ON DISCRETE EVASION GAMES WITH A TWO-MOVE INFORMATION LAG

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1. Introduction

This paper deals with extensions of the following game. Although there are other interpretations of this game, we use the traditional one of a ship trying to evade a bomber. This problem is sometimes called the bomber-battleship problem.

A ship is constrained to travel on the integer lattice of the real line. In one time unit, he may move one unit distance to the right or one unit distance to the left. He must move each time; he is not allowed to stay still. A bomber with exactly one bomb flies overhead and wants to drop the bomb on the ship. He may drop the bomb on any point he desires, but it takes two time units for the bomb to fall. He knows that at the end of two units time the ship can only be at one of three places: exactly where it is when he lets go of the bomb, two steps to the right, or two steps to the left. There is no use in dropping the bomb at any but these three points. The ship starts at the origin, and he does not know when or where the bomb is dropped until it hits. The bomber may observe the movements of the ship for as long as he likes before dropping the bomb. He wins one unit from the ship if the bomb hits the ship; otherwise, there is no payoff.

This game-theoretic problem, suggested by Rufus Isaacs, was first solved by Dubins in 1953 and published in [2]. It was solved independently by Isaacs and Karlin [4] using a different method. Further results were obtained by Isaacs [3] and by Karlin [5]. A general theory of games with information lag was studied by Scarf and Shapley [7].

In papers [2] and [4], it is shown that this game has a value $v_1 = (3 - \sqrt{5})/2 = .382\ldots$, that the ship has an optimal strategy which is explicitly described, and that the bomber does not have an optimal strategy (merely ϵ-optimal ones). In other words, there is a strategy for the ship such that no matter what strategy the bomber uses, the ship will not be hit with probability greater than v_1, and, for every $\epsilon > 0$, there is a strategy for the bomber such that no matter what strategy the ship uses, the ship will be hit with probability at least $v_1 - \epsilon$.

A strategy for the ship is a rule telling him at each step with what probability he should go to the right as a function of all his past moves. Such a strategy is said to be *Markov* if this probability depends on the past moves only through

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the preceding step. If this probability is also independent of time, the strategy is said to be *stationary Markov*. If the ship uses a (stationary) Markov strategy, the sequence of choices of left or right forms a (stationary) Markov chain. In the above problem, the ship has an optimal strategy which is stationary Markov and invariant under interchange of left and right. This strategy is as follows. At time zero, go right or left, each with probability $\frac{1}{2}$. At all future times, go in the same direction as the last move with probability $(\sqrt{5} - 1)/2 = .618 \ldots$, and go in the opposite direction with probability $(3 - \sqrt{5})/2 = .382 \ldots$.

This strategy insures that the ship will not be hit with probability greater than $v_1 = (3 - \sqrt{5})/2$, as is easily checked. To show that v_1 is in fact the value of the game is a more difficult problem. Dubins solved it by showing that the game has a value and that v_1 is the upper value of the game. Isaacs and Karlin solved it by dealing with the fundamental functional equation of this game. A third method is presented below in which an ϵ-optimal strategy for the bomber is explicitly exhibited.

The main objective of this paper is to give an extension of this result, where the ship is allowed to travel on a more general graph. Except for some related problems treated by Blackwell [1] and Matula [6], I know of no other extensions of this problem which have been solved. In section 5, some unsolved problems, possibly more direct generalizations of the main problem, are stated.

2. Graphs

By a graph, we mean a pair (V, E) where V is a set of points or vertices, and E is a set of *unordered* pairs of points of V. The elements of E are called paths or edges. In the following two sections, we treat the extension of the problem mentioned in the first section to the situation where the ship is constrained to travel on such a graph from one vertex to another vertex along one of the available edges in one time unit. It is immaterial whether or not there is more than one edge joining two vertices; for simplicity, we assume that to any pair of vertices there is at most one edge joining them. However, it is important that there does not exist an edge which joins one point to itself. One may take it that the definition of a graph excludes such a possibility. We prefer to list this restriction explicitly with two other restrictions which we place on graphs.

We define a *restricted n-graph* to be a graph which satisfies the following three conditions.

(i) *There does not exist an edge which joins a vertex to itself.*

(ii) *There are no four-sided figures.* In other words, there do not exist four distinct points A, B, C, and D such that $\{A, B\}$, $\{B, C\}$, $\{C, D\}$, and $\{A, D\}$ are edges.

