HORIZON IN DYNAMIC PROGRAMS

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1. Introduction

The theory of dynamical programs deals with undertaking decisions in time. Usually we have a functional over a set of sequences (or functions), and the task consists in finding a minimum of this functional. The components of the sequences (or the value of the functions—when time is considered to be continuous) represent the decisions, which are to be carried out at the appropriate point of time. As the solution—minimizing the functional—we get a sequence of decisions, which tells us what to do at all future times.

This is a considerable simplification of problems we face in applications. Usually in applications we are not interested in all sequences of decisions, but indeed, we are interested in the particular one which we must carry out at the present stage. However, the functional to be minimized is not completely known to us. This means that many data are needed to define a functional. These data will occur in time, finally allowing selection of one functional from a family of many possible functionals. But when making the first decision, we do not know which one will finally be selected.

In several cases, to compute the optimal first step decision, we do not need all the data of the functional, but only a part of them; for instance, those which will occur up to a specific point of time h in the future. Such a point is called the horizon of the problem. This is the point up to which one has to know the future in order to compute the optimal decision at the present stage.

The idea of horizon goes back to Modigliani, who in [6] and [7] defined it in an intuitive manner. But the ideas of Modigliani were not worked out to a precise form, and therefore, the term "horizon," which may be found in many papers concerned with dynamical programs, is used with various meanings.

In this paper we present a rigorous definition of the notion of horizon. An auxiliary notion is that of a dynamical parameter, which serves to express the information concerning data of the functional occurring in time.

There are two groups of problems basic to the theory of horizon. One of them deals with the properties of solutions computed with the help of a given horizon ("horizonal solutions"); the other one is concerned with the existence of the horizon in specific cases. Since this paper has an introductory character, both groups of problems are represented here, but by weak theorems only.

Stronger results may be obtained by additional assumptions on the families of problems concerned.

Notations. Throughout this paper we shall use a standard notation, with a few exceptions, which will be mentioned here.

Usually a lower case letter, like x or ζ, denotes infinite sequence:

$$x = \langle x_1, x_2, \cdots \rangle, \qquad \zeta = \langle \zeta_1, \zeta_2, \cdots \rangle. \tag{1}$$

By $x|k$ we denote the finite sequence $\langle x_1, x_2, \cdots, x_k \rangle$ of k first coordinates of x. By $k|x$ we denote the infinite sequence $\langle x_{k+1}, x_{k+2}, \cdots \rangle$. If A_n is a function of n variables, then $A_n(\overline{x}|k, x|n-k) = A_n(\overline{x}_1, \cdots, \overline{x}_k, x_1, \cdots, x_{n-k})$. The same notation applies to functional A over sequences x:

$$A(\overline{x}|k, x) = A(\overline{x}_1, \cdots, \overline{x}_k, x_1, x_2, \cdots). \tag{2}$$

The symbol R^+ denotes the set of nonnegative real numbers and $+\infty$. The symbol $\chi(\quad)$ denotes the characteristic function of the relation in parentheses. For instance,

$$\chi(0 < \alpha) = \begin{cases} 1, & \text{if } 0 < \alpha, \\ 0, & \text{if } 0 \geq \alpha. \end{cases} \tag{3}$$

2. Simple dynamic programming problems

A simple dynamic programming problem (d.p.p.) is defined by two sequences: $X_1, X_2, \cdots, A_1, A_2, \cdots$. The first one is a sequence of sets, the second one is a sequence of functions A_n: $X_1 \times \cdots \times X_n \to R^+$. By the "problem" we mean the problem of finding a minimum of the function A: $X \to R^+$, where $X = X_1 \times X_2 \cdots$ and $A(x) = \sum_{i=1}^{\infty} A_i(x|i)$. A d.p.p. is denoted (X_n, A_n).

EXAMPLE 2.1. The problem lies in finding a minimum of the function $C(x) = \sum_{i=1}^{\infty} c_i \cdot x_i$ for x's satisfying $x_i \geq 0$ and $\sum_{i=0}^{n} x_i \geq \sum_{i=1}^{n} d_i$. Here c_i (cost coefficients), d_i (demands), and x_0 (initial stock) are nonnegative numbers. One can assume that $\sum_{i=1}^{\infty} c_i \cdot d_i < +\infty$. To convert this problem into a d.p.p. we will set $X_n = R^+$ and

$$A_n(x_1, \cdots, x_n) = \begin{cases} c_n \cdot x_n, & \text{if } \sum_{i=0}^{k} x_i \geq \sum_{i=1}^{k} d_i \text{ for } k = 1, \cdots, n; \\ +\infty, & \text{in the opposite case.} \end{cases} \tag{4}$$

