RANDOM MEASURE PRESERVING TRANSFORMATIONS

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1. Introduction

It is the purpose of this note to show that it is impossible to define a probability measure on the group $\mathcal{G}$ of invertible measure-preserving transformations from the unit interval onto itself, if it is demanded that the measure on $\mathcal{G}$ obey two fairly "natural" conditions. One of these is an invariance condition on the measure, and the other asserts that certain distinguished subsets of $\mathcal{G}$ are measurable.

One reason for trying to construct such a probability measure is the following: the group $\mathcal{G}$ has been topologized in at least two different ways (see Halmos [3]); in one of those topologies (the "weak" topology) it has been proved that the set $\mathcal{E}$ of ergodic transformations (and in fact, the set $\mathcal{W}$ of weakly mixing transformations) is of the second category, and the set $\mathcal{S}$ of strongly mixing transformations is of the first category (see [3], p. 77 ff.). Corresponding to this information about the "topological size" of $\mathcal{E}$, $\mathcal{W}$, and $\mathcal{S}$, it would have been natural to seek information about the measures of these (and possibly other) subsets of $\mathcal{G}$. One could have hoped, for example, that "almost every transformation is ergodic." However, one needs first to have an appropriate measure on $\mathcal{G}$.

Another motivation comes from game theory. One of the characterizations of the Shapley value [4] of a cooperative n-person game involves a random ordering of the players. Recently games in which the player set may be a (possibly atomless) measure space have attracted attention, in part because of their applications to economics and politics. (For a comprehensive list of references, see Debreu [2].) One approach to defining the Shapley value for such games would involve the notion of a "random ordering" of the measure space of players. Replacing "ordering" with "measure-preserving transformation," leads to the question that we have answered (negatively) in this note.

The theorem of this paper provides additional evidence of the comparative intractability of function spaces when viewed from the measure-theoretic rather than from the topological viewpoint (compare with [1]).

A precise statement of the theorem is given in section 2, and it is proved in

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section 3. The "naturalness" of the invariance condition is discussed in the last section.

2. Statement of the theorem

Let $\mathbf{I}$ be the measure algebra of Lebesgue measurable subsets of the unit interval I, modulo the sets of measure 0 (that is, the algebra in which sets differing by sets of measure 0 are not distinguished). Let $\mathcal{G}$ be the group of Lebesgue measure preserving automorphisms of $\mathbf{I}$; the members of this group may be thought of as invertible measure-preserving transformations from I onto itself, where two transformations are identified if they differ on a set of measure 0 only. We will treat members of $\mathbf{I}$ as if they were subsets of I, speaking of unions, intersections, inclusions, and so on. Lebesgue measure will be denoted by λ throughout. No confusion will result.

THEOREM. *There is no pair* (Γ, μ), *where* Γ *is a* σ*-field of subsets of* $\mathcal{G}$, *and* μ *is a probability measure on* Γ, *for which*

(2.1) *for each* $\mathcal{H} \in \Gamma$ *and* $T \in \mathcal{G}$, *we have* $\mathcal{H}T \in \Gamma$, *and* $\mu(\mathcal{H}T) = \mu(\mathcal{H})$;

(2.2) *for all* E *and* F *in* $\mathbf{I}$, *the function from* $\mathcal{G}$ *to the reals defined by* $f(T) = \lambda(E \cap TF)$ *is* Γ*-measurable.*

A few words of explanation are in order. "Probability measure," of course, means that $\mu(\mathcal{G}) = 1$. Condition (2.1) is the right-invariance condition; it says that if a Γ-measurable set of transformations is multiplied on the right by a single transformation, then it remains Γ-measurable, and its μ-measure (probability) remains unchanged. Without some such condition it would be trivially possible to construct a probability measure on $\mathcal{G}$, for example by concentrating all the probability on one transformation T. Condition (2.2) is a measurability assumption which seems very reasonable.

The theorem remains true if $\mathcal{H}T$ is replaced by $T\mathcal{H}$ in condition (2.1), that is, if right invariance is replaced by left invariance. Condition (2.2) remains unchanged.

3. Proof of the theorem

It will be assumed throughout that there is given a pair (Γ, μ) obeying the specifications of the theorem, and this will lead eventually to a contradiction.

Often it will be convenient to use the language of probability, that is, to replace μ by "Prob," $\int_{\mathcal{G}} \mu(dT)$ by "Exp" (for "Expectation"), and so on. "Variance" will be abbreviated by "Var," and "Covariance" by "Cov"; like "Exp," these two operators will be applied exclusively to random variables defined on the probability space $(\mathcal{G}, \Gamma, \mu)$.