(iii) *There are exactly $n + 1$ distinct vertices joined to each vertex by an edge.* It is assumed that n is a positive integer.

It should be noted that three-sided figures are allowed, but that (ii) requires that no two distinct three-sided figures have a common edge.

The lattice of integers on the real line is a restricted 1-graph. Other examples when $n = 1$ may easily be constructed for any finite number k of vertices in V, provided $k \geq 3$ and $k \neq 4$.

The hexagonal lattice in the plane provides an example of a restricted 2-graph (figure 1). Another example of a restricted 2-graph is provided by the vertices and edges of a dodecahedron, the regular polyhedron with twelve faces made of regular pentagons, with twenty vertices and with thirty edges.

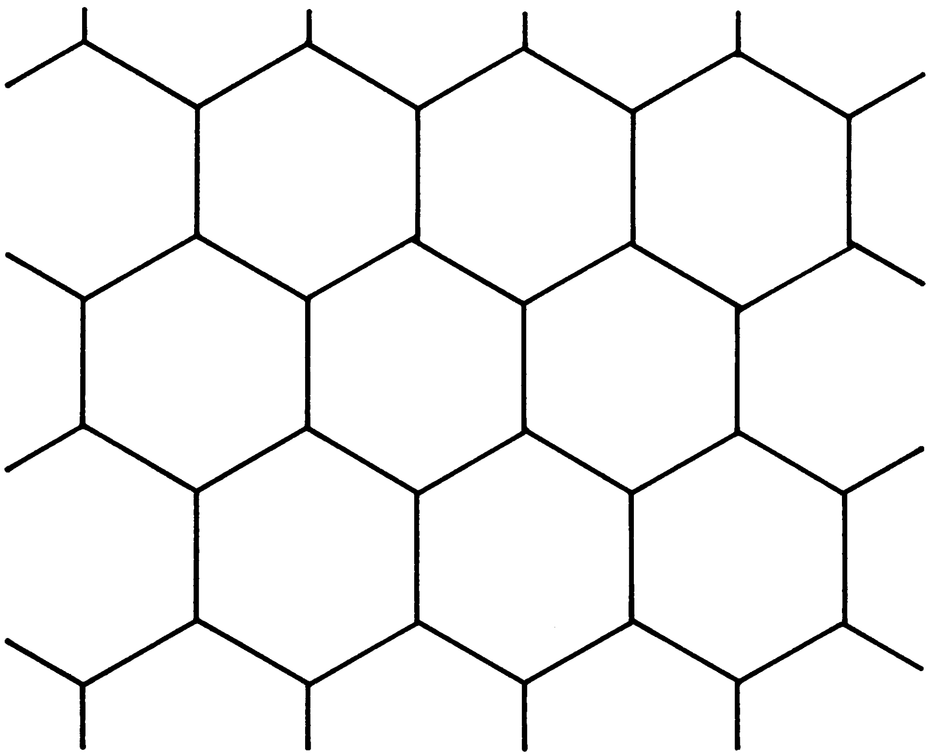

FIGURE 1

For arbitrary n, analogues of the hexagonal lattice in the plane can be described in n-dimensional space to provide examples of restricted n-graphs. Finally, figure 2 provides an example of a regular graph in the plane which is a restricted 3-graph.

3. An optimal strategy for the ship

In this section, we define a class of symmetric stationary Markov strategies for a ship traveling on a restricted n-graph. The minimax strategy within this class is then derived. That this strategy is in fact optimal within the class of all

