ON SOME QUESTIONS CONNECTED WITH TWO-SAMPLE TESTS OF SMIRNOV TYPE

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1. Introduction

1.1. In the following we shall consider some questions concerning the comparison of two samples. The test around which our investigations will center is the Kolmogorov-Smirnov two-sample test, restricted always to the case of equal sample sizes.

In the first part we shall treat the power function for certain alternatives and make some remarks on the efficiency of the test considered in the case of small samples. In the second part some remarks will be given on distributions and limiting distributions occurring in connection with the treated problems. The investigations given here are closely connected with the author's work presented at the Fourth Berkeley Symposium.

1.2. For diminishing the difference in efficiency between parametric and nonparametric tests, the author has in his papers [9], [11] proposed the use of a pair of statistics instead of one statistic. In consequence of the Neyman-Pearson lemma, this results, for given alternatives, in a better test than the one based on either single test statistic. We apply the maximum deviation of the two empirical distribution functions as the first statistic, which ensures the asymptotic consistency of the test. Then we can add to this for several types of alternatives a suitable corresponding pair, for example, the first maximum index, the number of intersections, the Galton statistic, and so on. In order to examine the increase in the efficiency of the two-sample Smirnov test, we shall treat the situation in the case of a special alternative, for which the computation is relatively easy.

In our treatment we make use of the power functions of the original test and of the two-statistic test as well. The power function can be constructed easily in case of a (continuous) alternative containing piecewise linear parts. With such alternatives we can approximate any given alternative. Following Z. W. Birnbaum [1], these kinds of alternatives (for instance, the maximum and minimum alternatives) were treated by many authors in the one-sample case. As we shall see in section 1, this power function can be easily obtained in the two-sample case for all tests for which the distribution of the test statistic under null hypothesis is known; the idea used is the extension of the method used by

Z. W. Birnbaum in his mentioned paper. Nevertheless, this is simple enough for further considerations only when the number of linear sections of the distribution function is small. Further, we need the power function for the more complicated case of two-statistics tests, so we shall work in this first occasion with a very simple alternative knowing that our aim is to discover what we can expect at all from the use of pairs of statistics. We shall consider the power for moderate or small sample sizes which occur very often in the applications. Our formulae give the possibility of carrying out the program of D. Chapman [2] which was done for the one-sample case.

2. The power of two-sample tests for piecewise linear alternatives

2.1. *Notations.* Let ξ and η be random variables with continuous distribution functions $F(x)$ and $G(x)$ resp. for which the null hypothesis H_0: $G(x)$ and $F(x)$ are equally and uniformly distributed in interval (0, 1). We shall treat piecewise linear alternatives depending on vectors:

$$z = (z_0, z_1, \cdots, z_r), \qquad (0 = z_0 < z_1 < z_2 < \cdots < z_r = 1), \tag{2.1}$$

$$g = (g_1, g_2, \cdots, g_r), \tag{2.2}$$

$$(g_i \geq 0, i = 1, 2, \cdots, r) \quad \text{with} \quad \sum_{i=1}^{r} g_i(z_i - z_{i-j}) = 1:$$

$$H_1^{(r)}: \begin{cases} F(x) \text{ as in } H_0, \\ G'(x) = g(x) = \begin{cases} g_i & \text{if } z_{i-1} < x < z_i, \ i = 1, 2, \cdots, r, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{2.3}$$

In this case $\int_{-\infty}^{\infty} g(x)\,dx = \int_0^1 g(x)\,dx = 1$.

Let further $\xi_1, \xi_2, \cdots, \xi_n$ and $\eta_1, \eta_2, \cdots, \eta_n$ be independent observations on ξ and η resp. We denote the elements of the ordered samples by ξ_i^* and η_i^* resp., and the union of the two samples in order of magnitude by

$$\tau_1^* < \tau_2^* < \cdots < \tau_{2n}^*. \tag{2.4}$$

Let us define further the random variables for $i = 1, 2, \cdots, 2n$,

$$\vartheta_i = \begin{cases} +1, & \text{if } \tau_i^* = \xi_j \\ -1, & \text{if } \tau_i^* = \eta_\ell' \end{cases} \tag{2.5}$$

for some j and ℓ. With the usual notation $s_0 = 0$, $s_i = \vartheta_1 + \vartheta_2 + \cdots + \vartheta_i$, $(i = 1, 2, \cdots, 2n)$, $s_{2n} = 0$. The points (i, s_i) in the plane give the path of a random walk starting at the origin and returning after $2n$ steps to the point $(2n, 0)$.