LEMMA 1. *Let* $D, F_1, F_2 \in \mathbf{I}$, *and* $\lambda(F_1) = \lambda(F_2)$. *Then*

$$\operatorname{Exp} \lambda(D \cap TF_1) = \operatorname{Exp} \lambda(D \cap TF_2). \tag{1}$$

PROOF. Let S be a member of $\mathcal{G}$ such that $SF_1 = F_2$. Define measures η_1 and η_2 on the closed unit interval $[0, 1]$ by

$$\eta_i[0, \alpha] = \mu\{T: \lambda(D \cap TF_i) \leq \alpha\} \tag{2}$$

for $i = 1, 2$. Then η_i is the distribution of the random variable $\lambda(D \cap TF_i)$, and

$$\begin{aligned} \eta_2[0, \alpha] &= \mu\{T: \lambda(D \cap TSF_1) \leq \alpha\} \\ &= \mu\{TS: \lambda(D \cap TSF_1) \leq \alpha\} \\ &= \mu\{U: \lambda(D \cap UF_1) \leq \alpha\} \\ &= \eta_1[0, \alpha], \end{aligned} \tag{3}$$

where the second equality follows from (2.1) and the third by setting $U = TS$ and noting that as T runs over $\mathcal{G}$, so does U. From this it follows that

$$\operatorname{Exp} \lambda(D \cap TF_2) = \int_0^1 \alpha\eta_2(d\alpha) = \int_0^1 \alpha\eta_1(d\alpha) = \operatorname{Exp} \lambda(D \cap TF_1), \tag{4}$$

which is the assertion of the lemma.

LEMMA 2. *For all $D, F, \in \mathbf{I}$,*

$$\operatorname{Exp} \lambda(D \cap TF) = \lambda(D)\lambda(F). \tag{5}$$

PROOF. For an arbitrary but fixed positive integer m, let $F_1, \cdots, F_m$ be disjoint members of $\mathbf{I}$ with equal measure, whose union is I. Then $\lambda(F_i) = 1/m$ for all i. From lemma 1 it follows that $\operatorname{Exp} \lambda(D \cap TF_i)$ does not depend on i; let us denote it by γ. Now

$$\begin{aligned} \lambda(D) &= \operatorname{Exp} \lambda(D) = \operatorname{Exp} \lambda(D \cap TI) \\ &= \operatorname{Exp} \lambda\left(D \cap T \bigcup_{i=1}^{m} F_i\right) = \operatorname{Exp} \sum_{i=1}^{m} \lambda(D \cap TF_i) \\ &= \sum_{i=1}^{m} \operatorname{Exp} \lambda(D \cap TF_i) = m\gamma. \end{aligned} \tag{6}$$

Hence, $\gamma = \lambda(D)(1/m) = \lambda(D)\lambda(F_i)$ for $i = 1, \cdots, m$.

Now whenever $\lambda(F) = 1/m$ for some m, it is possible to set $F_1 = F$ and to find $m - 1$ sets $F_2, \cdots, F_m$ satisfying the above conditions; hence, whenever $\lambda(F)$ is the reciprocal of an integer, the assertion of the lemma is established. But each measurable set $F \in \mathbf{I}$ is a countable union of sets whose measures are reciprocals of integers; and since Exp is countably additive for nonnegative random variables, the assertion of the lemma follows in the general case as well.

Before stating the next lemma, we introduce the following notation: for $D, E, F \in \mathbf{I}$ and $E \cap F = \varnothing$, we write

$$g(E, F) = g_D(E, F) = \operatorname{Exp} [\lambda(D \cap TE)\lambda(D \cap TF)]. \tag{7}$$

LEMMA 3. *Let $D, E, F_1, F_2 \in \mathbf{I}$, and $F_1 \cap E = F_2 \cap E = \varnothing$, $\lambda(F_1) = \lambda(F_2)$. Then $g_D(E, F_1) = g_D(E, F_2)$.*

PROOF. The proof is similar to that of lemma 1. This time, let S be a member of $\mathcal{G}$ such that both $SF_1 = F_2$ and $SE = E$. Define measures η_1 and η_2 on $[0, 1]$ by $\eta_i[0, \alpha] = \mu\{T: \lambda(D \cap TE)\lambda(D \cap TF_i) \leq \alpha\}$; then because of (2.1), $\eta_2[0, \alpha] = \eta_1[0, \alpha]$ for all α, and hence

$$\int_0^1 \alpha\eta_2(d\alpha) = \int_0^1 \alpha\eta_1(d\alpha); \tag{8}$$

but that is precisely what the lemma asserts.

