x) dx \leq \int_s^w be^{-bx} dx/[1 - e^{-bw}]$ , and since  $be^{-bx} > f(x)$  for  $s < x < w$ , we easily see that  $\int_s^t f(x) dx \leq \int_s^t be^{-bx} dx/[1 - e^{-bw}]$  for all  $t > s \geq \zeta^{-1}(\nu)$ .||

3.4. *Decreasing hazard rates.* Let  $\mathfrak{F}$  be the class of distributions  $F$  such that  $F(0-) = 0$ ,  $F$  is DHR, and  $\int_{0-}^\infty \zeta(x) dF(x) = \nu < \infty$ , where  $\zeta \geq 0$  is a strictly increasing function on  $[0, \infty)$ . Let  $w^*$  be defined by  $\int_{0-}^\infty \zeta(x)e^{-x/w^*} dx = w^*\nu$ , and let  $\mathfrak{G}_1 = \{G_w: 0 \leq w \leq w^*\}$  where

$$(3.34) \quad \bar{G}_w(x) = \begin{cases} 1, & x < 0, \\ e^{-x/w}, & x \geq 0; \end{cases}$$

let  $\mathfrak{G}_2 = \{G_w: w \geq w^*\}$  where

$$(3.35) \quad \bar{G}_w(x) = \begin{cases} 1, & x < 0, \\ \alpha e^{-x/w}, & x \geq 0, \end{cases}$$

and  $\alpha$  is determined by  $\int_{0-}^\infty \zeta(x) dG_w(x) = \nu$ . It can be shown that  $\mathfrak{G} = \{G_w: 0 \leq w \leq \infty\}$  is extremal for  $\mathfrak{F}$ ,  $v_0 = \infty$ , and  $u_{w^*}$  is the unique positive crossing point of  $F$  and  $G_{w^*}$  (which depends on  $F$ ).

In this case,  $\mathfrak{G} \not\subset \mathfrak{F}$  because  $\int_{0-}^\infty \zeta(x) dG_w(x) \neq \nu$  for  $w < w^*$ . However, it is easily seen that  $G_w$  can be approximated by distributions in  $\mathfrak{F}$  that are piecewise exponential, with two pieces.

**THEOREM 3.4.1.** *Let  $F$  be DHR,  $F(0-) = 0$ , and  $\int_{0-}^\infty x^r dF(x) = \mu_r$ . Denote  $[\mu_r/\Gamma(r + 1)]^{1/r}$  by  $\theta$  and  $t/s$  by  $\rho$ . If  $0 < s < t$ , then*

$$(3.36) \quad 0 \leq F(t) - F(s) \leq \begin{cases} \rho^{-s/(t-s)} - \rho^{-t/(t-s)}, & (t-s)/\theta \leq \log \rho, \\ e^{-s/\theta} - e^{-t/\theta}, & \log \rho \leq (t-s)/\theta \leq \log [(r\theta - t)/(r\theta - s)], \\ \rho^r z^r (e^{-sz} - e^{-tz}), & \log [(r\theta - t)/(r\theta - s)] \leq (t-s)/\theta, \end{cases}$$

where  $z$  is defined by  $\log [(r - tz)/(r - sz)] = (t - s)z$ .

**PROOF.** The lower bound is easily obtained since  $\lim_{w \rightarrow \infty} G_w(t) - G_w(s) = 0$  when  $s > 0$ . To obtain the upper bound, first consider

$$(3.37) \quad \sup_{\mathfrak{G}_1} [G(t) - G(s)] = \max_{w \leq w^*} [e^{-s/w} - e^{-t/w}]$$

where  $w^*$  is determined by  $\mu_r w^* = \int_0^\infty x^r e^{-x/w^*} dx = \Gamma(r + 1)w^{*r+1}$ , or  $w^* = \theta$ .

Therefore  $\max_{w \leq w^*} [e^{-s/w} - e^{-t/w}] = \max_{z \geq \theta^{-1}} [e^{-sz} - e^{-tz}]$ . By differentiating  $e^{-sz} - e^{-tz}$ , we see that this quantity has a maximum at  $z = \log \rho / (t - s)$ . Hence,

$$(3.38) \quad \max_{z \geq \theta^{-1}} [e^{-sz} - e^{-tz}] = \begin{cases} \rho^{-s/(t-s)} - \rho^{-t/(t-s)}, & \log \rho / (t - s) \geq \theta^{-1}, \\ e^{-s/\theta} - e^{-t/\theta}, & \log \rho / (t - s) < \theta^{-1}. \end{cases}$$

Next, consider

$$(3.39) \quad \sup_{\mathfrak{g}_z} [G(t) - G(s)] = \max_{w \geq \theta} \alpha [e^{-s/w} - e^{-t/w}]$$

where  $\alpha$  is determined by

$$(3.40) \quad \mu_r = \int_0^\infty x^r dG_w(x) = r \int_0^\infty x^{r-1} \alpha e^{-x/w} dx = \alpha w^r \Gamma(r + 1),$$

or  $\alpha = (\theta/w)^r$ . Thus

$$(3.41) \quad \max_{w \geq \theta} \alpha [e^{-s/w} - e^{-t/w}] = \max_{z < \theta^{-1}} (\theta z)^r [e^{-sz} - e^{-tz}].$$

We compute

$$(3.42) \quad \frac{d}{dz} z^r [e^{-sz} - e^{-tz}] = z^{r-1} \{e^{-sz}(r - sz) - e^{-tz}(r - tz)\}.$$

To investigate this derivative, consider  $e^{-xz}(r - xz)$  as a function of  $x$ . The derivative  $(d/dz)e^{-xz}(r - xz) = ze^{-xz}[xz - (r + 1)]$  is  $< 0$  for  $x < (r + 1)/z$  and  $= 0$  for  $x = (r + 1)/z$ , and  $> 0$  for  $x > (r + 1)/z$ .