EXAMPLE 2.2 (Modigliani, Hohn [7]). Let us consider the function

$$C(x) = \sum_{i=1}^{\infty} \beta^{i-1} \left[c(x_i) + \alpha \left(\sum_{j=0}^{i-1} x_j - \sum_{j=1}^{i-1} d_j \right) \right], \tag{5}$$

where c is a convex, monotone-increasing function, positive for $x_i > 0$ (cost function), x_0 (initial stock), α (storage cost) are nonnegative numbers, and where $0 \leq \beta \leq 1$ (discount factor) and $d_j \geq 0$ (demands). To transform the problem of finding a minimum of C over the set of x's satisfying $x_i \geq 0$ and $\sum_{i=0}^{n} x_i \geq \sum_{i=1}^{n} d_i$ into a d.p.p., we set $X_n = R^+$ and

$$A_n(x_1, \cdots, x_n) = \begin{cases} \beta^{n-1}\left[c(x_n) + \alpha\left(\sum_{j=0}^{n-1} x_j - \sum_{j=0}^{n-1} d_j\right)\right], \\ \text{if } \sum_{j=0}^{k} x_j \geq \sum_{j=1}^{k} d_j, \text{ for } k = 1, \cdots, n, \\ +\infty, \text{ in the opposite case.} \end{cases} \tag{6}$$

EXAMPLE 2.3 (Bellman, Glicksberg, Gross [3]). We are given two positive numbers c (cost coefficient) and α (cost of increasing rate of production). For $x_0 \geq 0$, $(i = 1, 2, \cdots)$ we define

$$A_n(x_{n-1}, x_n) = \begin{cases} a(x_{n-1}, x_n), & \text{if } x_n \geq d_n, \\ +\infty, & \text{if } x_n < d_n \end{cases} \tag{7}$$

where $a(x_{n-1}, x_n) = c \cdot x_n + \alpha(x_n - x_{n-1})\chi(x_n > x_{n-1})$. The sequence of sets $X_n = R^+$ and the sequence of functions A_n define a d.p.p. which is called the problem of production planning without storage.

EXAMPLE 2.4 (Wagner, Whitin [8]). Let us define for the nonnegative numbers $x_0, s_i, m_i, d_i,$

$$A_n(x_1, \cdots, x_n) = \begin{cases} s_n\chi(0 < x_n) + m_n\left(\sum_{j=0}^{n-1} x_j - \sum_{j=1}^{n-1} d_j\right), \\ \text{if } \sum_{j=0}^{k} x_j \geq \sum_{j=1}^{k} d_j \quad \text{for} \quad k = 1, \cdots, n; \\ +\infty, \quad \text{in the opposite case.} \end{cases} \tag{8}$$

The sequence of function A_n together with the sequence of sets $X_n = R^+$ form a d.p.p.

EXAMPLE 2.5 (Blackwell [4]). We are given two finite sets S (states) and A (actions), and moreover, two real functions $r\colon S \times A \to R^+$ and $p\colon S \times A \times S \to R^+$; the latter, $p(s'; \text{if } a, s)$, is a probability distribution in s'.

For every n, let X_n be the set of functions $x_n\colon S \to A$. Given an s_0 in S and $x = \langle x_1, x_2, \cdots \rangle$ in $X = X_1 \times X_2 \times \cdots$, we define

$$p_0(s; \text{if } s_0, x|0) = \begin{cases} 1, & \text{if } s = s_0, \\ 0, & \text{if } s \neq s_0, \end{cases} \tag{9}$$

$$p_{n+1}(s; \text{if } s_0, x|n+1) = \sum_{s' \in S} p(s'; \text{if } x_1(s_0), s_0) \cdot p_n(s; \text{if } s', (1|x) \mid n) \tag{10}$$

where $1|x$ denotes $\langle x_2, x_3, \cdots \rangle$ (and therefore, $(1|x) \mid n = \langle x_2, \cdots, x_{n+1} \rangle$).

We define

$$A_n(x|n) = \beta^{n-1} \sum_{s \in S} r(s, x_n(s)) p_{n-1}(s; \text{if } s_0, x|n-1). \tag{11}$$

This gives us a simple d.p.p. composed of the sequences $X_1 = X_2 = \cdots$ and $A_1, A_2, \cdots$.

3. Families of d.p.p. The dynamic parameter

If to every element ζ of a set P corresponds a simple d.p.p., $(X_n, A_n(\cdot\,; \zeta))$, $n = 1, 2, \cdots$, then we have a family of d.p.p. over the set of parameters P.

If the set P is a subset of a product $Z = Z_1 \times Z_2 \times \cdots$, and for every ζ in P and every n, $A_n(\cdot\,;\zeta)$ do not depend on the entire sequence ζ but only upon their first n coordinates $\zeta|n$, then we will call the family $(X_n, A_n(\cdot\,;\zeta|n))$, $\zeta \in P$, a family with a dynamic parameter.

EXAMPLE 3.1. In this example, fixing x_0 and c_i's, the sequence of d_i's is a dynamic parameter. The function A_n, in fact, does depend on the first n of the d_i's, and so we can write $A_n(x_1, \cdots, x_n; d_1, \cdots, d_n) = A_n(x|n; d|n)$. The set of parameters P is in this case the whole product $R^+ \times R^+ \times \cdots$, but for various reasons, it may be restricted to its subset.