In the present paper we shall consider the following statistics:

$$D_{n,n}^+ = \max_{(x)} (F_n(x) - G_n(x)) = \frac{1}{n} \max_{(i)} s_i, \tag{2.6}$$

$$D_{n,n} = \max_{(x)} |F_n(x) - G_n(x)| = \frac{1}{n} \max_{(i)} |s_i|, \tag{2.7}$$

(2.8) $R^+_{n,n} = \min\left\{\frac{i}{n}: \quad s_i = nD^+_{n,n}\right\}$

(2.9) $\Lambda_{n,n}$ = *the number of intersection points in the above mentioned random path, that is, the number of* i*'s for which* $s_i = 0$ *and* $s_{i-1}s_{i+1} = -1$ *occurs, adding the point* $(2n, 0)$.

As to the numerical determination of the values of these statistics we mention the following. As it was pointed out in [3], $nD^+_{n,n}$ agrees with that index, for which in the translation scheme

$$\begin{aligned} &\xi_1^*, \cdots, \xi_\kappa^*, \xi_{\kappa+1}^*, \cdots, \xi_{\kappa+s}^*, \cdots, \xi_n^* \\ &\qquad\qquad \eta_1^*, \cdots, \eta_s^*, \cdots, \eta_{n-\kappa}^*, \cdots, \eta_n^* \end{aligned} \tag{2.10}$$

it *first* occurs that each $\xi^*_{\kappa+i}$ exceeds the corresponding η^*_i. Translation in the opposite direction leads to $D^-_{n,n}$ and in this way to $D_{n,n}$. Further $nR^+_{n,n} = \kappa + 2s$ where $s = \min\{i: \xi^*_{\kappa+i} < \eta^*_{i+1}\}$, which can be seen easily.

Having the two ordered samples, let us define the random variables

$$\epsilon_i = \begin{cases} +1, & \text{if} \quad \xi_i^* > \eta_i^*, \\ -1, & \text{if} \quad \xi_i^* < \eta_i^*. \end{cases} \tag{2.11}$$

Then $\Lambda_{n,n} - 1$ equals the number of changes of sign in the sequence $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ (see [12]).

2.2. *The power function of the two-sample Smirnov test under the hypothesis* H_0 *against* $H_1^{(r)}$. As is known the α-size critical region for one-sided alternatives $F(x) = G(x)$—this case will be treated in more detail later in this article—is determined by the relation

$$P\left(D^+_{n,n} \geq \frac{k}{n} | H_0\right) = \frac{\binom{2n}{n-k}}{\binom{2n}{n}} = \alpha \tag{2.12}$$

with $k = k(\alpha, n)$.

Let us denote by the vectors $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_r)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_r)$ the events that out of the n ξ_i's exactly ν_j and (independently) out of the n η_i's exactly μ_j are contained in the intervals (z_{j-1}, z_j) $j = 1, 2, \cdots, r$. The probabilities of these events are clearly

$$\begin{aligned} P(\boldsymbol{\nu}|H_0) &= P(\boldsymbol{\nu}|H_1) = n! \prod_{j=1}^r \frac{(z_j - z_{j-1})^{\nu_j}}{\nu_j!}, \\ P(\boldsymbol{\mu}|H_0) &= n! \prod_{j=1}^r \frac{(z_j - z_{j-1})^{\mu_j}}{\mu_j!}, \end{aligned} \tag{2.13}$$

$$P(\boldsymbol{\mu}|H_1^{(r)}) = n! \prod_{j=1}^r \frac{[(z_j - z_{j-1})g_j]^{\mu_j}}{\mu_j!}; \tag{2.14}$$

further, for $i = 0, 1$,

$$P(\boldsymbol{\nu}, \boldsymbol{\mu}|H_i) = P(\boldsymbol{\nu}|H_i)P(\boldsymbol{\mu}|H_i). \tag{2.15}$$

Denoting now for $j = 1, 2, \cdots, r$ by n_j and m_j the partial sums $\nu_1 + \nu_2 +$

$\cdots + \nu_j$ and $\mu_1 + \mu_2 + \cdots + \mu_j$ resp., $(n_0 = m_0 = 0)$, then the following holds.