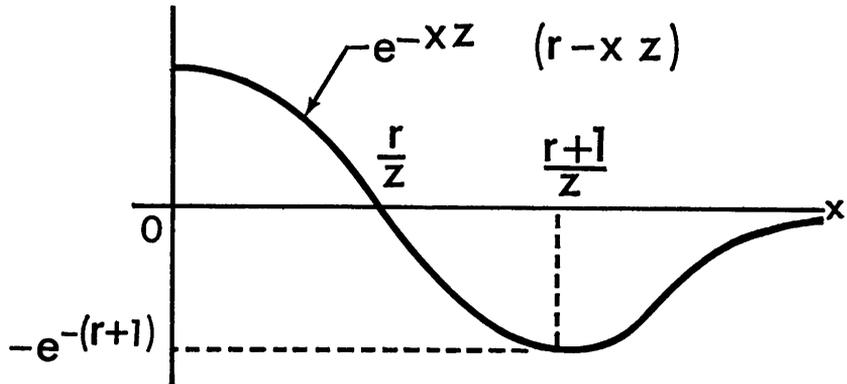


FIGURE 3.4.1

Suppose that  $\exp \{-t\theta^{-1}\} (r - t\theta^{-1}) \geq \exp \{-s\theta^{-1}\} (r - s\theta^{-1})$ . Then it is clear from figure 3.4.1 with  $z = \theta^{-1}$  that  $t > (r + 1)\theta$ . Since  $e^{-xz}(r - xz)$  is symmetric in  $x$  and  $z$ , its graph for fixed  $x$  as a function of  $z$  is as in figure 3.4.1 with  $x$  and  $z$  interchanged. By decreasing  $z$  from  $\theta^{-1}$  to  $(r + 1)/t$ , we see from such a figure with  $x = t$  and using  $t > (r + 1)\theta$  that  $e^{-tz}(r - tz)$  decreases to  $-e^{-(r+1)}$  from  $\exp \{-t\theta^{-1}\} (r - t\theta^{-1})$ . Similarly, from figure 3.4.1 with  $x$  and  $z$  interchanged, and  $x = s$  we see that  $e^{-sz}(r - sz)$  moves to  $e^{-(r+1)s/t} [r - (r + 1)s/t] >$

$-e^{-(r+1)}$  from  $\exp \{-s\theta^{-1}\}(r - s\theta^{-1})$ . Thus by continuity, there exists  $z_0 \in [(r + 1)/t, \theta^{-1}]$  such that  $e^{-tz_0}(r - tz_0) = e^{-sz_0}(r - sz_0)$ .

Next, suppose that  $\exp \{-t/\theta^{-1}\}(r - t\theta^{-1}) < \exp \{-s\theta^{-1}\}(r - s\theta^{-1})$ . Then if a solution  $z_0$  to the equation  $e^{-sz}(r - sz) = e^{-tz}(r - tz)$  exists in  $[0, \theta^{-1}]$ ,  $s < (r + 1)/z_0 < t$  by figure 3.4.1. As  $z$  decreases from  $\theta^{-1}$  to  $(r + 1)/t$ ,  $\exp \{-t\lambda^{-1}\}(r - t\lambda^{-1})$  decreases and  $\exp \{s\lambda^{-1}\}(r - s\lambda^{-1})$  increases so that no solution exists, and in fact  $e^{-sz}(r - sz) > e^{-tz}(r - tz)$ ,  $0 \leq z \leq \theta^{-1}$ . In this case,  $\max_{z \leq \theta^{-1}} \lambda z(e^{-sz} - e^{-tz})$  occurs at  $z = \theta^{-1}$ , and equals  $e^{-s/\theta} - e^{-t/\theta}$ . Referring to the results for  $\sup_{\mathfrak{G}_1} [G(t) - G(s)]$  we see that  $e^{-s/\theta} - e^{-t/\theta} \leq \rho^{-s/(t-s)} - \rho^{-t/(t-s)}$ . Hence, if  $(t - s)/\log \rho \leq \theta$ , then

$$(3.43a) \quad \sup_{\mathfrak{G}} [G(t) - G(s)] = \rho^{-s/(t-s)} - \rho^{-t/(t-s)}.$$

If  $(t - s)/\log \rho \geq \theta$  and  $\exp \{-t\theta^{-1}\}(r - t\theta^{-1}) < \exp \{-s\theta^{-1}\}(r - s\theta^{-1})$ , then

$$(3.43b) \quad \sup_{\mathfrak{G}} [G(t) - G(s)] = e^{-s/\theta} - e^{-t/\theta}.$$

If  $\exp \{-t\lambda^{-1}\}(r - t\lambda^{-1}) \geq \exp \{-s\lambda^{-1}\}(r - s\theta^{-1})$ , then

$$(3.43c) \quad \sup_{\mathfrak{G}} [G(t) - G(s)] = \lambda z(e^{-sz} - e^{-tz})$$

where  $z$  in  $((r + 1)/t, \lambda^{-1})$  uniquely satisfies  $e^{-tz}(r - tz) = e^{-sz}(r - sz)$ . Since  $\exp \{-t\theta^{-1}\}(r - t\theta^{-1}) \geq \exp \{-s\theta^{-1}\}(r - s\theta^{-1})$  implies  $s > r/\theta^{-1}$  (see figure 3.4.1 with  $z = \theta^{-1}$ ),  $\exp \{-t\theta^{-1}\}(r - t\theta^{-1}) \geq \exp \{-s\theta^{-1}\}(r - s\theta^{-1})$  if and only if  $(r - t\theta^{-1})/(r - s\theta^{-1}) \leq \exp \{\theta^{-1}\}(t - s)$  if and only if

$$(3.44) \quad (t - s)\theta^{-1} \geq \log [(r - t\theta^{-1})/(r - s\theta^{-1})].$$

The condition  $\log \rho \leq (t - s)/\theta \leq \log (r\theta - t)/(r\theta - s)$  is nonvacuous when  $s > r\theta$ ; otherwise,  $\log \rho > \log (r\theta - t)/(r\theta - s)$ .

3.5. *Increasing hazard rate averages.* Let  $\mathfrak{F}$  be the class of distributions  $F$  such that  $F(0) = 0$ ,  $F$  is IHRA and  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$ , where  $\zeta \geq 0$  is a monotone function on  $[0, \infty)$ . Let  $w^* = \zeta^{-1}(\nu)$ , and let  $\mathfrak{G}_1 = \{G_w: 0 \leq w \leq w^*\}$  where

$$(3.45) \quad \bar{G}_w(x) = \begin{cases} 1, & x < w, \\ e^{-bx}, & x \geq w, \end{cases}$$

and  $b$  is determined by the moment condition  $\int_0^\infty \zeta(x) dG_w(x) = \nu$ . Let  $\mathfrak{G}_2 = \{G_w: w \geq w^*\}$  where  $\bar{G}_w$  is given by (3.6).

