TOWARD AN OBJECTIVISTIC THEORY OF PROBABILITY

EDWARD W. BARANKIN
UNIVERSITY OF CALIFORNIA

CONTENTS

1. Introduction

It is the purpose of this article to set forth the initial ideas and implications of a mathematical theory first expounded in the RAND Corporation Research Memorandum RM-900. In that paper, and in the abstract [3], we refer to the theory as a *theory of behavior;* and this is a more adequate description. For, not only does the theory purpose to locate the concept of probability properly in the context of reality; it also provides a truer precisement of the concept of human motivation—what in the literature of economics and econometrics is designated by the word "utility"; and it implies an inherent discreteness attending all real behavior, in agreement with the quantum theory of physics. In short, the theory defines no boundaries for itself; it has, quite to the contrary, the force of asserting that when probability is correctly conceived in its intimate connection, nay, identification with other fundamental notions of science, then there emerges the structural unity of all reality, the conceptual oneness of all behavior, whether of physical particles or of machines or of human beings.

Reality is a going affair. It is the sum total of the acts of systems that our senses lead us to isolate and posit, and of the underlying determinants of these acts. Hence our designation: theory of behavior. And the core of the theory is the characterization of these underlying determinants and the specification of the law of actualization of the mentioned acts.

But the import of the theory goes much further. It forcefully suggests that the *systems* just spoken of—that is, for example, an electron, a baseball, an automobile, a lady's hat, a dog, a human being, a group of human beings, etc.—are indeed *complex* posits of our classical five senses. That these "material" things are *not* the pristine stuff of reality.

The author is indebted to the RAND Corporation for its support in the early stages of this research. This paper was prepared with partial support of the Office of Naval Research.

That the primitive real entities are other things altogether, and systems are structured complexes of these primitive entities.

The intuitions (i) that human behavior must admit the precise analysis of mathematics, and (ii) that there is an underlying unity to reality—these are vigorously represented today by the theory of games of von Neumann and Morgenstern [22] and the cybernetics of Wiener [24], respectively. (And it is no mere coincidence that probability plays an integral part in both.) Weyl has stated the case well [23, p. 214]: "There is no reason to see why the theoretical symbolic construction should come to a halt before the facts of life and of psyche. It may well be that the sciences concerned have not as yet reached the required level. But that this limitation is neither fundamental nor permanent is already shown by psychoanalysis, in my opinion. The fact that in nature 'all is woven into one whole,' that space, matter, gravitation, the forces arising from the electromagnetic field, the animate and inanimate are all indissolubly connected, strongly supports the belief in the unity of nature and hence in the unity of scientific method. There are no reasons to distrust it." The prominence of mathematicians among the supporters and developers of these intuitions should come as no surprise. For, alone on the face value of mathematics it is clear that the peculiar advantage of the mathematician, or at least of those who do not get bogged down in the provincialisms of their field, is his capacity to see through vastnesses of complication to the essential form and structure of phenomena.

But this turning of attention by a few far-sighted mathematicians to the problems "of life and of psyche" may have a significance that in fact goes very deep. Mathematics is a discipline that is almost universally taken for granted. Most of those who have not enjoyed extensive training in it look upon it as a completely sustained other world. To most natural scientists it is like an old familiar, but overabundant, spice rack to a chef, with some spices occasionally hard to find and with many others that he is sure he will never have need of. And even for many mathematicians it is quite sufficient unto itself, by reason of the wealth of well-defined, resolvable problems that exists within the framework of its concepts. Yet, the fact is that the foundations of mathematics are, still today, not satisfactorily clarified. The difficulties revolve immediately around the notion of "set" or "class." In the face of this fact, is it not reasonable to conjecture that these men mentioned here above (as well as others) whose profound mathematical insight has been proved to the satisfaction of everyone, have been led by that *self-same* insight to take up the analysis of animate behavior? That this study must *not* be a digression from the vital evolution of mathematics? That it is, rather, an indispensable stage in the coming to a complete understanding of the foundations and of the meaning of mathematics? Of all this there is not a single doubt in the opinion of this author. If the hierarchy of our scientific concepts is to undergo a drastic reorganization, there is no reason to believe that the concepts of mathematics, and of logic (which is comprehended by mathematics), must remain undisturbed in the shake-up. On the contrary, the extensive and forever expanding "applicability" of mathematics in the "applied" sciences challenges us to discover an—the—organic union of mathematics with all the rest of science, wherein mathematics would then appear, in its true light, as a "branch of the theoretical construction of the one real world" (Weyl [23, p. 235]).