LEMMA 4. *Let D, E, $F \in$ **I**, and $E \cap F = \varnothing$. Then*

$$g_D(E, F) \leq \lambda^2(D)\lambda(E)\lambda(F). \tag{9}$$

PROOF. If $\lambda(E)$ or $\lambda(F)$ vanish, there is nothing to prove; assume, therefore, that $\lambda(E) > 0$, $\lambda(F) > 0$, and so $\lambda(E) < 1$, $\lambda(F) < 1$. For an arbitrary but fixed positive integer m, let $F_1, \cdots, F_m$ be disjoint members of **I** with equal measure, whose union is $I - E$. Then $\lambda(F_i) = (1 - \lambda(E))/m$ for all i. From lemma 3 it follows that $g(E, F_i)$ does not depend on i; denote it by γ. Now

$$g(E, I\backslash E) = g\left(E, \bigcup_{i=1}^{m} F_i\right) = \sum_{i=1}^{m} g(E, F_i) = m\gamma. \tag{10}$$

Hence,

$$g(E, F_i) = \gamma = g(E, I\backslash E)/m = \lambda(F_i)g(E, I\backslash E)/(1 - \lambda(E)). \tag{11}$$

Whenever $F \subset I\backslash E$ and $\lambda(F) = (1 - \lambda(E))/m$ for some m, it is possible to set $F_1 = F$ and to find $m - 1$ sets $F_2, \cdots, F_m$ satisfying the above conditions; hence, for such F, we have

$$g(E, F) = \lambda(F) \frac{g(E, I\backslash E)}{(1 - \lambda(E))}. \tag{12}$$

But each set $F \subset I\backslash E$ is a countable union of such F; and so (12) follows for all F with $E \cap F = \varnothing$. Now

$$\begin{aligned} g(E, I\backslash E) &= \operatorname{Exp}\left[\lambda(D \cap TE)(\lambda(D) - \lambda(D \cap TE))\right] \\ &\leq \operatorname{Exp}\left[\max\nolimits_{0 \leq \beta \leq \lambda(D)} \beta(\lambda(D) - \beta)\right] \\ &= \operatorname{Exp}\left[\lambda^2(D)/4\right] = \lambda^2(D)/4. \end{aligned} \tag{13}$$

Then choosing E_0 so that $\lambda(E_0) = \frac{1}{2}$, and applying (12), we find

$$g(E_0, F) \leq \lambda(F) \frac{\lambda^2(D)/4}{\frac{1}{2}} = \lambda(F)\lambda(E_0)\lambda^2(D) \tag{14}$$

whenever $E_0 \cap F = \varnothing$. Now by using (12) and the symmetry of g in its two variables, we obtain

$$g(E, F) = \lambda(E) \frac{g(F, I\backslash F)}{(1 - \lambda(F))} \tag{15}$$

whenever $E \cap F = \varnothing$. Setting $E = E_0$ in (15) and combining with (14), we deduce

$$g(F, I\backslash F)/(1 - \lambda(F)) \leq \lambda(F)\lambda^2(D) \tag{16}$$

whenever $E_0 \cap F = \varnothing$. Combining this with (15), we obtain

$$g(E, F) \leq \lambda(E)\lambda(F)\lambda^2(D) \tag{17}$$

whenever $E_0 \cap F = \varnothing$ and $E \cap F = \varnothing$. Now whenever $\lambda(F) \leq \frac{1}{2}$, it is possible to choose E_0 so that $E_0 \cap F = \varnothing$; since $E \cap F = \varnothing$ by the hypothesis of the lemma, the lemma is proved in those cases. When $\lambda(F) > \frac{1}{2}$, we may express F as the union of two disjoint subsets each of measure $\leq \frac{1}{2}$: the lemma then follows from the additivity in F both of $g(E, F)$ and of $\lambda(F)$.