Note that  $\mathfrak{G}_2, w^*$  and  $u_{w^*}$  are the same as in the IHR case; this means that the upper bounds for  $\phi(\bar{F}(s), \bar{F}(t))$  obtained from theorem 2.3 with  $t > u_{w^*}$  are the same as in the IHR case.

Contrary to the IHR case, it is possible that  $F$  in  $\mathfrak{F}$  and  $G$  in  $\mathfrak{G}$  coincide over an interval where  $0 < F(x) = G(x) < 1$ . Thus, crossing "points" may actually be intervals; in particular,  $v_0$  may be an interval. To avoid notational complications, we write the proofs below as though crossing points are well-defined; by  $s = v_0$  we mean  $s$  is in the crossing interval  $v_0$ , and by  $s < v_0$  ( $s > v_0$ ) we mean

that  $s$  lies to the left (right) of each point in the interval. (See the remark following definition 2.1.)

**THEOREM 3.5.1.** *If  $F(0) = 0$ ,  $F$  is IHRA and  $\int_0^\infty x^r dF(x) = \mu_r$  ( $r \geq 0$ ), then*

$$(3.46) \quad F(t) - F(s) \geq \begin{cases} 0, & s < t < \mu_r^{1/r} \text{ or } \mu_r^{1/r} \leq s < t, \\ \min [e^{-b_s s} - e^{-b_s t}, e^{-b_s s} - e^{-b_t t}], & s < \mu_r^{1/r} \leq t, \end{cases}$$

where  $b_s$  is determined by  $s^r(1 - e^{-b_s s}) + \int_s^\infty x^r b_s e^{-b_s x} dx = \mu_r$  and  $b_t$  is determined by  $r \int_0^t x^{r-1} e^{-b_t x} dx = \mu_r$ .

**PROOF.** The lower bound of 0 is attained by the degenerate distribution. Suppose that  $s < \mu_r^{1/r} \leq t$ , and let  $w = t$  so that  $G_w \in \mathcal{G}_2$ . If  $\bar{F}(s) \geq \bar{G}_t(s)$ , then since  $\bar{G}_t(t-) \geq \bar{F}(t)$  (see figure 3.2.1),

$$(3.47) \quad F(t) - F(s) \geq G_t(t-) - G_t(s) = e^{-b_s s} - e^{-b_t t}.$$

If  $\bar{F}(s) < \bar{G}_t(s)$ , and if  $s \leq v_0$  (see figure 3.2.2), then there exists  $w \geq t$  such that  $v_w = s$ , and we conclude that

$$(3.48) \quad F(t) - F(s) \geq \inf_{w \geq t} [e^{-b_s s} - e^{-b_t t}]$$

where  $b$  satisfies

$$(3.49) \quad r \int_0^w x^{r-1} e^{-bx} dx = \mu_r.$$

If, on the other hand,  $s > v_0$ , then there exists  $w < s$  such that  $v_w = s$  (figure 3.5.1).

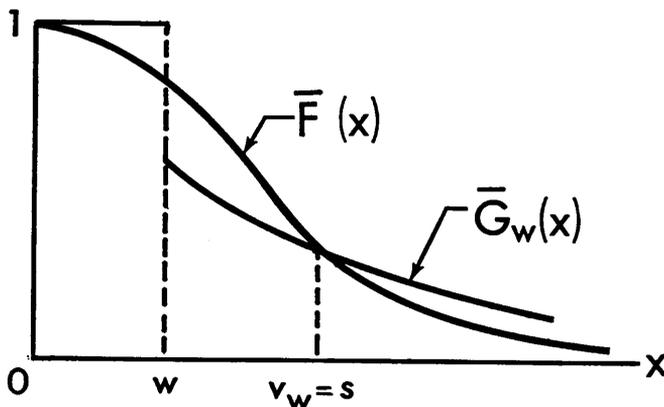


FIGURE 3.5.1

Then  $\bar{G}_w(t) \geq \bar{F}(t)$ , and we conclude that

$$(3.50) \quad F(t) - F(s) \geq \inf_{w \leq s} [e^{-b_s s} - e^{-b_t t}]$$

where  $b$  satisfies

$$(3.51) \quad w^r(1 - e^{-bw}) + \int_w^\infty x^r b e^{-bx} dx = \mu_r.$$

Now  $G_w(t) - G_w(s) = e^{-bs} - e^{-bt}$  both for  $G_w$  in  $\mathcal{G}_1$  and  $G_w$  in  $\mathcal{G}_2$ ; also in both cases,  $b$  is an increasing function of  $w$  (two extremal distributions must cross to have  $r$ -th moment  $\mu_r$ ). Hence,

$$(3.52) \quad \inf_{w \geq t} [e^{-bs} - e^{-bt}] = \inf_{b_t \leq b_w \leq b_\infty} [e^{-b_w s} - e^{-b_w t}]$$

where  $b = b_w$  is determined by (3.49), and

$$(3.53) \quad \inf_{w \leq s} [e^{-bs} - e^{-bt}] = \inf_{b_0 \leq b_w \leq b_s} [e^{-b_w s} - e^{-b_w t}]$$

where  $b = b_w$  is determined by (3.51). Since  $b_0 = b_\infty$ , we conclude that

$$(3.54) \quad \min \{ \inf_{w \leq s} [e^{-bs} - e^{-bt}], \inf_{w \geq t} [e^{-bs} - e^{-bt}] \} = \inf_{b_t \leq b \leq b_s} [e^{-bs} - e^{-bt}].$$

Now  $e^{-bs} - e^{-bt}$  is increasing in  $b \leq (t - s)^{-1} \log(t/s)$  and decreasing in  $b \geq (t - s)^{-1} \log(t/s)$ . Hence,  $\inf_{b_t \leq b \leq b_s} [e^{-bs} - e^{-bt}]$  occurs at an endpoint.||

**THEOREM 3.5.2.** *If  $F(0) = 0$ ,  $F$  is IHRA and  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$  where  $\zeta$  is a strictly monotone nonnegative function on  $[0, \infty)$ , then*

$$(3.55) \quad F(t) - F(s) \leq \begin{cases} 1 - e^{-b_t t}, & s < t < \zeta^{-1}(\nu) \\ 1, & s < \zeta^{-1}(\nu) \leq t, \\ e^{-b_s s}, & \zeta^{-1}(\nu) \leq s < t, \end{cases}$$

where  $b_s$  is determined by  $\int_0^s \zeta(x) b e^{-bx} dx + \zeta(s) e^{-bs} = \nu$  and  $b_t$  is determined by  $\zeta(t)[1 - e^{-b_t t}] + \int_t^\infty \zeta(x) b e^{-bx} dx = \nu$ .