By fixing only x_0 and assuming not only d_i's but also c_i's as being variable, we obtain a larger family with d_i's and c_i's occurring as a dynamic parameter. Strictly speaking, in order to conform to the definition, we have to accept as a dynamic parameter the sequence of pairs

$$\zeta = \langle\langle c_1, d_1\rangle, \langle c_2, d_2\rangle, \cdots\rangle. \tag{12}$$

By varying x_0 we can change the considered d.p.p.'s. But x_0 is not a dynamic parameter.

EXAMPLE 3.2. In this example α, β, x_0 and d_i's are parameters. Only d_i's may be considered as a dynamic parameter. If instead of the discount factor β we adopt varying factors β_i, then the sequence of β_i's may be also considered as a dynamic parameter. This may have some meaning when studying discount fluctuations on a market.

Note that the choice of the dynamic parameter depends on the problem we plan to study.

EXAMPLE 3.3. Here, the numbers c, α, x_0 and all d_i's are parameters, but only d_i's form a dynamic parameter. Following the definition of A_n which we accepted, these functions depend only on the two last coordinates of $x|n$ and on the last coordinate of $d|n$. It neither affects the definition of d.p.p. nor that of the dynamic parameter. Defining functions $\overline{A}_n$ as

$$\overline{A}_n(x_1, \cdots, x_n) = \begin{cases} a(x_{n-1}, x_n), & \text{if} \quad x_k \geq d_k, \quad \text{for} \quad k = 1, \cdots, n, \\ +\infty, & \text{in the opposite case,} \end{cases} \tag{13}$$

we obtain another d.p.p. These d.p.p.'s considered as a family with a dynamic parameter $d = \langle d_1, d_2, \cdots\rangle$ no longer have the property mentioned earlier. All functions $\overline{A}_n$ depend essentially on $x|n$ and $d|n$. In spite of the identity $A(x) = \sum_{n=1}^{\infty} A_n(x|n) = \overline{A}(x) = \sum_{n=1}^{\infty} \overline{A}_n(x|n)$ (for a fixed d), both d.p.p. (X_n, A_n) and $(X_n, \overline{A}_n)$ must be considered as different d.p.p.'s, because generally $A_n \neq \overline{A}_n$.

EXAMPLE 3.4. All sequences s_i, m_i, and d_i may be considered as a dynamic parameter. The number x_0 is not one.

EXAMPLE 3.5. The element s_0 is the only varying factor in the A_n's. It cannot be considered as a dynamic parameter.

4. Truncated and partially completed d.p.p.'s. The initial parameter

Let us define for a given d.p.p. (X_n, A_n),

$$A_n(x|n; 1) = A_n(x|n) \quad \text{and} \quad A_n(x|n; 0) = 0. \tag{14}$$

The family $(X_n, A_n(\cdot\,; t))$, $t \in T_0$, where T_0 is the set of all sequences t with $t_1 = t_2 = \cdots = t_N = 1$, $t_{N+1} = t_{N+2} = \cdots = 0$ for some natural number N, is called the family of truncated d.p.p.'s of (X_n, A_n).

For a d.p.p. (X_n, A_n), $\bar{x}$ in $X = X_1 \times X_2 \times \cdots$ and natural number k, we define the d.p.p. partially completed by $\bar{x}|k$ as the d.p.p. $(X_n, \bar{A}_n)$ with $\bar{X}_n = X_{n+k}$ and $\bar{A}_n(x|n) = A_{n+k}(\bar{x}|k, x|n)$ for every x in $X_{k+1} \times X_{k+2} \times \cdots$.

Let us consider a family of d.p.p.'s $(X_n, A_n(\cdot\,; \zeta|n))$, $\zeta \in P$ with the (only) dynamic parameter ζ. For a given $\bar{x}$ in $X = X_1 \times X_2 \times \cdots$, $\bar{\zeta}$ in P and a natural number k, we define the family partially completed by $\bar{x}|k$ and $\bar{\zeta}|k$ as the family of all d.p.p.'s $(X_n, A_n(\cdot\,; \zeta|n))$ with $\zeta \in P$ and $\zeta|k = \bar{\zeta}|k$, partially completed by $\bar{x}|k$. This is again a family with a dynamic parameter ξ which runs over the set $P(\bar{\zeta}|k)$ of all $\xi = \langle \xi_1, \xi_2, \cdots \rangle$ such that $\langle \bar{\zeta}_1, \cdots, \bar{\zeta}_k, \xi_1, \xi_2, \cdots \rangle$ belongs to P.

Let us now assume that the family $(X_n, A_n(\cdot\,; \zeta|n, x_0))$, $\zeta \in P$, $x_0 \in X_0$ is such that $X_1 = X_2 = X_3 = \cdots$ and $P(\bar{\zeta}|k) = P$ for every $\bar{\zeta}$ in P and natural k.