Such an organic union appears to be connoted by the theory we are presenting. We cannot yet offer it as a demonstrated fact. But with section 9 we hope to have imparted to the reader the strength of its necessity, so inseparably bound up is it with the other implications of the theory.

The severely circumscribed conception of mathematical science that many people have—scientists and nonscientists alike—causes in them a fearful reaction to the phrase "mathematical theory of human behavior." They preserve a vague, unprecise feeling that the presence of mathematical theory means something like certain predictability, strict determinism. Against this dark background they hold up the light of their self-determination, their free will, and they tremble at the presumed suggestion that this light is to be extinguished, to be proved a delusion. *It is the fear itself that is the delusion!* Consider what the fear rests on: a *feeling* that mathematical description implies a stern rigidity. This feeling analyzes into two components: a feeling about mathematical descriptions as such, and a feeling about mathematics *per se*. As to the former, it is not sound at all in consideration, for example, of quantum theory mathematics, which engenders the Heisenberg uncertainty, or, for that matter, of all the many successful applications of mathematical probability in physics. In regard to the latter component feeling, the point to be made is a much sharper one; namely, that it is utterly without defense in the absence of a full clarification of the foundations of mathematics. Thus, the feeling nowhere stands on firm ground. It is neither true that actual experience with mathematical description points to a pervading rigidity, nor is it true that the content and import of mathematics are sufficiently well understood to support any suspicion of an ultimate inflexibility of mathematical description. Like so many other fears, this one vanishes into thin air when we examine with a little care what we know and what we don't know.

The free will of the human being is indeed a light, and it must be the guiding light in any serious attempt to formulate a precise, general theory of behavior. The theory to be presented here has grown out of the effort to truly represent that human freedom of choice in a mathematical precisement of the notion of utility. The original mathematical idea was that this could be done with a *set function*, in contrast to the *point function* that had been used in all previous attempts. In section 2 we discuss this matter in detail, preparatory to giving our exact formulation. One of the most important steps in this section is the clarification of the words "rational" and "irrational" in regard to human behavior. These words have been the same stumbling block to understanding in the domain of human behavior that the words "magical" and "miraculous" once were to our understanding in the domain of physical behavior.

In section 3 we give the mathematical characterization of what we call a *personality process*. This is a structured complex of elements that we designate, in behavioral terms, as *acts*, *eventualities* and *utilities*. Thus, utility appears—and indeed as a set function, in the mathematical language—not superimposed on the usual conception of a human being, but in equal conceptual status with elements of two other kinds, composed in a structure—a personality process—which is to be viewed, at this stage of the development, as only a part of the human being in question. Several examples are given to illustrate the notions introduced.

In section 4 the law of behavior is enunciated, which governs the *actualization* of eventualities into acts, or (we say also) which governs the *evolution* of a process. The assertions of the law are, of course, in terms of the utilities of the process. Illustrative examples are followed through.

In section 5 we state three alternative forms of the law of behavior. The equivalence of the four forms of the law is proved in the paper [2], and we omit the proof from this article.[1]

[1] This proof will be published in the near future.

Section 6 is concerned with the delicate task of establishing the conceptual oneness of personality processes and stochastic processes. It is in this section that the universality of the theory begins to unfold. It is here that the identity of utility and probability, of free will and uncertainty, is propounded. This done, the theory of probability is an immediate corollary of the theory of utility: they are one and the same theory! And from this point on we speak in the language of probability. In this section we give critical attention to the ideas of past authors on the concept of probability.

In section 7 the over-all structure of reality is presented. And here it appears that a human being, or any system, *is* a stochastic (personality) process. In conjunction with this our proposal is set forth, that acts, eventualities and probabilities (utilities) are the primitive objective elements of reality. In this section we discuss the immediate implications of our theory for quantum mechanics. In particular, we present the very simple proof of the inherent discreteness of all behavior. In this section, also, we exhibit how (in anticipation of the successful resolution of the still-open questions at the foundations of our theory) the "causal" laws of classical physics are comprehended by the theory.