**PROOF.** If  $s < t < \zeta^{-1}(\nu)$ , then  $\bar{G}_t(s) \geq \bar{F}(s)$  and  $\bar{G}_t(t+) \leq \bar{F}(t)$ ; otherwise  $F$  and  $G_t$  would not cross (see figure 3.5.1 with  $w = t$ ). In case  $t \geq \zeta^{-1}(\nu)$ , the bounds follow from theorem 3.2.2 and the remark preceding theorem 3.5.1.||

**3.6. Bounds on integrals.** Bounds were obtained by Barlow [1] on integrals of the form  $\int_0^t \bar{F}(x) dx$ , assuming that  $F \in \mathcal{F}$  is IHR, with specified mean and variance. In this case, the extremals  $G_w \in \mathcal{G}$  were piecewise exponentials; these extremals can cross  $F \in \mathcal{F}$  at most three times, but are not extremal in the sense of definition 2.1. However,  $\bar{F}^*(x) = \int_x^\infty \bar{F}(u) du$  and  $\bar{G}_w^*(x) = \int_x^\infty \bar{G}_w(u) du$  can cross at most twice, since they agree at  $x = 0$ . Hence, we can show that  $\mathcal{G}^* = \{G_w^* : G_w \in \mathcal{G}\}$  is extremal in the sense of definition 2.1 for  $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$ , and theorem 2.3 can be applied. Hence, for example,

$$(3.56) \quad \inf_w \int_s^t \bar{G}_w(x) dx \leq \int_s^t \bar{F}(x) dx \leq \sup_w \int_s^t \bar{G}_w(x) dx.$$

From another point of view, if we let  $f_1(x) = \bar{F}(x)/\mu_1$ , then we have actually obtained bounds for the class of distributions having decreasing  $PF_2$  densities, constrained at the origin with specified mean.

#### 4. Bounds on densities and hazard rates

Generally speaking, bounds on densities do not exist, even under restrictions which guarantee that the densities exist; a density  $f$  need only satisfy  $P\{X \in A\} = \int_A f(x) dx$  for measurable  $A$ , so can be arbitrarily defined at a

fixed point to violate any nontrivial bound. However, when  $F$  is differentiable, the most natural version of the density is  $f(t) = F'(t)$ , and this often can be bounded nontrivially.

If  $\mathcal{G}$  is extremal for  $\mathcal{F}$ , then for each  $t > 0$  and each  $F \in \mathcal{F}$ , there exists  $G \in \mathcal{G}$  such that  $\bar{G}$  crosses  $\bar{F}$  from above at  $t$ . If  $F'(t) = f(t)$  and  $G'(t) = g(t)$  exist, then clearly  $f(t) \leq g(t)$ . Similarly, there exists  $G$  in  $\mathcal{G}$  such that  $\bar{G}$  crosses  $\bar{F}$  from below at  $t$ ; if  $F'(t) = f(t)$  and  $G'(t) = g(t)$  exist, then  $f(t) \geq g(t)$ . Hence, barring differentiability problems, we conclude that if  $\mathcal{G}$  is extremal for  $\mathcal{F}$ , then

$$(4.1) \quad \inf_{\mathcal{G}} g(t) \leq f(t) \leq \sup_{\mathcal{G}} g(t).$$

Even though  $F$  is not differentiable at  $t$ , both the right and the left derivatives  $f_+(t) = \lim_{\Delta \downarrow 0} [F(t + \Delta) - F(t)]/\Delta$  and  $f_-(t) = \lim_{\Delta \downarrow 0} [F(t) - F(t - \Delta)]/\Delta$  may exist at least for some  $t$ . In this case, we consider bounds valid for any version  $f(t)$  of the density lying between  $f_+(t)$  and  $f_-(t)$ . Similarly,  $G \in \mathcal{G}$  need not be differentiable at  $t$ ; we use  $\max(g_+(t), g_-(t))$  for the upper bound and  $\min(g_+(t), g_-(t))$  for the lower bound. With these conventions, (4.1) still holds.

Of course if there exists  $G$  in  $\mathcal{G}$  discontinuous at  $t$ , then no upper bound exists for  $F$  in  $\mathcal{F}$  at  $t$ . Similarly, if there exists  $G$  in  $\mathcal{G}$  such that  $G(t) = 0$  or  $1$ , then the lower bound for  $f(t)$  is  $0$ .

From the definition of an extremal family and the location of  $t$  with respect to  $u_{v^*}$  and  $v_0$ , one can easily ascertain whether the extremizing  $g$  is in  $\mathcal{G}_1$  or  $\mathcal{G}_2$ .

Bounds on interval probabilities yield bounds on densities via limiting arguments in an obvious way, and similarly, bounds on the conditional probability  $P\{s < X \leq t | X > s\}$  yield bounds on the hazard rate  $q$ . We do not give a proof that such bounds are automatically sharp, even if the bounds on interval probabilities are sharp. However, in each case that we apply this method, it is not difficult to verify that the inequality obtained is sharp.

4.1. *Decreasing densities.* If  $F(0-) = 0$  and  $\bar{F}(x)$  is convex in  $x \geq 0$ , then the right and left derivatives of  $F$  exist finitely except possibly at  $0$ . Let  $f$  be a version of the density bounded by these quantities. Then by passing to the limit in (3.5), we obtain

$$(4.2) \quad f(t) \leq \begin{cases} (r + 1)\mu_r/t^{r+1}, & t \leq [(r + 1)\mu_r]^{1/r}, \\ t^{-1}, & t \geq [(r + 1)\mu_r]^{1/r}. \end{cases}$$

Lower bounds for  $f(t)$  are trivial except when  $t = 0$ . In this case, we obtain from (3.5) with  $t = \infty$  that  $F(s) \geq s[(r + 1)\mu_r]^{-1/r}$ , and hence that

$$(4.3) \quad f(0) \geq [(r + 1)\mu_r]^{-1/r}.$$

4.2. *IHR distributions.* If  $F$  is IHR, then Marshall and Proschan [11] have shown that  $F$  is absolutely continuous, except possibly for a jump at the right-hand endpoint of its support. Thus  $q(x) = f(x)/\bar{F}(x)$  exists for all  $x$  such that  $\bar{F}(x) < 1$ , and there exists a version of  $f$  for which  $q$  is increasing. The following bounds apply to any such version, which, since  $q$  is increasing, must satisfy  $f_-(t) \leq f(t) \leq f_+(t)$ .