In this case the partial completion of a subfamily $(X_n, A_n(\cdot\,; \zeta|n, x_0))$, $\zeta \in P$, with a fixed x_0, by $\bar{x}|k$ and $\bar{\zeta}|k$, results in a family defined onto the same sets X_n and with the same dynamic parameter $\bar{\zeta}$ in P. It may happen that this family is one of the subfamilies of the whole family, only with another nondynamic parameter x_0'. If this occurs for every $\bar{x}|k$, $\bar{\zeta}|k$ and x_0 in X_0, then the parameter x_0 is called the initial parameter of the whole family.

EXAMPLES 4.1–4.4. In all these examples x_0 is an initial parameter. Let us consider, for instance, example 1. If the initial stock x_0 demands $\bar{d}_1, \cdots, \bar{d}_k$ and productions $\bar{x}_1, \cdots, \bar{x}_k$ are given for the first k periods, then at the $k+1$-period the initial stock is $x_0' = x_0 + \sum_{i=1}^{k} \bar{x}_i - \sum_{i=1}^{k} \bar{d}_i$, and this is the only influence of the past on the coming periods. In order to cover the case when $\bar{x}_1, \cdots, \bar{x}_k$ is not feasible for $\bar{d}_1, \cdots, \bar{d}_k$ (for instance, $x_0' < 0$), we may assume that x_0 takes nonnegative values and -1.

EXAMPLE 4.5. In this example s_0 is not an initial parameter, even in the case when $\beta = 1$. If we complete the program by $\langle \bar{x}_1, \cdots, \bar{x}_k \rangle = \bar{x}|k$, starting with a given s_0 in S, then the initial s in the partially completed program is known to us only through the distribution $p_k(s; \text{if } s_0, \bar{x}|k)$.

In order to have an initial parameter in our family of problems, we may extend the set of parameters S to the set Π of (unconditional) distribution over S. Then for a given π in Π we will have

$$\bar{p}_n(s; \text{if } \pi, x|n) = \sum_{s_0 \in S} p_n(s; \text{if } s_0, x|n)\pi(s_0). \tag{15}$$

Now for a given π and $\bar{x}|k$, the parameter π of the partially completed problem will be $\bar{p}_k(\cdot\,; \text{if } \pi, \bar{x}|k)$ which is in Π.

The parameter π may be considered as an initial parameter also in the case when $\beta < 1$, provided that we agree to consider two d.p.p.'s which differ only by a positive coefficient (that is, $A_n = \alpha A_n'$, for all n, with $\alpha > 0$) as equal. (This remark also applies in example 2, where the discount factor β is introduced.)

Finally, let us note that the family considered in this example has no proper dynamic parameter. In order to fit it to the definition we may always introduce a dynamic parameter P consisting of a constant sequence. Such a P fulfills the requirement of the definition.

5. The horizon and horizonal solutions

From this point on we will assume that all d.p.p.'s with which we are concerned attain a minimum at some point of the product of their sets.

Now let us fix a family $(X_n, A_n(\cdot; \zeta|n))$, $\zeta \in P \subset Z = Z_1 \times Z_2 \times \cdots$. The function connected with a ζ in P shall be denoted by $A(x; \zeta) = \sum_{n=1}^{\infty} A_n(x|n; \zeta|n)$. By $v(\zeta)$ we shall denote the value of the d.p.p. with the parameter $\zeta: v(\zeta) = \min_{x\in X} A(x; \zeta)$. A d.p.p. with $v(\zeta) < +\infty$ is called convergent.

We define a relation between a natural number h and an element $\bar{\zeta}$ in Z (but not necessarily in P) in the following way: h is the horizon for $\bar{\zeta}$ if there exists an element x_1^* in X_1, such that for every ζ in P and such that $\zeta|h = \bar{\zeta}|h$, there exists an x in X with $A(x, \zeta) = v(\zeta)$ and $x_1 = x_1^*$.

Roughly speaking, h is a horizon for $\bar{\zeta}$ if there exists a "first step decision" x_1^*, which may be extended to the minimal solution of every d.p.p. of the family concerned, provided the dynamic parameter ζ of that program agrees with $\bar{\zeta}$ in h first coordinates.

If h is a horizon for $\bar{\zeta}$, then the element x_1^*, which the definition asserts to exist, is called the horizonal element for $\bar{\zeta}$.

We should point out that the notion of horizon depends on the family of d.p.p.'s under consideration. The correct way of expressing the defined relation is the following: "h is the horizon for $\bar{\zeta}$ in the family. . . ." The notion of a horizonal element is strongly dependent on h. If h is a horizon for $\bar{\zeta}$, then every $h_1 > h$ is also a horizon for $\bar{\zeta}$. But an element x_1^* which satisfies the definition for h_1 does not necessarily satisfy it for h.

A sequence x^* in X is called a horizonal solution for $\bar{\zeta}$, iff for every k, x_{k+1}^* is a horizontal element for $\langle \bar{\zeta}_{k+1}, \bar{\zeta}_{k+2}, \cdots \rangle$ in the family of d.p.p.'s partially completed by $x^*|k$ and $\bar{\zeta}|k$.

A very important lemma on the horizonal solutions is the following.