Section 8 is devoted to a brief explanation of induction and statistics, as they appear in the light of the theory.

In section 9 we give some indications toward the deeper foundations of the theory, and we indicate here the unavoidable implications of the theory for the basic problems of mathematics.

Finally, in section 10 we speak briefly of the rather striking assertions of the theory in regard to language.

2. Toward a new conception of utility

It has quite naturally fallen to economics, where the concern is with a fairly narrow and well-defined segment of human behavior, to sponsor the most intensive mathematical investigations into the nature of human motivation. There is a complete theoretical format within which this work has gone on: a mathematical description tying the individual to his potential acts and to the acts, past and potential, of the environment; and a principle or law operative on this description to say which of the individual's acts will eventuate, or which acts will not eventuate. The mathematical description in question is to embody the individual's propensities or motivations to his various potential acts, relative to the acts of the environment; this part is called the individual's *utility* (pattern or function)[2] relative to the environment, the word "utility" connoting an interested valuation *by* the individual of his potential acts. On this account it is a good word, for it carries an appreciation of the human freedom of choice.

The particular mathematical form given to utility, and the concomitant law of behavior, have remained mostly the same[3] through the entire history of econometric investigation: utility is a point function on the space of acts concerned, and the law asserts (to put it briefly and therefore roughly) that the individual selects such an act that, in consideration of the acts of the environment and of his knowledge of them, no other

[2] Econometric terminology is far from uniform—nor could it be expected to be, while the sorting out of concepts is still going on. Our presentation here of the word "utility" in a straight over-all sense (that is, pertaining to acts, not consequences) follows the spirit of von Neumann and Morgenstern [22].

[3] An exception is the suggestion of G. C. Evans (see [10]) that utility be considered more generally as a function of paths in the commodity space.

available act has higher utility.[4] The terrible struggle of recent years to make something effective out of this kind of theory is well represented in the pages of *Econometrica* since 1950. The struggle has not brought us very far. Small modification has followed upon small modification, and every proposed form again admits the most immediate and abundant counterinstances in the actual behavior of human beings.[5] The kind of situation toward which these theoretical attempts have been directed is not one in which real-life conditions have been summarily hypothesized away. On the contrary, it is a wholly realistic one and on that account exacting to the extreme for the theorist. It is, namely, the situation in which the individual faces environmental uncertainty. (The impetus to this was the theory of games, more exactly the publication of the book [22].) This meant that theory could not confine its attention to utility in isolation, but had necessarily to try to understand the interplay between utility and probability. It is against this challenge that the point-utility type of theory has proved so ineffectual.

These past attempts at theory have been dominated by the words "rational" and "irrational." There is the underlying presumption that "rational behavior" is a fundamental, objective category of actual behavior, and the theory is aimed at analyzing this. At first it would appear that this manner of proceeding represents a close adherence to the scientific method. And it certainly does when there is ample evidence for the objectivity of a proposed category of the behavior under study. But this cannot be said of "rational behavior."

Consider how the words rational and irrational are used in statements regarding people's behavior. In consideration of certain data relating to the individual (and to his environment), an act or sequence of acts is called rational or irrational. It is called rational if it is understandable on the basis of the data, and irrational if it cannot be comprehended as an implication of the data. Now, to speak of comprehensibility and incomprehensibility is to say that a theory is at hand; and to classify the behavior into one of these two categories is to say that the given data fulfill the data requirements of the theory and therefore the theory is definitely verified or definitely not verified. Thus, rationality = comprehensibility = verification of the theory. In short, *rationality is not a concept pertaining to behavior, it is a concept pertaining to theories of behavior.*

The historical failure to grasp this point would seem to be due entirely to men's impatient and unquestioning fervor for their ideals. It is said, for example, that a man is irrational if he kills without provocation. One is avoiding the issue to maintain that this means simply that the man killed without provocation—that it is intended to convey just this fact and nothing more. In its general use, the word is well known to convey, in the instance cited here and in the many others in which it occurs, the idea of a deeper, pervading quality in the being of the individual. It is considered "natural" that a man does not kill without provocation. An act of killing without provocation is therefore "unnatural," "contrary to the nature of things," "wrong in its very essence,"—"irrational." The quotations here are the loaded ideals of everyday thinking. But now let us go back over these two sentences ending with the word irrational and restate them after

[4] We have already lapsed into using the word utility for both the function and a value of the function. Where there is a possibility of confusion we shall be careful to make ourselves explicit.