LEMMA 5. *For all* $D, F \in \mathbf{I}$,

$$\operatorname{Var} \lambda(D \cap TF) = 0. \tag{18}$$

PROOF. If $D = I$ there is nothing to prove. Therefore, assume that $\lambda(D) < 1$. Let $F_1, \cdots, F_n$ be disjoint members of $\mathbf{I}$, with equal measures, such that $\bigcup_{i=1}^n F_i = F$; then $\lambda(F_i) = \lambda(F)/n$ for all i. Assume $n > 1$. Define random variables $\mathbf{x}_1, \cdots, \mathbf{x}_n$ by $\mathbf{x}_i = \lambda(D \cap TF_i)$. Then

$$\operatorname{Var} \lambda(D \cap TF) = \sum_{i=1}^{n} \operatorname{Var} \mathbf{x}_i + 2 \sum_{i>j} \operatorname{Cov}(\mathbf{x}_i, \mathbf{x}_j). \tag{19}$$

Now by lemmas 2 and 4,

$$\begin{aligned} \operatorname{Cov}(\mathbf{x}_i, \mathbf{x}_j) &= g(F_i, F_j) - (\operatorname{Exp} \lambda(D \cap TF_i))(\operatorname{Exp} \lambda(D \cap TF_j)) \\ &\leq \lambda(F_i)\lambda(F_j)\lambda^2(D) - (\lambda(D)\lambda(F_i))(\lambda(D)\lambda(F_j)) \\ &= 0. \end{aligned} \tag{20}$$

On the other hand, $\mathbf{x}_j$ is clearly bounded by $\lambda(F_i) = \lambda(F)/n \leq 1/n$, so $\operatorname{Var} \mathbf{x}_i \leq 1/n^2$. Hence

$$\operatorname{Var} \lambda(D \cap TF) \leq n/n^2 = 1/n. \tag{21}$$

Letting $n \to \infty$, we deduce the conclusion of the lemma.

Suppose now that $F = [0, \frac{1}{2}]$. Then it follows from lemmas 2 and 5 that with probability one, TF intersects every rational interval D in a set of measure $\frac{1}{2}\lambda(D)$. But then with probability one, TF is a set of density $\frac{1}{2}$ at each point; whereas, it is known that there are no such Lebesgue measurable sets. This is the contradiction that establishes our theorem.

The corresponding theorem when right invariance is replaced by left invariance can be proved in a similar manner. Alternatively, if (Δ, ν) is a pair satisfying (2.2) and the left-invariant analogue of (2.1), define

$$\Gamma = \{\mathcal{H} \subset \mathcal{G}\colon \mathcal{H}^{-1} \in \Delta\}, \tag{22}$$

where

$$\mathcal{H}^{-1} = \{T \in \mathcal{G}\colon T^{-1} \in \mathcal{H}\}; \tag{23}$$

and define μ on Γ by $\mu(\mathcal{H}) = \nu(\mathcal{H}^{-1})$. Then it may be verified that (μ, Γ) satisfies (2.1) and (2.2), and so contradicts the main theorem; this establishes the left-invariant version.

4. Discussion of the invariance condition

The invariance condition (2.1) seems rather strong. One may ask whether weaker conditions might not be devised, under which it *would* be possible to define a probability measure on $\mathcal{G}$, while still retaining the intuitive notion that the measure is distributed "uniformly" over $\mathcal{G}$. Certainly, the current result does not entirely exclude such a possibility, and we will not pretend that the last word on the subject has been said.

However, it should be pointed out that we have proved more than appears at first sight. The invariance condition (2.1) is used only twice in the proof, namely in the proofs of lemmas 1 and 3. Thus one could substitute these lemmas for condition (2.1) and still obtain the same result. Although the statements of these lemmas are more involved and less concise than (2.1), their direct intuitive appeal is perhaps greater than that of (2.1): both lemmas assert that the measure on $\mathcal{G}$ does not "discriminate" between sets F_1 and F_2 in $\mathbf{I}$ of equal Lebesgue measure. It is really only these "non-discrimination" conditions, in addition to condition (2.2), that are needed to prove the nonexistence of a measure on $\mathcal{G}$.

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Note added in proof. Following is an extremely short proof of the theorem of this paper, for which I am indebted to Professor Harry Furstenberg. Let S in $\mathcal{G}$ be strongly mixing. Fix D and F in $\mathbf{I}$, and define random variables $\mathbf{y}_n = \mathbf{y}_n(T)$ by $\mathbf{y}_n = \lambda(D \cap TS^nF)$. Then for each T, $\mathbf{y}_n(T) = \lambda(T^{-1}D \cap S^nF) \rightarrow \lambda(D)\lambda(F)$ as $n \rightarrow \infty$. Because of the invariance condition (2.1), all the $\mathbf{y}_n$ have the same distribution; since they tend to the constant $\lambda(D)\lambda(F)$ pointwise, it follows that $\mathbf{y}_n = \lambda(D)\lambda(F)$ for each n with probability 1. By setting $n = 0$ we complete the proof.

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