LEMMA 5.1. *If x^* is a horizonal solution for $\bar{\zeta}$, then for every k there exists an h_k such that, for every ζ in P satisfying $\zeta|k + h_k = \bar{\zeta}|k + h_k$, there exists an x in X with $A(x; \zeta) = v(\bar{\zeta})$ and $x|k + 1 = x^*|k + 1$.*

The proof of this lemma is by induction on k, and we will not give it here.

We should note that a $\bar{\zeta}$ for which there exists a horizonal solution cannot be completely arbitrary. In order to have the family partially completed by $x^*|k$ and $\bar{\zeta}|k$, a $\zeta^{(k)}$ in P with $\zeta^{(k)}|k = \bar{\zeta}|k$ is needed. In this case we say that $\bar{\zeta}$ is a limit of the sequence $\zeta^{(k)}$, and we write $\zeta^{(k)} \to \bar{\zeta}$. In order to have a horizonal solution, $\bar{\zeta}$ has to be a limit of parameters in P.

We say that a simple d.p.p. has a horizon if the constant sequence $t = \langle 1, 1, \cdots \rangle$ has a horizon in the family of truncated d.p.p.'s of the given d.p.p. In the same way, we say that x^* is a horizonal solution of a simple d.p.p. This means that it is a horizonal solution for $t = \langle 1, 1, \cdots \rangle$ in the family of truncated d.p.p.'s.

EXAMPLE 5.1. In this example, h is a horizon for $\langle\langle\bar{c}_1, \bar{d}_1\rangle, \langle\bar{c}_2, \bar{d}_2\rangle, \cdots\rangle$ in the family with an initial parameter x_0, iff $\bar{c}_h \le \bar{c}_1$ and $\bar{c}_k > \bar{c}_1$, for $k = 2, \cdots, h-1$. Then if $\sum_{i=1}^{h-1} \bar{d}_i \le x_0$, $x_1^* = 0$ is the horizonal element and if $\sum_{i=1}^{h-1} \bar{d}_i > x_0$, then $x_1^* = \sum_{i=1}^{h-1} \bar{d}_i - x_0$ is the horizonal element.

If, for example, lim inf $\bar{c}_i = 0$ (which is a reasonable assumption, as usually the cost $\bar{c}_i$ will be the real costs reduced by a discount factor), then the horizonal solution exists.

EXAMPLE 5.2. This is the classical example for the horizonal solution. If we allow the parameters $d = \langle d_1, d_2, \cdots\rangle$ to run through the set of all nonnegative sequences, then there obviously is no horizon for any sequence. But if we restrict d to a set P of uniformly bounded sequences (that is, $d_i \le M$, for all i and d in P), then in that family with the arbitrary initial parameter there exists a horizon for every sequence in P. Since the whole family is a family with an initial parameter, then for every $\bar{\xi}$ in P a horizonal solution for $\bar{\xi}$ exists. Some generalizations of this theorem have recently been proved (see section 6).

EXAMPLE 5.3. It may be easily shown that for a given $d = \langle d_1, d_2, \cdots\rangle$, h is a horizon for d in the family with an initial parameter x_0, if $d_h \ge x_0$ and $d_k < x_0$ for $k = 1, \cdots, h-1$. A better result states that independently of x_0 the natural number h' with $(\alpha/c) < h' \le (\alpha/c) + 1$ is a horizon for every d. The first horizon h, as a function of x_0 and d, is neither defined everywhere, nor bounded on the set where it is defined. The second one is defined everywhere and bounded, but it may happen that $h < h'$. Hence, the horizon h' is not always the shortest one.

Following the method presented in Arrow, Karlin [2], it may be shown that there exists a horizon even in the continuous case with a convex cost function (see section 6).

EXAMPLE 5.4. There is no horizon for all sequences of the dynamic parameters. (This fact was established by A. Brauner.) It is proved by showing that if $m_i = 2$, $d_i = 1$, and $s_i = 3 + a_i$, where $0 < a_i < a_{i+1} < 1$, then the minimal solution of the d.p.p. truncated on N is either $\langle 2, 0, 2, 0, \cdots, 2, 0\rangle$ if N is even, or $\langle 1, 2, 0, 2, 0, \cdots, 2, 0\rangle$ if N is odd.

It can be shown that a d.p.p. with all dynamic parameters constant, namely $s_i = s_1 > 0$, $m_i = m_1 > 0$, $d_i = d_1 > 0$, has a horizon in the family of its truncated programs.

EXAMPLE 5.5. Not every d.p.p. with $\beta < 1$, belonging to the family presented in section 1, has a horizon in the family of its truncated d.p.p.'s. The example is the following. The set S contains four states: s_0, s_1, s_2, s_3. All actions a in A lead from one state to another in a deterministic manner (that is, $p(s'; \text{if } a, s) = 1$ or 0). We are given actions leading from s_0, s_1, s_2 to every other state, but s_3 is an absorbing state. This means that every action leads from s_3 to s_3 only. For transitions $s_0 \to s_1$, $s_0 \to s_2$, $s_1 \to s_2$, $s_2 \to s_1$, the loss r is equal to 1. For transition $s_2 \to s_3$ there is no loss, that is, r is equal to 0. All other transitions have the loss $r > 1$; in particular, the transition $s_3 \to s_3$ has a loss which is large in comparison to β, let us say $2/\beta$.