[5] One can begin the perusal of the literature reasonably with the articles [1], [5], [15]. Among the most recent publications is the book [19], which, in advocating the point-utility ideas in the strongest way it knows how, reveals their utter failure: it sets down postulates, in the scientific manner, but all the while urging that the principal use for these "postulates" is the policing of one's own actions! The methodology of science is humiliated and debased to the service of masking a proposal of an equivocal code of conduct.

searching our consciences for the meanings of the phrases in quotes. They would read as follows: If a man kills with some provocation, then we understand why he has killed. An act of killing without provocation is incomprehensible. This is the honest statement of the case. The theory in question is this: a man does not kill without provocation. Further, the theory will carry, among other things, some indications for recognizing provocation. The circumstances surrounding a particular act of killing are the pertinent data. If these data permit us to say that there was definitely provocation or there was definitely no provocation, according to the indications set out in the theory, then the data requirements of the theory are fulfilled and it can be said either that the theory is verified or that it is not verified.

Recognized for what they are, the words rational and irrational become entirely unnecessary in the field of behavior theories. Not only unnecessary, but also dangerously misleading by virtue of their prevalent absolutistic connotation. The word irrational is the exact historical counterpart for human behavior of the word miraculous (or magical) in relation to physical behavior. At this day we have long since emerged from the sway of the latter word. In the physical sciences, when a theory is put forth, we do not append to it the qualification "provided the behavior is not miraculous." For, this would be merely to utter the useless phrase "provided the behavior is in accordance with the theory." Instead, it is simply understood that if a new phenomenon is observed which is within the domain of the theory but is not explained by it, then new, deeper theory must be awaited and sought for. Likewise there will come a day when the words rational and irrational have gone out completely from all discussions of human behavior.

There is no need for us to enter here into a detailed critique of point-utility theories. We shall, before beginning the presentation of our own ideas, only make some general observations. Throughout the literature we have been referring to, one finds statements such as the following: "The tangled web of the problem of human free will does not really have to be unravelled for our purpose; surely, in any case, our ignorance of the world is so much greater than the 'true' limits to possible knowledge that we can disregard such metaphysical questions," and "The departure [of people's behavior from accord with the proposed theory] is sometimes flagrant, in which case our attitude toward it is much like that we hold toward a slip in logic, calling the departure a mistake and attributing it to such things as accident and subconscious motivation." These examples illustrate the spirit that has governed past research. The separation of conscious and subconscious is taken as self-evidently sensible, and utility is considered without question to belong strictly to the conscious level. Those who persist in maintaining that there is a sharp differentiation between conscious and subconscious behavior must acknowledge a serious weakening of their position as a result of the lack of success of point-utility to produce theory of any adequacy. On the other hand, it is no surprise to those of us who sense the fully integrated nature of all behavior—for which there is abundant evidence. For us there is the confident intuition that such attempts at theory were doomed to failure from the start; and if there is any point of general interest for us emerging from this theoretical experience, it is that the failure has been so complete. There is the strongest indication that success can be met with only in concert with a sweeping away of the boundaries between physical and metaphysical, conscious and subconscious.

It is interesting to see that as soon as the artificial boundary between conscious and subconscious (accidental, unintentional, etc.) is dropped, then the idea of point-utility is immediately cast under serious suspicion. Let us take the extremely simple example of three potential acts: the individual may choose an apple, an orange or a banana. Each of these acts corresponds to a point of a selection space consisting (consequently) of three points in all. With each of these points is associated a utility: the individual's utility (propensity, motivation) for choosing the apple, etc. This is as far as the conscious-point-utility conception takes us. But now (to illustrate the salient point in one of many possible ways) consider that the individual may choose the orange and—without paying particular conscious attention to the matter—pick it up from the table with either his left hand or his right hand. Thus, the potential act of choosing the orange may[6] in fact be the following: the act of picking up the orange with his left hand or picking up the orange with his right hand. The act of choosing the orange is then seen to be the *union* of the two acts (1) picking up the orange with his left hand, and (2) picking up the orange with his right hand. Shall we believe that the union of these two acts has a utility and the individual acts themselves do not have utilities? Surely there is no reason to believe this to be the case if we consider that there is no separation between conscious and subconscious. Another question, then: Shall some unions of potential acts be considered to have utilities and others not? It is difficult to see what support there could be for such a contention. Thus it emerges that for example the following act must be considered to have a utility: the act of choosing either an apple or an orange. That is, every subset of our selection space must have a utility associated with it. Thus we find ourselves denying that utility is a point function—and more: we find ourselves coming to see that *utility is a set function.*