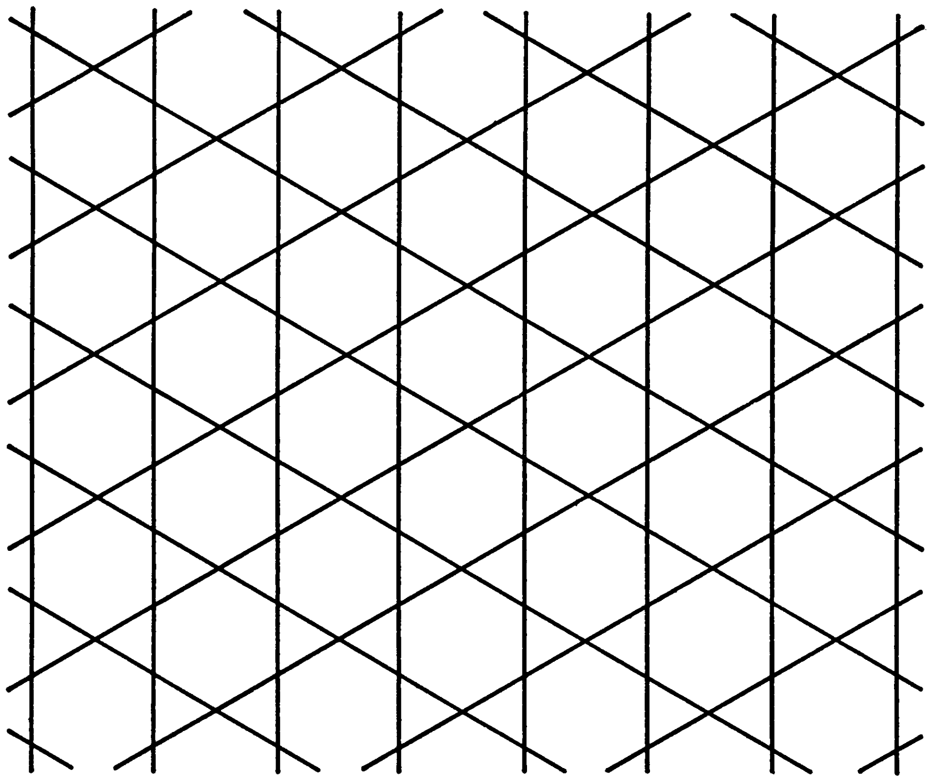

FIGURE 2

strategies is shown in the following section by exhibiting an ϵ-optimal strategy for the bomber.

Consider the following class of stationary Markov strategies for the ship. The first step is chosen at random among the $n + 1$ points one step away, each point having probability $1/(n + 1)$. From then on, the probabilities depend only on the point just vacated, the probability of a return to that point being $1 - np$ where $0 \leq p \leq 1/n$, and the probability of a trip to any of the other n available points being p. We shall find that value of p which minimizes the maximum probability of being at any point two steps ahead among this class of strategies.

At the very start, there is probability $p/(n + 1)$ of being at each of the $n^2 + n$ points two steps away from the starting point (exclusive of the starting point). The restriction (ii) on the graphs is used here to imply that these $n^2 + n$ points are distinct. The probability of returning to the starting point in two steps is $1 - np$.

At any time later, the probability of advancing to any of the n^2 points two steps away not passing through the point just vacated is p^2 for each point. The probability of retreating to any of the n points two steps away passing through the point just vacated is $p(1-np)$ for each point. The probability of returning to the point occupied after two steps is $1-np$.

To find the minimum of the maximum of these three functions, p^2, $p(1-np)$, and $(1-np)$, we note first that $p(1-np) \leq (1-np)$ for all p. Then, since p^2 is increasing in p and $1-np$ is decreasing in p, the minimax strategy is found as that value of p for which $p^2 = 1-np$. This equation has two roots $p = (-n \pm \sqrt{n^2+4})/2$, of which one

$$p = (\sqrt{n^2+4} - n)/2 \tag{1}$$

is positive. The minimax value, denoted by v_n is easily found to be

$$v_n = p^2 = 1 - np = (n^2 + 2 - n\sqrt{n^2+4})/2. \tag{2}$$

We must check that the probabilities of being at the various points after the very first steps does not exceed v_n. That is, we must show $p/(n+1) \leq v_n$ and $1 - np \leq v_n$, where p and v_n satisfy (1) and (2). The inequality $1 - np \leq v_n$ is obviously satisfied with equality from (2). The inequality $p/(n+1) \leq v_n$ is equivalent to $(n+1)p \geq 1$ or to $(n+1)\sqrt{n^2+4} \geq n^2 + n + 2$, which is easily checked by squaring both sides.

When $n = 1$, we find that $p = (\sqrt{5} - 1)/2$ and $v_1 = (3 - \sqrt{5})/2$, the solution given by Dubins. When $n = 2$, we find $p = \sqrt{2} - 1 = .414\ldots$ and $v_2 = 3 - 2\sqrt{2} = .172\ldots.$ Thus, on the hexagonal lattice of figure 1, the optimal policy of the ship is to return to the point just vacated with probability $.172\ldots$ and to move to one of the other two points with probability $.414\ldots$ each.