THEOREM 2.1. *For the α-size one-sided two-sample Smirnov test the power function under $H_1^{(r)}$ against H_0 is given by*

$$(2.16)\qquad W_n(H_1^{(r)}, \alpha) = 1 - (n!)^2 \sum_{(\nu)}^* \sum_{(\mu)}^* \prod_{j=1}^{r} \times \left(1 - \frac{\binom{\nu_j + \mu_j}{\mu_j + k - n_{j-1} + m_{j-1}}}{\binom{\nu_j + \mu_j}{\mu_j}}\right) \frac{(z_j - z_{j-1})^{\nu_j + \mu_j} g_j^{\mu_j}}{\nu_j! \mu_j!}$$

where $\sum_{(\nu)}^*$ and $\sum_{(\mu)}^*$ denote r-summation for all possible ν and μ vectors with $0 \le \nu_j, \mu_j \le n$ and $\sum_{j=1}^r \nu_j = \sum_{j=1}^r \mu_j = n$ with the further restriction $n_j - m_j < k$, $j = 1, 2, \cdots, r$.

Before outlining the proof we make some remarks.

(a) The expression of the power function seems to be suitable for asymptotic considerations or computational work in general for small values of r. For r large and for small intervals (z_{i-1}, z_i) we are interested in the mutual order of sample elements inside the intervals only for very peculiar alternatives. In these cases the problem can be reduced into the consideration of the null hypothesis

$$(2.17)\qquad H_0^1:\ P(\xi \in (z_{i-1}, z_i)) = P(\eta \in (z_{i-1}, z_i)) \qquad \text{for} \quad i = 1, 2, \cdots, r.$$

This means that our problem is the comparison of two multinomial distributions, which is treated recently by Hoeffding [6].

(b) If n remains finite and r tends to infinity, then we come essentially to the evaluation of the probabilities of each different array of the sample elements, arrays which belong to the critical region. This kind of expression was given by Hoeffding [5] and considered by Lehmann [7].

(c) In the two-sided case the combinatorial quantity just after the product sign is to be replaced by the expression corresponding to the two-barrier case

$$(2.18)\qquad \frac{1}{\binom{\nu_j + \mu_j}{\mu_j}} \sum_{\gamma=-\infty}^{\infty} \left[\binom{\nu_j + \mu_j}{\nu_j + 2\gamma k} - \binom{\nu_j + \mu_j}{\nu_j + n_{j-1} - m_{j-1} + (2\gamma + 1)k}\right],$$

where for the summation the restriction $|n_j - m_j| < k$, $j = 1, 2, \cdots, r$ is to be made.

The proof is the consequence of some simple arguments which are used often for similar purposes and which are the following.

LEMMA 2.1. *Let ξ' be a continuous random variable in the interval $(0, 1)$ whose density function is constant in a subinterval (a, b). Then ξ' is uniformly distributed in (a, b) under the condition $\{a < \xi' < b\}$.*

A consequence of this is the following lemma.

LEMMA 2.2. *Let ξ' and η' be continuous random variables in the interval $(0, 1)$, the density functions of which are (not necessarily equal) constants in a subinterval (a, b). Let $\xi_1', \xi_2', \cdots, \xi_\nu'$ and $\eta_1', \eta_2', \cdots, \eta_\mu'$ be independent observations on ξ' and*

η' resp. falling in the interval (a, b). Under this condition, all possible arrays of the mentioned $\nu + \mu$ sample elements have the common probability $\binom{\nu+\mu}{\mu}^{-1}$.