We have pointed out that the problem faced has not been that of utility alone, but that of the pair of concepts: utility and probability. And for this reason: that the situations under scrutiny were those of individuals confronting uncertain environments. It is well appreciated today that each of the words "individual," "uncertain" and "environment" is without unequivocal clarification. The question that raises itself is: can the conception of an individual in an uncertain environment be effectively analyzed without aiming directly at a full explication of all three of the constituent concepts involved? In the spirit of the scientific method, it would be agreed by most that at any rate the attempt ought to be made. The attempt has been made. Current mathematical theorization takes for self-evident the compartmentalization: (1) the conscious individual, (2) the subconscious individual, and (3) the environment, together with their usual perceptual and physical interconnections; and it proceeds by superimposing on this base, in one way or another, the two additional interconnective elements, utility and probability. These theories thus continue the classical line of attack, modestly viewing motivation (utility) and uncertainty (probability) as only two more in a long sequence of concepts whose cumulative aggregate would, as time goes on, gradually converge toward full resolution of the nature of reality; a resolution which, by its structure, would thus verify the presumed soundness of the classical intuitive compartmentalization. *In contrast to this, it is the contention of the theory we shall expound that, with the notions of utility and probability, science has at last come upon the single key to the full resolution of the nature of reality.* There is no question of an infinitely enduring placement of new concepts one upon another, asymptotically revealing all the interconnective features between the compartments of our classical conceptual picture. What takes place

[6] We use the word "may" here to allow for the possibility that the nature of the individual and the circumstances are such that there is not the potential act of the individual's picking up the orange with, say, his left hand.

instead is a reorganization of the conceptual picture, wherein utility and probability take a ground level position. And, in particular, a human being and his environment are concepts to be *defined* in the terms of this new conceptual structure. It is the burden of our theory that, in the first place, the modesty of the classical mode of attack on the concepts of utility and probability is no virtue in itself, and secondly that the overwhelming evidence today for the conceptual integration of *all* knowledge is ignored in one way or another by all present theorization, and it is this that is at the root of its inadequacy.

We can usefully draw this section to a close by looking at some instructive indications. We pointed out in the preceding paragraph that current theorizing proceeds on the classical basis of viewing the human being and his environment as distinct, differentiable essences. We would suggest for the reader's consideration that this basis is compellingly controverted by the present status and impetus of the art of prosthetic surgery. The old peg leg is not a very far throw from the walking cane, and the statement, that it is a part of his environment which the man fashions to his use, need not leave us with any particular reservations about our classical differentiation of the man and his environment. But an artificial heart valve, on which the man's very life depends, must disturb our complacency in our old conceptions. The interpenetration of the man and the environment (which is, incidentally, also strikingly exemplified by alimentation, or by infectious disease and internal medication) that we see in such examples is strong evidence toward the need for a thoroughgoing revision of our conceptual framework. In the very next section it will become apparent that our theory does not go forth from any presumptions of self-evidence concerning the man-environment couple.

Consider a man who, in the period of time we shall think of, is sitting in on a poker game, purchasing some lottery tickets and contemplating in what direction to push his business. This is an epitomized example of a human being facing an uncertain environment, as treated in the econometric literature. The man has certain *utilities* for his various potential acts in these circumstances. Now let us enlarge the situation: let there be on hand, recording the man's acts, a psychologist who is interested in the "factors" that govern a man's behavior in risky situations. This psychologist will look forward to making statistical analyses of his data, which is to say that he is concerned with the *probabilities* of the man's various potential acts. An immediate question presents itself: what is the connection between these utilities and these probabilities? Current theory is not precise on this question. On the other hand, our theory presents a completely definitive answer. We have already given the reader an indication of what to expect in the following pages, by displaying an argument that points up the set function character of utility. The reader will see that this is borne out when we assert that *these utilities and these probabilities are one and the same thing.*