**THEOREM 4.2.1.** *Let  $F$  be IHR, and  $F(0) = 0$ . If  $\int_0^\infty x dF(x) = \mu_1$ , then*

$$(4.4) \quad f(t) \leq q(t) \leq \begin{cases} 1/(\mu_1 - t), & t < \mu_1, \\ \infty, & t \geq \mu_1. \end{cases}$$

*If  $\int_0^\infty x^2 dF(x) = \mu_2$ , then*

$$(4.5) \quad f(t) \leq q(t) \leq \begin{cases} [t + (2\mu_2 - t^2)^{1/2}]/(\mu_2 - t^2), & t < \mu_2^{1/2}, \\ \infty, & t \geq \mu_2^{1/2}. \end{cases}$$

Equation (4.4) easily follows from theorems 3.2.2 and 3.2.6. Equation (4.5) follows from theorem 4.2.3 below.

Explicit sharp bounds for general  $r$ -th moment given do not seem to be obtainable. The following theorem gives explicit bounds that are sharp only for  $r = 1$  or  $t = 0$ .

**THEOREM 4.2.2.** *If  $F$  is IHR,  $F(0) = 0$ , and  $\int_0^\infty x^r dF(x) = \mu_r$  ( $r \geq 1$ ), then*

$$(4.6) \quad f(t) \leq q(t) \leq [\Gamma(r + 1)]^{1/r}/(\mu_r^{1/r} - t), \quad 0 \leq t < \mu_r^{1/r}.$$

**PROOF.** Since  $q(x)$  is increasing in  $x$ , for  $t \leq \mu_r^{1/r}$  it follows that

$$(4.7) \quad q(t)(\mu_r^{1/r} - t) \leq \int^{\mu_r^{1/r}} q(z) dz.$$

The right-hand inequality in statement (4.6) follows from this and the bound  $F(\mu_r^{1/r}-) \geq \exp \{-[\Gamma(r + 1)]^{1/r}\}$  (see Barlow and Marshall [2], p. 1242).||

Equality is attained in (4.6) with  $t = 0$  by the exponential distribution; with  $r = 1$ , the result coincides with (4.4).||

The method of proof we illustrate in the following theorems easily admits a generalization of the IHR property; we assume that for some given  $\theta(x) \geq 0$ ,  $a(x) \equiv \theta(x)q(x)$  is increasing in  $x \geq 0$ . A special case of interest is  $\theta(x) = 0$ ,  $x < x_0$ ,  $\theta(x) = 1$ ,  $x \geq x_0$ , in which case the hypothesis that  $a(x)$  is increasing becomes the hypothesis that  $q(x)$  is increasing in  $x \geq x_0$ . Thus,  $q$  is allowed to be initially decreasing. In this case, nontrivial upper bounds for the density are obtainable.

In order to state these results, we fix  $t$ , suppose that  $\theta(x) > 0$  for  $x \geq w$ , and let

$$(4.8) \quad \bar{G}_w(x; a) = \begin{cases} 1, & x \leq w, \\ \exp \left\{ -a \int_w^x dz/\theta(z) \right\}, & x > w. \end{cases}$$

In case  $\theta(x) > 0$  for all  $x \leq w$ , let

$$(4.9) \quad H_w(x; a) = \begin{cases} \exp \left\{ -a \int_0^x dz/\theta(z) \right\}, & 0 \leq x < w, \\ 0, & x > w. \end{cases}$$

**REMARK.** If  $1/\theta(x)$  is finitely integrable over all intervals and if  $a$  is determined by the moment condition  $\int_0^\infty \zeta(x) dG_w(x; a) = \int_0^\infty \zeta(x) dH_w(x; a) = \nu$ , then distributions of the form  $G_w$  and  $H_w$  form an extremal family for the distributions to be considered in theorems 4.2.3 and 4.2.4. The case that  $1/\theta(x)$

is not finitely integrable over all intervals is more complex. However, in proving the following theorems, we do not adopt this point of view.

**THEOREM 4.2.3.** *Let  $\zeta \geq 0$  be a strictly monotone function on  $[0, \infty)$  such that  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$ . Let  $\theta$  be such that  $\int_t^\infty dz/\theta(z) < \infty$  for all finite  $x > t$ . If  $a(x)$  is increasing in  $x \geq 0$ , there exists a unique solution  $a_1$  of  $\nu = \int_0^\infty \zeta(x) dG_t(x; a_1)$  whenever  $t < \zeta^{-1}(\nu)$ . Furthermore,*

$$(4.10) \quad f(t) \leq q(t) \leq \begin{cases} a_1/\theta(t), & t < \zeta^{-1}(\nu), \\ \infty, & t \geq \zeta^{-1}(\nu). \end{cases}$$

**THEOREM 4.2.4.** *Let  $\zeta \geq 0$  be a strictly monotone function on  $[0, \infty)$  such that  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$ . Let  $\theta$  be such that  $\int_0^t dz/\theta(z) < \infty$  for all  $x \leq t$ . If  $a(x)$  is increasing in  $x \geq 0$ , there exists a unique solution  $a_2$  of  $\nu = \int_0^\infty \zeta(x) dH_w(x; a_2)$  whenever  $t > \zeta^{-1}(\nu)$ . Furthermore,*

$$(4.11) \quad q(t) \geq \begin{cases} a_2/\theta(t), & t > \zeta^{-1}(\nu), \\ 0, & t \leq \zeta^{-1}(\nu), \end{cases}$$

and  $f(t) \geq 0$ .