If we start with s_0 and the program is infinite, then the best we can do is to go to either s_1 or s_2, and then to change at every step from s_1 to s_2 and from s_2 to s_1. Proceeding this way we incur a minimal loss equal to $v = \sum_{i=1}^{\infty} \beta^{i-1}$. If the program is finite, let us say of the length N, then the best policy, when starting with s_0, is to go in $N - 1$ first steps through transitions with loss 1 and then to finish with the transition $s_2 \rightarrow s_3$. The total loss in such a case will be $v_N = \sum_{i=1}^{N-1} \beta^{i-1}$, and it is the minimal one. But to achieve this we must make in the first step the transition $s_0 \rightarrow s_2$, if N is even, and the transition $s_0 \rightarrow s_1$, if N is odd. This shows that there is no horizon for the infinite problem in the family of its truncated d.p.p.'s.

In spite of the nonexistence of the horizon for some d.p.p.'s in our family, we can show that in some cases the horizon does exist.

We want to recall that

$$A_n(x|n; s_0) = \beta^{n-1} \sum_{s \in S} r(s, x_n(s)) p_{n-1}(s; \text{if } s_0, x|n-1). \tag{16}$$

Let us form the family of truncated d.p.p.'s. By $t^{(N)}$ we shall denote the sequence with $t_n^{(N)} = 1$ or 0, according to $n \leq N$ or $n > N$. Then the functions of truncated problems are

$$A_n(x|n; t_n^{(N)}, s_0) = t_n^{(N)} \cdot A_n(x|n; s_0) \tag{17}$$

and

$$A(x; t^{(N)}, s_0) = \sum_{n=1}^{N} A_n(x|n; s_0). \tag{18}$$

It follows from the inductive definition of the conditional distribution p_n that

(i) $A(x; t^{(N)}, s_0) = r(s_0, x_1(s_0)) + \beta \sum_{s \in S} p(s; \text{if } x_1(s_0), s_0) \cdot A(1|x, t^{(N-1)}, s)$.

Starting with this formula it can easily be proved by induction that

(ii) For every N, there exists an $x^{(N)}$ such that $A(x^{(N)}; t^{(N)}, s_0) = v(t^{(N)}, s_0)$, for every s_0 in S.

Another easy preparatory lemma is the following.

(iii) For every s_0 in S, $v(t^{(N)}, s_0) \rightarrow v(s_0)$, when $N \rightarrow \infty$.

We associate with each function f: $S \rightarrow A$ (then f is in X_1) and every function ϕ: $S \rightarrow R^+$, a function $L(f, \phi)$: $S \rightarrow R^+$, which is defined by

$$L(f, \phi)(s_0) = r(s_0, f(s_0)) + \beta \sum_{s \in S} p(s; \text{if } f(s_0), s_0) \phi(s). \tag{19}$$

Then we have

(iv) $A(x; t^{(N)}, s_0) = L(x_1, A(1|x; t^{(N-1)}, \cdot))(s_0)$.

Let us call an x_1^* in X_1 a minimal element with respect to the function ϕ: $S \rightarrow R^+$, iff $L(f, \phi)(s_0) \geq L(x_1^*, \phi)(s_0)$ for all f in X_1 and s_0 in S.

(v) For an x_1^* in X_1, to have an extension $x^{(N)}$ (that is, $x_1^{(N)} = x_1^*$) with $A(x^{(N)}; t^{(N)}, s_0) = v(t^{(N)}, s_0)$ for all s_0 in S, it is necessary and sufficient to be a minimal element with respect to $v(t^{(N-1)}, \cdot)$.

It is easy to show that if x_1^* fulfills the condition, then for every $x^{(N-1)}$ which minimizes the problem of the length $N - 1$, $\langle x_1^*, x_1^{(N-1)}, x_2^{(N-1)}, \cdots \rangle$ minimize the problem of the length of N.

On the other hand, if $x^{(N)}$ minimizes the problem of the length of N and $x_1^* = x_1^{(N)}$, then, as $A(1|x^{(N)}, t^{(N-1)}, s) \geq v(t^{(N-1)}, s)$ for every s and L is monotonic, we have

$$(20) \qquad L(x_1^*, A(1|x^{(N)}, t^{(N-1)}, \cdot))(s_0) = A(x^{(N)}, t^{(N)}, s_0)$$
$$\geq L(x_1^*, v(t^{(N-1)}, \cdot))(s_0) = v(t^{(N)}, s_0) \qquad \text{for every } s_0.$$

It follows that $L(x_1^*, v(t^{(N-1)}, \cdot))(s_0) \leq L(f, v(t^{(N-1)}, \cdot))(s_0)$ for every s_0 and f, which means that x_1^* is a minimal element with respect to $v(t^{(N-1)}, \cdot)$.