4. An ϵ-optimal strategy for the bomber

Let $w < v_n = (n^2 + 2 - n\sqrt{n^2+4})/2$. We will describe a strategy for the bomber which guarantees him that he will hit the ship with probability at least w.

Let the initial position of the ship be called point A. At time zero, drop the bomb on point A with some small probability p_0, the exact value of which will be determined later. The first step of the ship carries him to some point which we shall call B. At time one, drop the bomb on point B and each of the n^2 points two steps away from B not passing through A with equal probabilities p_1 each. (The p_i are unconditional probabilities, *not* conditional probabilities, given that the bomb has not been dropped.)

If the next step of the ship is to a point other than A, he is bound on step three to go to a vertex hit with probability p_1. Hence, the conditional probability that he will be hit, given that the bomb is dropped at times zero or one, is $p_1/(p_0 + (n^2+1)p_1)$. We require of p_0 and p_1 that

$$w = p_1/(p_0 + (n^2+1)p_1). \tag{3}$$

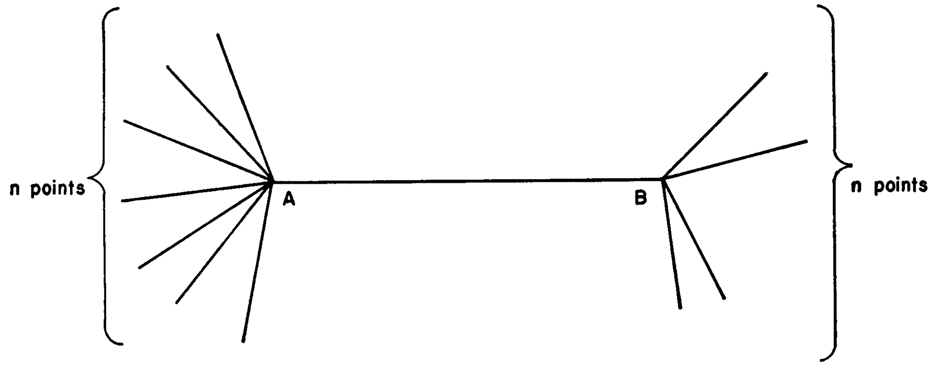

FIGURE 3

Thus, if the ship does not return to point A at step two, he will be hit with conditional probability w, given that the bomb was dropped at times zero or one. If, in this case, the bomb was not dropped, we may start over. Eventually (with probability one), the bomb will be dropped, and we will have probability w of hitting the ship.

Suppose now that the ship returns to A after two steps. He immediately gets hit with probability p_0, or with conditional probability $p_0/(p_0 + (n^2 + 1)p_1)$, given that the bomb was dropped at times zero or one. If this conditional probability is at least w, we are finished. Ordinarily, however, it will be impossible to choose p_0 and p_1 satisfying (3) and $p_0/(p_0 + (n^2 + 1)p_1) \geq w$. In such a case, we bomb A and the n^2 points two steps away from A not passing through B with equal probabilities p_2 each. If the ship goes to a point other than B at step three, he is bound at step four to go to a vertex hit with probability p_2. The overall conditional probability that he will be hit is $(p_0 + p_2)/(p_0 + (n^2 + 1)(p_1 + p_2))$. We require

$$w = (p_0 + p_2)/(p_0 + (n^2 + 1)(p_1 + p_2)). \tag{4}$$

Thus, if the ship does not return to B at step three, he will be hit with conditional probability w, given the bomb has been dropped.