The proofs of these two theorems depend upon the fact that if  $\bar{F}(x) \underset{\leq}{\geq} \bar{G}(x)$  for all  $x$ , and  $\zeta(x)$  is increasing in  $x \geq 0$ , then

$$(4.12) \quad \int_0^\infty \zeta(x) dF(x) \underset{\leq}{\geq} \int_0^\infty \zeta(x) dG(x).$$

If  $a(x)$  is increasing in  $x \geq 0$ , then

$$(4.13) \quad a(x) \leq \begin{cases} a(t), & x \leq t \\ \infty, & x > t \end{cases} \quad \text{and} \quad a(x) \geq \begin{cases} 0, & x < t \\ a(t), & x \geq t \end{cases}$$

so that

$$(4.14) \quad q(x) \leq \begin{cases} a(t)/\theta(x), & x \leq t \\ \infty, & x > t \end{cases} \quad \text{and} \quad q(x) \geq \begin{cases} 0, & x < t \\ a(t)/\theta(x), & x \geq t \end{cases}$$

Hence,

$$(4.15) \quad Q(x) \equiv \int_0^x q(z) dz \leq \begin{cases} \int_0^x a(t) dz/\theta(z), & x \leq t, \\ \infty, & x > t, \end{cases}$$

and

$$(4.16) \quad Q(x) \geq \begin{cases} 0, & x < t, \\ a(t) \int_t^x dz/\theta(z), & x \geq t, \end{cases}$$

or

$$(4.17) \quad H_t(x; a(t)) \leq \bar{F}(x) \leq \bar{G}_t(x; a(t)).$$

**PROOF OF THEOREM 4.2.3.** Assume that  $\zeta(x)$  is increasing in  $x$ , so that by (4.12) and (4.17),

$$(4.18) \quad \nu = \int_0^\infty \zeta(x) dF(x) \leq \int_0^\infty \zeta(x) dG_t(x; a(t)) \equiv \phi_1(a(t)).$$

Clearly,  $\phi_1(a)$  is strictly decreasing and continuous in  $a$ ,  $\lim_{a \rightarrow 0} \phi_1(a) = \lim_{x \rightarrow \infty} \zeta(x) > \nu$ ,  $\lim_{a \rightarrow \infty} \phi_1(a) = \zeta(t)$ .

Thus, if  $\nu > \zeta(t)$ , there exists a unique solution  $a_1$  of  $\phi_1(a_1) = \nu$ ; furthermore,  $a_1 \geq a(t)$  yields theorem 4.2.3. The proof for decreasing  $\zeta$  is analogous. ||

PROOF OF THEOREM 4.2.4. Again assume  $\zeta(x)$  is increasing, in which case it follows from (4.12) and (4.17) that

$$(4.19) \quad \nu = \int_0^\infty \zeta(x) dF(x) \geq \int_0^\infty \zeta(x) dH_{a(t)}(x) \equiv \phi_2(a(t)).$$

Clearly,  $\phi_2(a)$  is strictly decreasing and continuous in  $a$ ,  $\lim_{a \rightarrow 0} \phi_2(a) = \zeta(t)$ ,  $\lim_{a \rightarrow \infty} \phi_2(a) = \zeta(0) < \nu$ . Thus, if  $\zeta(t) > \nu$ , there exists a unique solution  $a_2$  of  $\phi_2(a) = \nu$ ; furthermore,  $a_2 \leq a(t)$ , and this yields theorem 4.2.4. ||

It is true that the inequalities of theorems 4.2.3 and 4.2.4 are sharp, but we omit the proof.

4.3. *PF<sub>2</sub> densities.* Bounds on *PF<sub>2</sub>* densities can be obtained from theorem 3.3.1 using limiting arguments. However, we assume that  $\zeta(x) = x^r$  and obtain more explicit results by different methods.

THEOREM 4.3.1. *If  $f$  is  $PF_2$  on  $[0, \infty)$ ,  $f(x) = 0$  for  $x < 0$  and  $\int_0^\infty x^r f(x) dx = \mu_r$  ( $r \geq 1$ ), then*

$$(4.20) \quad f(t) \leq \begin{cases} a_1, & t < \mu_r^{1/r}, \\ \infty, & t = \mu_r^{1/r}, \\ be^{-bt}/[1 - e^{-bt}], & t > \mu_r^{1/r}; \end{cases}$$

$$(4.21) \quad f(t) \geq \begin{cases} 0, & t < \mu_r^{1/r} \text{ or } t > \mu_r^{1/r}, \\ [\Gamma(r + 1)/\mu_r]^{1/r} e^{-[\Gamma(r+1)]^{1/r} t}, & t = \mu_r^{1/r}, \end{cases}$$

where  $a_1$  is the unique solution to

$$(4.22) \quad \int_0^\infty x^r a_1 e^{-a_1(x-t)} dx = \mu_r,$$

and  $b$  is the unique solution to

$$(4.23) \quad \int_0^t x^r b e^{-bx} dx / (1 - e^{-bt}) = \mu_r.$$

Both inequalities are sharp.

From the bound on  $f(\mu_r^{1/r})$  we can obtain an explicit lower bound on  $\int_{\mu_r^{1/r}}^t f(x) dx$ , thus complementing the sharp but nonexplicit results of theorem 3.3.1. From (4.21),

$$(4.24) \quad \int_{\mu_r^{1/r}}^t f(x) dx \geq \int_{\mu_r^{1/r}}^t g(x) dx$$

for  $t - \mu_r^{1/r}$  sufficiently small, where

$$(4.25) \quad g(x) = [\Gamma(r + 1)/\mu_r]^{1/r} \exp \{-[\Gamma(r + 1)/\mu_r]^{1/r} x\}.$$

Since  $f$  crosses  $g$  from above and exactly once to the right of  $\mu_r^{1/r}$ , a strict reversal of (4.24) for some  $t$  would imply that

$$(4.26) \quad \int_{\mu_r^{1/r}}^\infty f(x) dx < \int_{\mu_r^{1/r}}^\infty g(x) dx,$$

which contradicts theorem 3.8 of Barlow and Marshall [2]. Hence (4.24) holds for all  $t \geq \mu_r^{1/r}$ .