A consequence of this lemma is that in each interval the conditional probability that $s_i < k$ given $\{\mu, \nu\}$, can be calculated under H_0.

Turning now to the determination of the probability

$$1 - W_n(H_1^{(r)}, \alpha) = P\left(D_{n,n}^+ < \frac{k}{n} | H_1^{(r)}\right), \tag{2.19}$$

we calculate this conditionally given $\{\mu, \nu\}$ and multiply this by $P(\nu, \mu | H_1^{(r)})$; then we have to sum over all ν and μ. But under the mentioned condition the random walk falls into parts with division points $(i_j = n_j + m_j, s_{i_j} = n_j - m_j)$, $(j = 1, 2, \cdots, r)$ and according to the Markov property of the random walk, the relation

$$\begin{aligned} P\left(D_{n,n}^+ < \frac{k}{n} \mid \mu, \nu, H_1^{(r)}\right) \\ = \prod_{j=1}^{r} P(s_i < k, \text{ for } n_{j-1} + m_{j-1} < i < n_j + m_j | \mu, \nu, H_1^{(r)}) \end{aligned} \tag{2.20}$$

holds. Taking into account the elementary formula

$$\begin{aligned} P(s_i < k, \text{ for } n_{j-1} + m_{j-1} < i < n_j + m_j | \mu, \nu, H_1^{(r)}) \\ = 1 - \frac{\binom{\nu_j + \mu_j}{\mu_j + k - n_{j-1} + m_{j-1}}}{\binom{\nu_j + \mu_j}{\mu_j}}, \end{aligned} \tag{2.21}$$

and the relations (2.13), (2.14), and (2.15), we come to theorem 2.1.

2.3. *A special case.* For our comparative considerations we shall treat the following very simple alternative

$$H_1^{(2)}: \begin{cases} F(x) & \text{uniformly distributed in } (1, 1), \\ G'(x) = g(x) = \begin{cases} 0 & \text{if } 0 \le x < z, \\ \dfrac{1}{1-z} & \text{if } z \le x \le 1. \end{cases} \end{cases} \tag{2.22}$$

For this kind of alternative the first index of the maximum will occur with higher probability for smaller values than in the case of the null hypothesis. This alternative being simple enough, I have chosen it for a first comparison of tests based on 1 and on 2 statistics respectively.

Making use of the notation $\nu_1 = \nu$, then $\nu_2 = n - \nu$, and further, knowing that $P(\mu_1 = 0, \mu_2 = n | H_1^{(2)}) = 1$, we obtain for the power function

$$W_n(H_1^{(2)}, \alpha) = 1 - \sum_{\nu=0}^{k-1} \binom{n}{\nu} z^\nu (1-z)^{n-\nu} \left(1 - \frac{\binom{2n-\nu}{n-\nu+k}}{\binom{2n-\nu}{n-\nu}}\right). \tag{2.23}$$

This simple case shows that in determining the power, the normal approximation of the binomial terms is not suitable, since $\nu < k(\alpha, n) \sim y_\alpha \sqrt{2n}$ when $n \to \infty$. A similar remark was made by J. Rosenblatt [8].

After some modification we may obtain the following form:

$$W_n(H_1^{(2)}, \alpha) = 1 - B_{1-z}(n - k + 1, k) + \frac{\alpha}{(1 - z)^k} B_{1-z}(n + 1, k), \tag{2.24}$$

where B is the beta-function

$$B_z(p, q) = \frac{\int_0^z t^{p-1}(1 - t)^{q-1}\, dt}{\int_0^1 t^{p-1}(1 - t)^{q-1}\, dt}. \tag{2.25}$$

This is suitable for immediate computation of the power against this simple kind of alternative. Now we give some numerical values for small samples using (n, k) pairs for which α is near 10% and $z = \max_{(x)} (F(x) - G(x)) = 0.1$; 0.2; 0.3. The following table gives the errors of the second kind for these values of z.