What we have just stated serves to suggest why we can proceed in the next section to develop the concept of utility for an individual without any essential mention of the probabilities of his environment. In sketchy form, the reason is this: the concepts of utility and probability are universally one and the same. The individual's utilities and the environment's probabilities are interlaced in a single, larger probability (utility) complex. We must proceed by talking about some complex. Let it be that pertaining to the individual. In so doing, and using the name utility for the concept in question, we shall explicate the intuitive notion of utility for an individual; but at the same time we shall be developing the kind of structure that describes an individual-environment couple as well.

One last word. We have been forced to use, in the second paragraph back, a reference to statistical analysis. This early allusion to statistics should not be misinterpreted to mean that statistics stands by itself, firmly foundationed apart from the theory here being presented. The contrary is true. Statistics, by its very nature, is meaningless in the absence of a meaningful theory of probability. Statistics today rests entirely on the *intuitive* content of probability, and will have a floor under it only with the development of a—the—rigorous and exact theory of probability. We shall speak again of statistics in section 8.

3. Personality processes

The word "personality" appears to have come to connote for psychologists the fully integrated physical, psychical and social aspects of an individual; actually synonymous with the "individual" himself, but in addition bespeaking an analytic attitude on the part of the speaker. This is the sense that the word will have in the present context.

To convey a mathematical theory of behavior, a bridge between reader and writer must be established. It is necessary to select and work with a concept (or complex of concepts) within the theory which has two qualities: its sense can be imparted by example, and its mathematical precisement can be stated and is sufficient for the further mathematical development. Such a concept in our case is that of what we shall call a *personality process*. It is, however, not a primitive concept of the theory. And in discussing it we shall therefore come to analyze it in terms of other concepts.

A personality process is a dynamic thing. It is a steadily going temporal accumulation of acts, together with the underlying fabric motivating these acts. Some examples, as suitably expressed as the language seems to permit, are (1) making as much money as possible, (2) doing a good job of carpentering, (3) playing a game of chess to win, (4) playing a game of ticktacktoe to lose, (5) pleasing one's mother, (6) pleasing one's mother to outdo one's sister, (7) suffering an attack of influenza, (8) not getting along with others, (9) walking, (10) physically growing. A process is a specific, not a generic thing; for example, a specific instance of an individual's walking is a process, not the abstraction of walking that pertains to the individual. The range of these examples shows that we are admitting no underlying separation of conscious, subconscious, physical, etc. in the personality. Furthermore, the fact that the environment enters in an essential way in all these examples signals our denial also of an underlying separation of the individual from his environment.

A personality process falls immediately into two parts: the accumulating temporal sequence of acts, which we shall call the *behavior* (of the individual); and what we have referred to above as the underlying fabric motivating this behavior. This latter part we shall call a *personality strain* (of the individual). Again let us note, concerning behavior and personality strains, that they are specific things, not generic types. The notion of behavior is clear, and we need not discuss it further at this time. It is to the personality strain that we now turn our attention, and to which we shall give an explicit mathematical form. But first we wish to remark that the personality strain is what we called a "modified drive" in our article [2]. We have decided against continued use of this latter term because of its unfavorable analytic connotation.