PROOF OF (4.20). The inequality for  $t \leq \mu_r^{1/r}$  follows from theorem 4.2.3. For  $t > \mu_r^{1/r}$ , let

$$(4.27) \quad g_t(x) = \begin{cases} be^{-bx}/(1 - e^{-bt}), & 0 \leq x \leq t, \\ 0, & x > t, \end{cases}$$

and suppose that  $f \neq g_t$ . Since  $\log f(x)$  is concave and  $\log g_t(x)$  is linear in  $x \in [0, t]$ , there are at most two crossings of  $f$  by  $g_t$ . Since  $f$  and  $g_t$  are densities with  $r$ -th moment  $\mu_r$ , they cross at least twice. Hence  $f$  and  $g_t$  cross exactly twice in  $[0, t]$ ; moreover, the second crossing of  $f$  by  $g_t$  must be from below, and we conclude that  $f(t) \leq g_t(t)$  as asserted. Of course, equality in (4.20) for  $t > \mu_r^{1/r}$  is attained by  $g_t$ . ||

To prove (4.21), we need the following lemma and theorem.

LEMMA 4.3.2. *If  $\int \phi(x)f_1(x) dx = \int \phi(x)f_2(x) dx < \infty$ , and if the support of  $f_1$  is contained in the support of  $f_2$ , then*

$$(4.28) \quad \int \phi(x)f_1(x) \log [f_1(x)/f_2(x)] dx \geq 0.$$

PROOF. We have

$$(4.29) \quad \begin{aligned} \int \phi(x)f_1(x) \log [f_1(x)/f_2(x)] dx &= - \int \phi(x)f_1(x) \log [f_2(x)/f_1(x)] dx \\ &\geq \int \phi(x)f_1(x)[1 - f_2(x)/f_1(x)] dx \\ &= \int \phi(x)f_1(x) dx - \int \phi(x)f_2(x) dx \\ &= 0. \end{aligned}$$

The inequality follows directly from  $\log z \leq z - 1, z > 0$ . ||

REMARK. With  $\phi(x) \equiv 1$ , this is the well-known "information inequality."

THEOREM 4.3.3. *Let  $\phi$  be a nonnegative function and  $\lambda$  be a number such that*

$$(4.30a) \quad 0 < \int_0^\infty \phi(x)f(x) dx = \int_0^\infty \phi(x)\lambda e^{-\lambda x} dx < \infty.$$

*If  $f$  is PF<sub>2</sub> and  $f(x) = 0, x < 0$ , then  $f(a) > \lambda e^{-\lambda a}$  where*

$$(4.30b) \quad a = \left( \int x\phi(x)f(x) dx \right) / \left( \int \phi(x)f(x) dx \right).$$

REMARK. In general,  $\lambda$  satisfying  $\int_0^\infty \phi(x)f(x) dx = \int_0^\infty \phi(x)\lambda e^{-\lambda x} dx$  does not necessarily exist. However, if  $\phi$  is monotone, then such a  $\lambda$  always exists.

PROOF. Since  $f$  is log concave,  $\log f(x)$  lies below its tangent at  $a$ , that is,  $(x - a)f'(x)/f(x) + \log f(a) \geq \log f(x)$ . If  $\phi(x) \geq 0$ ,

$$(4.31) \quad \phi(x)(x - a)f'(x)/f(x) + \phi(x) \log f(a) \geq \phi(x) \log f(x),$$

and upon integrating, we obtain

$$\begin{aligned}
 (4.32) \quad \frac{f'(a)}{f(a)} \int_0^\infty \phi(x)(x - a)f(x) dx + \log f(a) \int_0^\infty \phi(x)f(x) dx \\
 \geq \int_0^\infty \phi(x)f(x) \log f(x) dx \\
 \geq \int_0^\infty \phi(x)f(x)[\log \lambda - \lambda x] dx \\
 = (\log \lambda - a\lambda) \int_0^\infty \phi(x)f(x) dx.
 \end{aligned}$$

The second inequality follows from lemma 4.3.2. By the definition of  $a$ , the first term on the left of this inequality is zero, and we have||

$$(4.33) \quad \log f(a) \int_0^\infty \phi(x)f(x) dx \geq (\log \lambda - a\lambda) \int_0^\infty \phi(x)f(x) dx.||$$

PROOF OF (4.21). If  $r = 1$ , the result follows from theorem 4.3.3 with  $\phi(x) \equiv 1$ . If  $r > 1$ , let  $\phi(x) = x^r + (\mu_{r+1} - \mu_r^{(r+1)/r})/(\mu_r^{1/r} - \mu_1)$ . Then since  $\mu_s^{1/s}$  is increasing in  $s > 0$ , it follows that  $\phi(x) > 0$ . By straightforward algebra,  $a = \mu_r^{1/r}$ . Thus  $\lambda = [\Gamma(r + 1)/\mu_r]^{1/r}$ , and (4.21) follows.||

The bound of (4.21) for  $r = 1$  was originally communicated to us by Samuel Karlin.

THEOREM 4.3.4. *If  $f$  is  $PF_2$ ,  $f(x) = 0, x < 0$ , and  $\zeta$  is a function continuous and strictly monotone on  $[0, \infty)$  such that  $\int_0^\infty \zeta(x)f(x) dx = \nu$  exists finitely, then*

$$(4.34) \quad q(t) \leq \begin{cases} a_1, & t < \zeta^{-1}(\nu), \\ \infty, & t \geq \zeta^{-1}(\nu), \end{cases}$$

$$(4.35) \quad q(t) \geq \begin{cases} 0, & t < \zeta^{-1}(\nu), \\ \inf_{m \geq t} g_m(t) / \int_t^m g_m(x) dx, & t \geq \zeta^{-1}(\nu), \end{cases}$$

where  $g_m(x)$  is defined in (4.27) with  $b$  uniquely determined by  $\int_0^m \zeta(x)g_m(x) dx = \nu$ , and  $a_1$  is determined by  $\int_t^\infty \zeta(x)ae^{-a(x-t)} dx = \nu$ .

PROOF. The upper bound follows from theorem 4.2.3. To show the lower bound, let  $x^*(m)$  be the unique point where  $g_m$  crosses  $f$  from below, and suppose first that  $t < x^*(\infty)$ . Then there exists  $m_0 > t$  such that  $f(t) = g_{m_0}(t)$  (the proof of this in case  $\zeta$  is increasing is given by Barlow and Marshall [3] in the proof of theorem 5.1; the modifications necessary in case  $\zeta$  is decreasing are obvious and not extensive). But  $f(t) = g_{m_0}(t)$  together with  $1 - F(t) \leq \int_t^{m_0} g_{m_0}(x) dx$  (again, see [3], proof of theorem 5.1) yields the desired result.