If he does return to B at step three, he gets hit with probability p_1, so that if $(p_0 + p_1)/(p_0 + (n^2 + 1)(p_1 + p_2)) \geq w$, we are finished. But again, it is likely that equalities (3) and (4) imply $(p_0 + p_1)/(p_0 + (n^2 + 1)(p_1 + p_2)) < w$, so that in order to obtain probability w of hitting the ship, we again bomb A and the n^2 points two steps distant not going through B with equal probabilities p_3 each. We continue in this manner hoping that after some k steps we will have not only

$$\begin{aligned} \frac{p_1}{p_0 + (n^2 + 1)p_1} &= \frac{p_0 + p_2}{p_0 + (n^2 + 1)(p_1 + p_2)} = \cdots \\ &= \frac{p_0 + p_1 + \cdots + p_{k-2} + p_k}{p_0 + (n^2 + 1)(p_1 + \cdots + p_k)} = w, \end{aligned} \tag{5}$$

but also

$$\frac{p_0 + p_1 + \cdots + p_{k-2} + p_{k-1}}{p_0 + (n^2 + 1)(p_1 + \cdots + p_k)} \geq w. \tag{6}$$

Inequality (6) together with the last equality of (5) imply that $p_{k-1} \geq p_k$. If w were not less than v_n, then equalities (5) would imply, as we shall see, that the p_i are strictly increasing, so that there would not exist a k such that (6) is satisfied. But all we need to show is that if $w < v_n$, then (5) and (6) may be satisfied.

In the equations

$$\begin{aligned} w &= \left(\sum_0^j p_i - p_{j-1}\right) \Big/ \left(p_0 + (n^2+1) \sum_1^j p_i\right) \\ &= \left(\sum_0^{j+1} p_i - p_j\right) \Big/ \left(p_0 + (n^2+1) \sum_1^{j+1} p_i\right), \end{aligned} \tag{7}$$

the ratio of the difference of the numerators to the difference of the denominators is also w:

$$w = (p_{j+1} - p_j + p_{j-1})/((n^2+1)p_{j+1}) \qquad \text{for } j = 1, \cdots, k-1. \tag{8}$$

In another form,

$$p_{j+1} = \alpha(p_j - p_{j-1}) \qquad \text{for } j = 1, \cdots, k-1 \tag{9}$$

where

$$\alpha = (1 - (n^2+1)w)^{-1}. \tag{10}$$

The difference equation (9) is to be solved subject to the boundary condition (3), rewritten as

$$p_1 = (\alpha - 1)p_0/(n^2+1). \tag{11}$$

Since the equations (5) and (6) involve only ratios of the p_i, we may arbitrarily set $p_0 = 1$ for the purposes of the computation, and later norm the p_i so that $p_0 + (n^2+1)(p_1 + \cdots + p_k) = 1$. Thus we add the boundary condition,

$$p_0 = 1. \tag{12}$$

The general solution of (9) is

$$p_j = C_1\left(\frac{\alpha + \sqrt{\alpha^2 - 4\alpha}}{2}\right)^j + C_2\left(\frac{\alpha - \sqrt{\alpha^2 - 4\alpha}}{2}\right)^j \tag{13}$$

where C_1 and C_2 are arbitrary constants. Boundary condition (12) implies that $C_1 + C_2 = 1$. Boundary condition (11) implies

(14)

$$(\alpha - 1)p_0/(n^2+1) = C_1((\alpha + \sqrt{\alpha^2 - 4\alpha})/2) + (1 - C_1)((\alpha - \sqrt{\alpha^2 - 4\alpha})/2),$$

or equivalently,

$$2C_1 = 1 - \frac{(n^2-1)\alpha + 2}{(n^2+1)\sqrt{\alpha^2 - 4\alpha}}. \tag{15}$$

We want to show that all quantities involved are real, provided that $w < v_n$ and w is sufficiently close to v_n. This amounts to showing that $\alpha > 4$. But $\alpha > 4$ if and only if $\frac{3}{4} < (n^2 + 1)w < 1$. In other words, we must show $\frac{3}{4} < (n^2 + 1)v_n \leq 1$. But it is easily checked that $(n^2 + 1)v_n$ is an increasing function of $n \geq 1$, that $(n^2 + 1)v_n \to 1$ as $n \to \infty$, and that $(n^2 + 1)v_n$ for $n = 1$ is $3 - \sqrt{5}$, which is between $\frac{3}{4}$ and 1.

The question remaining is whether or not (6) can be satisfied for some k. As noticed, this is equivalent to asking if for some k, $p_k \leq p_{k-1}$. Since $\alpha > 4$, this question is answered affirmatively provided $C_1 < 0$, as can be seen from equations (9) and (13). But from equation (15), $C_1 < 0$ if and only if (when $\alpha > 4$)

$$n^2\alpha^2 - n^2(n^2 + 3)\alpha - 1 < 0 \tag{16}$$

or, equivalently, if and only if

$$w^2 - (n^2 + 2)w + 1 > 0 \tag{17}$$

using (10). This is satisfied if $w < v_n$ as is easily seen.