TABLE I

n	k	α	$z = 0.1$	$z = 0.2$	$z = 0.3$
20	7	0.0873	0.8177	0.6163	0.3363
30	8	0.1197	0.7230	0.4039	0.1164
40	9	0.1331	0.6587	0.2640	0.0377
50	11	0.7190	0.7190	0.2700	0.0200

2.4. *The power function of the test based on a pair of statistics.* Let us consider the two statistics $D^+_{n,n}$ and $R^+_{n,n}$ for the decision between H_0 and $H_1^{(2)}$, defined in 2.3. We shall introduce the random variable $S^+_{n,n} = R^+_{n,n} - D^+_{n,n}$. As $nD^+_{n,n}$ and $nR^+_{n,n}$ are of the same parity, $S^+_{n,n}$ is always even. Let us use the following notations:

$$P\left(D^+_{n,n} = \frac{k}{n}, S^+_{n,n} = \frac{s}{n} \mid H_0\right) = P^{(n)}_{k,s} \tag{2.26}$$

and

$$P\left(D^+_{n,n} = \frac{k}{n}, S^+_{n,n} = \frac{s}{n} \mid H_1\right) = Q^{(n)}_{k,s}. \tag{2.27}$$

Denoting the best critical region of the α-size test (restricting ourselves to the (k, s) plane) by $\mathcal{K}_\alpha$, this is defined with the aid of a suitable constant c_α, and can be written in the form

$$\mathcal{K}_\alpha = \left\{(s, k):\ \frac{Q^{(n)}_{k,s}}{P^{(n)}_{k,s}} > c_\alpha\right\} \tag{2.28}$$

and

$$\sum_{(k,s)\in\mathcal{K}_\alpha} P_{k,s}^{(n)} = \alpha. \tag{2.29}$$

The power function is

$$W_n(H_1^{(2)}, \alpha) = \sum_{(k,s)\in\mathcal{K}_\alpha} Q_{k,s}^{(n)}. \tag{2.30}$$

We need the probabilities $P_{k,s}^{(n)}$ and $Q_{k,s}^{(n)}$.

As it was proved in [10] the following relations are valid:

$$P_{0,s}^{(n)} = \frac{1}{2(2s-1)(n-s+1)} \frac{\binom{2s}{s}\binom{2n-2s}{n-s}}{\binom{2n}{n}}, \qquad s = 1, 2, \cdots, n, \tag{2.31}$$

$$P_{k,s}^{(n)} = \frac{k(k+1)}{(k+2s)(n-s+1)} \frac{\binom{k+2s}{s}\binom{-k+2n-2s}{n-s}}{\binom{2n}{n}}, \tag{2.32}$$

$$s = 0, 1, 2, \cdots, n-k.$$

We turn now to the determination of the probabilities $Q_{k,s}^{(n)}$. The number of ξ_i's in the interval $(0, z)$ may be $\nu = 0, 1, 2, \cdots, k$. As was mentioned, $\mu_1 = 0$ with probability 1. For $k = 0$, ν must be 0 and $P(\nu) = P(\nu = 0, n - \nu = n) = (1-z)^n$. In this case it can be seen immediately that

$$Q_{0,s}^{(n)} = P_{0,s}^{(n)}(1-z)^n, \qquad s = 1, 2, \cdots, n. \tag{2.33}$$

The case $s = 0$ (that is, $R_{n,n}^+ = D_{n,n}^+$) can happen only when $\nu = k$. In this case we have the second part of the path starting with probability 1 from the point $(k, s_k = k)$ and ending at $(2n, 0)$. The probability that this path will never reach the height $s_i = k + 1$ multiplied by $P(\nu) = P(\nu = k) = \binom{n}{k} z^k(1-z)^{n-k}$ gives the required probability

$$Q_{k,0}^{(n)} = \frac{k+1}{2n-k+1} \frac{\binom{2n-k+1}{n+1}}{\binom{2n-k}{n}} \binom{n}{k} z^k (1-z)^{n-k} \tag{2.34}$$

$$= \frac{k+1}{n+1} \frac{\binom{2n-k}{n}}{\binom{2n}{n}} \binom{2n}{k} z^k (1-z)^{n-k}, \qquad k = 1, 2, \cdots, n.$$

At the end, for $s > 0$, $k > 0$, we can construct the power function as given in 2.2 and evaluate the joint probabilities for the maximum and first maximum index in the case of the several ν's.