It remains to consider the case that  $t \geq x^*(\infty) \equiv x^*$ . Then by an argument identical with the case  $t < x^*$ , we obtain

$$(4.36) \quad q(x^*) \geq g_\infty(x^*) / \int_{x^*}^\infty g_\infty(x) dx,$$

which together with  $q$  increasing yields the lower bound in this case.||

4.4. *DHR distributions.* If  $F$  is DHR, then  $F$  is absolutely continuous except possibly for mass at the origin (Marshall and Proschan [11]). The following

bounds apply to any version  $f$  of the density satisfying  $f_-(t) \geq f(t) \geq f_+(t)$ , in which case  $q(x) = f(x)/\bar{F}(x)$  is decreasing.

**THEOREM 4.4.1.** *If  $F$  is DHR and  $\zeta \geq 0$  is a monotone function on  $[0, \infty)$  such that  $\int_0^\infty \zeta(x) dF(x) = \nu < \infty$ , then*

$$(4.37) \quad f(t) \leq \max \left[ \sup_{0 < \alpha \leq 1} \alpha e^{-\alpha t}, \sup_{b \geq a^*} b e^{-bt} \right],$$

where for each  $\alpha$ ,  $a \equiv a(\alpha)$  satisfies

$$(4.38) \quad \alpha \int_0^\infty \zeta(x) e^{-\alpha x} dx + (1 - \alpha)\zeta(0) = \nu,$$

and  $a^* = a(1)$  is determined by  $a^* \int_0^\infty \zeta(x) e^{-a^* x} dx = \nu$ .

**PROOF.** We have  $\sup_{\mathcal{G}_1} g(t) = \sup_{b \geq a^*} b e^{-bt}$  and  $\sup_{\mathcal{G}_2} g(t) = \sup_{0 < \alpha \leq 1} \alpha e^{-\alpha t}$ , where  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are defined in section 3.4. The result thus follows from the remarks at the beginning of section 4. ||

**COROLLARY 4.4.2.** *If  $F$  is DHR and  $\int_0^\infty x^r dF(x) = \mu_r < \infty$ , then*

$$(4.39) \quad f(t) \leq \begin{cases} (te)^{-1}, & t \leq \lambda_r^{1/r}, \\ \lambda_r^{-1/r} e^{-t/\lambda_r^{1/r}}, & \lambda_r^{1/r} \leq t \leq (r+1)\lambda_r^{1/r}, \\ \lambda_r \left( \frac{r+1}{t} \right)^{r+1} e^{-(r+1)}, & t \geq (r+1)\lambda_r^{1/r}, \end{cases}$$

where  $\lambda_r = \mu_r/\Gamma(r+1)$ .

This result can be obtained from theorem 4.4.1 or from theorem 3.4.1.

**THEOREM 4.4.3.** *If  $F$  is DHR,  $\mu_r = \int_0^\infty x^r dF(x)$ , then*

$$(4.40) \quad f(0) = q(0) \geq \lambda_r^{-1/r}.$$

**PROOF.** Since  $Q(x) = -\log(1 - F(x))$  is concave,  $Q(x)/x$  is decreasing in  $x$ , and  $q(0) = \lim_{x \rightarrow 0} Q(x)/x \geq Q(\mu_r^{1/r})/\mu_r^{1/r}$ . But  $1 - F(\mu_r^{1/r}) \leq e^{-[\Gamma(r+1)]^{1/r}}$  (Barlow and Marshall [2]), and the result follows. ||

Upper bounds for  $q$  similar to the results of theorems 4.2.3 and 4.2.4 have been obtained by Barlow and Marshall [4] for cases that  $\zeta(x)$  is decreasing and  $\zeta(x)$  is increasing but bounded,  $\int_0^\infty dz/\theta(z) < \infty$  for all  $x > 0$ , and  $a(x)$  is decreasing. The impossibility of nontrivial lower bounds at  $t > 0$  is also demonstrated.

REFERENCES

[1] R. E. BARLOW, "Bounds on integrals with applications to reliability problems," *Ann. Math. Statist.*, Vol. 36 (1965), pp. 565-574.  
 [2] R. E. BARLOW and A. W. MARSHALL, "Bounds for distributions with monotone hazard rate, I," *Ann. Math. Statist.*, Vol. 35 (1964), pp. 1234-1257.  
 [3] ———, "Bounds for distributions with monotone hazard rate, II," *Ann. Math. Statist.*, Vol. 35 (1964), pp. 1258-1274.  
 [4] ———, "Bounds on densities and hazard rates," Boeing Scientific Research Laboratories Document D1-82-0338; Operation Research Center report ORC 64-9(RR), University of California, Berkeley.  
 [5] C. Berge, *Topological Spaces*, New York, Macmillan, 1963.

- [6] Z. W. BIRNBAUM, J. D. ESARY, and A. W. MARSHALL, "A stochastic characterization of wear-out for components and systems," *Ann. Math. Statist.*, Vol. 37 (1966), pp. 816–825.
- [7] M. Fréchet, *Généralités sur les Probabilités. Éléments Aléatoires*, Borel Series, Traité du calcul des probabilités et de ses applications, Div. I, Part III, Vol. 1, Paris, Gauthier-Villars, 1950 (2d ed.).
- [8] W. Hoeffding, "The extrema of the expected value of a function of independent random variables," *Ann. Math. Statist.*, Vol. 26 (1955), pp. 268–275.
- [9] C. L. MALLOWS, "A generalization of the Chebyshev inequalities," *Proc. London Math. Soc.* 3, Vol. 13 (1963), pp. 385–412.
- [10] ———, "Generalizations of Tchebycheff's inequalities," *J. Roy. Statist. Soc. Ser. B*, Vol. 28 (1956), pp. 139–176.
- [11] A. W. MARSHALL and F. PROSCHAN, "Maximum likelihood estimation for distributions with monotone failure rate," *Ann. Math. Statist.*, Vol. 36 (1965), pp. 69–77.
- [12] J. S. RUSTAGI, "On minimizing and maximizing a certain integral with statistical applications," *Ann. Math. Statist.*, Vol. 28 (1957), pp. 309–328.
- [13] H. L. SELBERG, "Zwei ungleichungen zur ergänzung des Tchebycheffschen lemmas," *Skand. Aktuarietidskr.*, Vol. 23 (1940), pp. 121–125.