Now let us remark that since $L(f, \phi)$ is continuous in ϕ, then

(vi) if f is not minimal with respect to ϕ, then there exists a neighborhood U of ϕ such that f is not minimal with respect to every ψ in U.

It follows from (vi) that,

(vii) there exists a neighborhood U of $v(\cdot)$ such that if x_1^* is minimal with respect to a given ψ in U, then x_1^* is minimal with respect to $v(\cdot)$.

Now we can prove the following theorem.

THEOREM 5.1. *If the minimal element x_1^* with respect to $v(\cdot)$ is unique, then there exists a horizon h for the infinite problem, and the horizonal element is x_1^*.*

Let U be the neighborhood as described in (vii) and h a number such that, if $N \geq h$, then $v(t^{(N)}, \cdot)$ belongs to U. It follows from the construction that the minimal element x_1^* with respect to $v(\cdot)$ is minimal with respect to $v(t^{(N)}, \cdot)$, and it follows from (v) that it can be extended to an $x^{(N)}$ with $A(x^{(N)}, t^{(N)}, s_0) = v(t^{(N)}, s_0)$ for all s_0 in S. This proves the theorem.

6. Optimal properties of horizonal solutions

One of the most important problems of the theory of the horizon is to establish when a horizonal solution is a minimal one. It is not always minimal, but the theorem presented in this section will cover some important cases when it is so.

As in the preceding section, we will fix a family $(X_n, A_n(\cdot; \zeta|n))$, $\zeta \in P \subset Z = Z_1 \times Z_2 \times \cdots$ and will use the notation $A(x; \zeta) = \sum_{n=1}^{\infty} A_n(x|n; \zeta|n)$, $v(\zeta) = \min_{x \in X} A(x; \zeta)$ and $\zeta^{(n)} \to \zeta$.

THEOREM 6.1. *If $\zeta^{(n)} \in P$, $\zeta^{(n)} \to \zeta$, $v(\zeta^{(n)}) \leq M$ and x^* is a horizonal solution for $\bar{\zeta}$, then $A(x^*; \bar{\zeta}) \leq M$.*

PROOF. Since $\zeta^{(n)} \to \bar{\zeta}$ and x^* is a horizonal solution for $\bar{\zeta}$, then, by the lemma in section 5, for every k we may find a number n and $x^{(n)}$ such that $\zeta^{(n)}|k = \bar{\zeta}|k$, $A(x^{(n)}, \zeta^{(n)}) = v(\zeta^{(n)})$ and $x^{(n)}|k = x^*|k$. Hence,

$$(21) \qquad \sum_{i=1}^{k} A_i(x^*|i; \bar{\zeta}|i) = \sum_{i=1}^{k} A_i(x^{(n)}|i; \zeta^{(n)}|i)$$
$$\leq A(x^{(n)}; \zeta^{(n)}) = v(\zeta^{(n)}) \leq M,$$

which proves the theorem.

THEOREM 6.2. *If $\zeta^{(n)} \in P$, $\zeta^{(n)} \to \bar{\zeta}$, $v(\zeta^{(n)}) \leq v(\bar{\zeta})$ and x^* is a horizonal solution for $\bar{\zeta}$, then $A(x^*; \bar{\zeta}) = v(\bar{\zeta})$.*

PROOF. By applying theorem 6.1 with $M = v(\bar{\zeta})$, we obtain $A(x^*; \bar{\zeta}) \leq v(\bar{\zeta})$.

THEOREM 6.3. *If $\bar{\zeta} \in P$ and x^* is a horizonal solution for $\bar{\zeta}$, then $A(x^*; \bar{\zeta}) = v(\bar{\zeta})$.*

PROOF. Since $\bar{\zeta} = \zeta^{(n)} \to \bar{\zeta}$, then this is the corollary of theorem 6.2.

Following theorem 6.3, a horizonal solution for a d.p.p. in the family with respect to which the horizonal solution has been constructed is a minimal one. This theorem was proved by Maria W. Łoś in 1962.

THEOREM 6.4. *If x^* is a horizonal solution of a simple d.p.p. (that is, in the family of its truncated problems), then it is a minimal solution.*

PROOF. If $t^{(N)}$ is the sequence with $t_n^{(N)} = 1$ for $n \leq N$ and $t_n^{(N)} = 0$ for $n > N$, then $t^{(N)} \to t = \langle 1, 1, \cdots \rangle$. Obviously, $v(t^{(N)}) \leq v(t) = v$. Therefore, this theorem follows from theorem 6.2.

EXAMPLE 5.1. Let us suppose we are given a family of d.p.p.'s with the parameter s_0 in S as described in section 2 and later studied in section 5. Moreover, let us assume that there is a horizon in this family. In order to have an initial parameter, we extend S to the set Π of all distributions over S, and we consider the functions A_n with average distributions $\bar{p}_n(s; \text{if } \pi, x|n)$, as shown in section 4.

It is easy to check that extending S to Π does not affect the existence of the horizon, and moreover, that both the horizon and the horizonal element may be chosen independently of the parameter π.