The resulting formula is

$$(2.35)\qquad Q_{k,s}^{(n)} = \frac{k+1}{n-s+1}\frac{\binom{-k+2n-2s}{n-s}}{\binom{2n}{n}}\sum_{\nu=0}^{k-1}\frac{k-\nu}{n-\nu}\binom{k-\nu+2s}{s}\binom{2n}{\nu}z^{\nu}(1-z)^{n-\nu}.$$

Using the above formulas for numerical calculation we can compare the second kind of errors for the use of $D_{n,n}^{+}$ alone and of the pair $(D_{n,n}^{+}, R_{n,n}^{+})$ respectively. The results are tabulated in table II. (The computation was carried out on a GIER electronic computer. I am indebted to A. Békéssy and G. Tusnády for their kind help in accomplishing these calculations.)

TABLE II

n	Error of first kind $D_{n,n}^{+}$	Error of first kind $(D_{n,n}^{+}, R_{n,n}^{+})$	Error of second kind, if $\Delta = z = 0, 1$: $D_{n,n}^{+}$	Error of second kind, if $\Delta = z = 0, 1$: $(D_{n,n}^{+}, R_{n,n}^{+})$	Error of second kind, if $\Delta = z = 0, 2$: $D_{n,n}^{+}$	Error of second kind, if $\Delta = z = 0, 2$: $(D_{n,n}^{+}, R_{n,n}^{+})$
10	0.0739	0.0839	0.8581	0.8071	0.7532	0.6198
	0.2005	0.2038	0.6815	0.5849	0.5215	0.3683
30	0.0675	0.0675	0.8263	0.6149	0.5573	0.2568
		0.0893		0.5636		
	0.1197	0.1197	0.7230	0.5036	0.4039	0.1703
50	0.0562	0.0562	0.8017	0.5105	0.3812	0.1179
	0.0893	0.0893	0.7190	0.4299	0.2700	0.0780
	0.1362	0.1362	0.6133	0.3488	0.1733	0.0473

3. The maximum and the number of intersections

3.1. *The nonconsistency of the number of intersections.* In our paper with E. Csáki [3] we considered the statistic $\Lambda_{n,n}$ (see 2.1), that is, the number of intersections. As a test statistic this has the following advantages. As we mentioned in 1.1, its value can be determined very easily. The distribution of $\Lambda_{n,n}$ under H_0 is very simple too:

$$(3.1)\qquad P(\Lambda_{n,n} = \ell | H_0) = \frac{2\ell}{n}\frac{\binom{2n}{n-\ell}}{\binom{2n}{n}}, \qquad \ell = 1, 2, \cdots, n.$$

In addition to these, the test based on $\Lambda_{n,n}$ has the same properties for one-sided and two-sided alternatives.

We conjectured that this statistic is consistent against all continuous alternatives. The grounds for this conjecture were the following: if $\max_{(x)} (F(x) - G(x)) = \Delta > 0$, then with probability greater than zero the statistic $nD_{n,n}^{+}$ will take values of order of magnitude n. Now the following theorem is valid.

THEOREM 3.1 (E. Csáki and I. Vincze). *Let $\vartheta_1, \vartheta_2, \cdots$ be independent random variables with $P(\vartheta_i = 1) = P(\vartheta_i = -1) = 1/2$. Let $S_0 = 0$, $S_i = \vartheta_1 + \vartheta_2 + \cdots + \vartheta_i$, $i = 1, 2, \cdots$, and let us define λ_n as the number of i's for which $S_i = 0$, $S_{i-1}S_{i+1} = -1$, $1 < i < n$. Then the following relation holds with $0 < c \leq 1$,*

$$\lim_{n\to\infty} P(\lambda_n = \ell | S_n \sim cn) = \frac{2c}{1+c}\left(\frac{1-c}{1+c}\right)^{\ell}, \qquad \ell = 0, 1, 2, \cdots. \tag{3.2}$$

Further if $\psi(n) \to \infty$ and $\psi(n)/n^{1/2} < 1$, then

$$\lim_{n\to\infty} P\left(\lambda_n < \frac{n^{1/2}}{\psi(n)}\, y | S_n \sim c\psi(n)n^{1/2}\right) = 1 - e^{-2cy}, \qquad y > 0. \tag{3.3}$$

Now from this argument we would think that for n large enough the case $F(x) \equiv G(x)$ will lead to the greatest number of intersections (in probability), and thus the critical region will be the small values of $\Lambda_{n,n}$.