In summary, for a given w satisfying $\frac{3}{4} < (n^2 + 1)w$ and $w < v_n$, compute α from (10) and C_1 from (15). You will find that $\alpha > 4$ and $C_1 < 0$. Let $C_2 = 1 - C_1$ and compute p_j from (13). Let k be the first integer for which $p_k \leq p_{k-1}$. Normalize these numbers by dividing by a constant so that $p_0 + (n^2 + 1) \sum_1^k p_i = 1$. With probability $(n^2 + 1)p_j$, the bomb is dropped at time j, provided the ship has alternated between the points A and B up to that time, and then the target is chosen at random among the point where the ship is and the n^2 points two steps away from that point not reversing direction, each of the $n^2 + 1$ points being equally likely. If, at any time, the ship goes to a point other than A or B, and the bomb has not been dropped, the whole procedure is repeated starting at the new point. Eventually with probability one, the bomb will be dropped giving probability at least w of hitting the ship.

5. Unsolved problems

There are (at least) three natural extensions of the problem mentioned in the first section which the present methods do not treat and which are still unsolved problems.

Problem 1. Give the ship the option of staying still. Thus at the end of two moves, he may be at one of five positions: where he is now, one or two steps to the left, or one or two steps to the right. The restricted n-graph could be modified to contain this problem by removing condition (i).

Problem 2. Consider the original problem on the square lattice on the plane. Such a graph does not satisfy condition (ii) of the restricted 3-graph.

Problem 3. Assume that the bomb takes three moves to fall. This is known as the three-move lag problem.

For each of these problems, it is known that the game has a value, that the ship has an optimal strategy, and that the bomber does not have an optimal strategy. The reason that these three problems seem more difficult than those treated in this paper is that those treated in this paper have optimal stationary Markov strategies for the ship, whereas for the above three problems it is conjectured that no stationary Markov strategy is optimal.

The outstanding unsolved problem in my opinion is number three, the three-move lag problem on the integer lattice on the real line. In this problem it is known that the optimal strategy of the ship is not Markov. But since the bomb takes three moves to fall, one would expect the ship to use 2-dependent strategies, those strategies which depend on the past only through the two previous positions. The general stationary 2-dependent strategies invariant under interchange of right and left may be described by two probabilities q_0 and q_1, where q_0 represents the probability of continuing in the same direction as the last move, given that the last two moves were in the same direction, and where q_1 represents the probability of continuing in the same direction as the last move, given that the last two moves were in opposite directions. For stationary Markov strategies $q_0 = q_1$. The best that can be done with these stationary Markov strategies is to choose $q_0 = q_1 = \frac{2}{3}$, and this ensures that the ship will not be hit with probability greater than $(\frac{2}{3})^3 = \frac{8}{27} \sim .296 \ldots$. I claim that the best that can be done with these stationary 2-dependent strategies is to choose $q_0 = (3 - \sqrt{3})/2 \sim .634 \ldots$ and $q_1 = \sqrt{3} - 1 \sim .732 \ldots$, and this ensures that the ship will not be hit with probability greater than

$$3(3\sqrt{3} - 5)/2 \sim .294 \ldots . \tag{18}$$

It is unknown whether or not this last strategy is optimal. In fact, it has been conjectured that no strategy with finite memory (that is, a strategy which depends only on the last m moves for some finite integer m) is optimal for the three-move lag problem.

Blackwell [1] treats a class of problems on the prediction of sequences, very similar to those treated in this paper. Matula [6], in extending some of these results, gives an example of a problem with a four-move lag in which he finds an optimal strategy without finite memory for the sequence chooser (the ship), and gives convincing arguments that no optimal strategy for the sequence chooser has finite memory. Such a state of affairs could hold for the three-move lag problem above. A lower bound on the value, v, of the three-move lag problem has been obtained by investigating strategies of the bomber similar to the ϵ-optimal strategies found in the previous section for the two-move lag case. Using methods found there and analyses twice as complicated, it may be shown that $v \geq \frac{23}{81} = .284 \ldots$. Combined with the upper bound of (18), we have $.284 \ldots \leq v \leq .294 \ldots$.

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