Since the family being considered has a horizon and an initial parameter, then there exists a horizonal solution for every simple d.p.p. in this family. As the horizonal element does not depend on π and, going step by step, the same horizonal element may be used, then the horizonal solution will be a sequence x with $x_1 = x_2 = x_3 = \cdots$. Such a solution is called a stationary solution.

It follows from theorem 4 that this horizonal stationary solution is a minimal one.

In the paper by Blackwell [4] it is shown that every family of d.p.p.'s as studied here has a stationary minimal solution. It may be shown by easy examples that not every stationary minimal solution is a horizonal one, even in the case when a horizonal solution does exist.

7. Horizon for d.p.p.'s with continuous time

By studying dynamical programming problems with continuous time, the theory of the horizon changes in several respects. Without going into detail we shall show by two examples how in these cases the notion may be applied. Both examples are indeed continuous versions of formerly presented examples.

We are given a nonnegative function c, defined for $x \geq 0$ and such that $c'(x) > 0$, $c''(x) \geq 0$. Furthermore, we are given two nonnegative constants α and x_0. The problem lies in minimizing the functional (see Arrow, Karlin [1])

$$\mathfrak{I}_0^T(x, \zeta) = \int_0^T z(x(t)) + \alpha\left[x_0 + \int_0^t (x(\tau) - \zeta(\tau))\, d\tau\right] dt \tag{22}$$

over the set of all nonnegative functions x, continuous and differentiable for all but a finite number of points, and such that

$$y(x, \zeta, t) = x_0 + \int_0^t (x(\tau) - \zeta(\tau))\, d\tau \geq 0 \qquad \text{for} \quad 0 \leq t \leq T. \tag{23}$$

We shall assume that the function ζ—which is indeed the dynamical parameter of the problem—belongs to a set Z_0 of nonnegative, continuous, and differentiable functions. This set will be more exactly specified later.

We say that the function $\bar{\zeta}$ has a horizon H for $T_0 > 0$ in the above-defined family with parameters in Z_0, iff the following is true.

There exists a function $\bar{x}$ such that for every $T \geq H$ and every ζ in Z_0 with $\zeta(t) = \bar{\zeta}(t)$, for $0 \leq t \leq H$, there exists a function x^* with $y(x^*, \zeta, t) \geq 0$ for $0 \leq t \leq T$, $\mathfrak{I}_0^T(x^*, \zeta) = \min_{y(x,\zeta,t)\geq 0} \mathfrak{I}_0^T(x, \zeta)$ and $x^*(t) = \bar{x}(t)$ for $0 \leq t \leq T_0$.

This is certainly not true if we do not restrict Z_0 to be a uniformly bounded set of functions. But if we do restrict Z_0 to be the set of functions with $0 \leq m \leq x(t) \leq M < \infty$, then, as is shown by Blikle [5], the above statement is true for $H = T_0 + (1/\alpha)[c'(M) - c'(m)]$.

Now let c be a nondecreasing and nonnegative differentiable function, and let ψ be a decreasing positive continuous function. Finally, let α be a positive constant. Let us suppose we are interested in minimizing the functional

$$\mathfrak{F}_0^T(x) = \int_0^T [c(x(t)) + \alpha \cdot x'(t)\chi(0 < x'(t))]\psi(t)\, dt \tag{24}$$

over the set of continuous nonnegative functions x, differentiable in all but a finite number of points t, and such that $x(t) \geq \zeta(t)$, for all t.

Here again ζ—the dynamical parameter—is a function which is assumed to belong to a set Z_0 of continuous differentiable and nonnegative functions.

Independent of what the set Z_0 is assumed to be, it may be shown, following methods given in Arrow, Karlin [2], that there exists a horizon for every ζ in Z_0. In particular we have the following theorem.

THEOREM 7.1. *For a given $\bar{\zeta}$ and $T_0 > 0$, there exists a function $\bar{x}$ such that, for every ζ in Z_0 with $\zeta(t) = \bar{\zeta}(t)$ for $0 \leq t \leq T_0 + \max_{0\leq t\leq T_0} \alpha/c'(\bar{\zeta}(t))$, there exists a function x^* with $x^*(t) \geq \zeta(t)$, $0 \leq t \leq T$, $\mathfrak{F}_0^T(x^*) = \min_{x(t)\geq\zeta(t)} \mathfrak{F}_0^T(x)$ and $x^*(t) = \bar{x}(t)$, for $0 \leq t \leq T_0$.*

This theorem may be stated briefly, as follows.

In every family with parameters in Z_0, $H = T_0 + \max_{0\leq t\leq T_0} \alpha/c'(\bar{\zeta}(t))$ is a horizon for an arbitrary function $\bar{\zeta}$.

Neither of the theorems we have given in this section is stated in its strongest form. In both cases the defined horizon is not the shortest one for a given parameter $\bar{\zeta}$. For the sake of simplicity we have taken their weaker form, since the aim in presenting them was only to give an example of horizons in dynamical programming problems with continuous time.

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