But E. M. Sarhan (unpublished) has given an example that $\Lambda_{n,n}$ is not consistent against the following alternative in (0, 1) for $z > 0$,

$$F(x) \equiv x, \qquad 0 \leq x \leq 1 \tag{3.4}$$

$$G(x) = \begin{cases} x, & \text{if } 0 \leq x < z, \\ \dfrac{x^2 + z}{1 + z}, & \text{if } z \leq x \leq 1. \end{cases} \tag{3.5}$$

On the other hand, he showed that the test based on $\Lambda_{n,n}$, using for the decision between H_0 and $H_1^{(2)}$ defined in 2.3, is more efficient than the one-sided Kolmogorov-Smirnov test. This way $\Lambda_{n,n}$ as a test statistic—by itself or in addition to the Smirnov statistic—seems not to be without interest.

3.2. *Joint distribution of the maximum deviation and the number of intersections.* In our paper [4] with E. Csáki the generating function of $D_{n,n}$ and $\Lambda_{n,n}$ is determined under H_0, which is the following

$$\sum_{n=1}^{\infty} \binom{2n}{n} P\left(D_{n,n} < \frac{k}{n}, \Lambda_{n,n} = \ell | H_0\right) z^n = 2\left(\frac{w - w^k}{1 - w^{k+1}}\right)^{\ell}, \qquad k, \ell = 1, 2, \cdots \tag{3.6}$$

where

$$w = \frac{1 - \sqrt{1 - 4z}}{1 + \sqrt{1 + 4z}}. \tag{3.7}$$

We can obtain without any difficulty the probabilities by series-expansion; these are the following:

$$P\left(D_{n,n} < \frac{k}{n}, \Lambda_{n,n} = \ell | H_0\right) = \frac{1}{\binom{2n}{n}} \sum_{i=0}^{\ell} \sum_{j=0}^{\infty} (-1)^i \binom{\ell}{i} \binom{\ell + j - 1}{j} \times \binom{2n}{n + i(k-1) + j(k+1) + \ell} \frac{i(k-1) + j(k+1) + \ell}{n}: \tag{3.8}$$

This formula is not suitable for the determination of the limiting distribution as $n \to \infty$, because in the sum each term tends to infinity. The way of solving the problem was the evaluation out of the integral form

$$\lim_{n\to\infty} \frac{\sqrt{2n}}{\binom{2n}{n}} \frac{2}{2\pi i} \oint \left(\frac{w - w^k}{1 - w^{k+1}}\right)^t \frac{dz}{z^{n+1}}, \tag{3.9}$$

where the integration path is a small circle around the origin. This was kindly done by N. G. de Bruijn, which we give in the following theorem.

THEOREM 3.2 (N. G. de Bruijn). *If $x > 0$, $y > 0$, then*

$$\lim_{n\to\infty} P\left(\sqrt{\frac{n}{2}}\, D_{n,n} < y, x \leq \frac{1}{\sqrt{2n}} \Lambda_{n,n} < x + \Delta x | H_0\right) = \frac{2}{i\sqrt{2\pi}} \int_{1-i\infty}^{1+i\infty} \exp\left\{-2xu \frac{e^{uy} + e^{-uy}}{e^{uy} - e^{-uy}} + \tfrac{1}{2}u^2\right\} u\, du\, \Delta x + \sigma(\Delta x) \tag{3.10}$$

hold.

To the proofs of theorems 3.1 and 3.2 and a detailed treatment of the questions of the joint distribution law, we should like to return later.

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