The relation between the behavior and the personality strain of a personality process may be described by saying that the behavior is the *development* or *evolution* of the strain. Thus we speak of a personality process *developing* or *evolving*. The strains of the processes cited as examples above can be described by replacing the gerunds by infinitives: (1) to make as much money as possible, etc. We are now about to make a personality strain mathematically precise. In this connection we may usefully remark that in such a procedure of exact theorizing, a concept may be given a large, detailed structure, to the extent that many of the elements of the structure are ordinarily unspecified in everyday reference to instances of the concept, and indeed very possibly for the reason that language does not meet the need. This is the case with the concept of a personality strain. Suppose, for example, that on leaving a room where we have been watching a man playing chess, we encounter a friend and we assert to him that the man has in him the personality strain to play that game of chess to win. Assuming that our friend has no basis for conjecturing on his own about the man, but is interested in him, he will press us further; for, what we have asserted does not go very far. He will ask if the man is a good player and is playing with ease toward winning, or if the man is only a moderately good player but is playing with feverish intent to win. Perhaps the latter is true, and we correctly assert this to our friend. Then our friend may go on to ask: is the man doing this out of some need or desire to win just this game, or does he have a neurotic compulsion to win? And again we may answer correctly that it is the latter. By such interrogation our friend is urging us to a finer and finer specification of a personality strain of the man in question. "Feverish intent" has refined the simple description "play to win," and "neurotic compulsion" has further refined "feverish intent." Our friend may feel still not satisfied, sensing that there is more to be known about the man, but finding that there are no pertinent posited notions, and hence no vocabulary, with which to carry on the questioning. In this situation, he will have recourse to a systematic questioning of us concerning the behavior of the man, and then proceed to make his own guesses about the personality strain—guesses that will take the form, for him, of a feeling about the man. In putting forth a full mathematical characterization of a personality strain, we are asserting that a complete specification of all elements of the mathematical structure leaves nothing more to be known (about that particular strain). Thus, for example, the unsatisfied feeling of our friend, before he took to questioning us on the man's behavior, would be resolved beyond all further possible inquiry by that exhaustive specification of the structural elements (provided, of course, he had faith in the correctness of the specification). And if our friend did go on to add his own guesses to ours, there would be no need for him to keep his finer guesses in the form of a feeling: all these guesses (his and ours) are directed at precisely the goal of specifying the structural elements; beyond these there is nothing else to be said or felt about the strain. Any deeper concepts, further qualifying neurotic compulsion, that our friend sensed but could find no authority and no words for, are all to be defined in terms of the structural elements of the strain. And by the same token, the vagueness of the descriptions "play to win," "feverish intent" and "neurotic compulsion" can be replaced by exact definitions in terms of the structural elements of the strain. But the language question actually goes much deeper, and we shall discuss it again in section 10.

Mathematically, a personality strain is a finite or denumerable unitary measure chain. (This countability should not be taken as an altogether rigid commitment: see footnote 10.) Since a finite strain is always part of a denumerable one, we shall, for simplicity, restrict our attention to the denumerable case. It is, then, first of all, a sequence of spaces, $\Omega_1, \Omega_2, \cdots$. For each space Ω_i there is a σ-algebra $\mathcal{Q}_i$ of its subsets, and $\mathcal{Q}_i$ contains every subset consisting of a single point. Finally, there is a chain of unitary measures on this sequence of spaces. To describe this, let us first introduce for each $n = 1, 2, \cdots$, the symbol $\Omega_{[n]}$ to denote the direct product space $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. A point of Ω_i will be denoted by ω_i, and a point $(\omega_1, \omega_2, \cdots, \omega_n)$ of $\Omega_{[n]}$ by $\omega_{[n]}$. The unitary measures on the spaces Ω_i are as follows. There is such a measure, u_1, on the sets of $\mathcal{Q}_1$ in Ω_1; for each point $\omega_1 \in \Omega_1$ there is such a measure $u_2(\cdot\,|\omega_1)$ on $\mathcal{Q}_2$; for each point $\omega_{[2]} \in \Omega_{[2]}$ there is such a measure $u_3(\cdot\,|\omega_{[2]})$ on $\mathcal{Q}_3$; and so on, there being in general, for each point $\omega_{[n-1]} \in \Omega_{[n-1]}$, a unitary measure $u_n(\cdot\,|\omega_{[n-1]})$ on $\mathcal{Q}_n$. Let us denote, for each $n = 1, 2, \cdots$, by $\mathcal{Q}_{[n]}$ the σ-algebra of sets in $\Omega_{[n]}$ generated by $\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_n$. Then, the above measures have the following property: for each $n = 1, 2, \cdots$, and each $A_n \in \mathcal{Q}_n$, the point function $u_n(A_n\,|\cdot)$ on $\Omega_{[n-1]}$ is measurable $\mathcal{Q}_{[n-1]}$.

The remaining elements in the structure of the personality strain can now be described in terms of their mathematical relationship to the elements listed above. There is the full direct product space $\Omega = \Omega_1 \times \Omega_2 \times \cdots$, and the σ-algebra of subsets of Ω, denoted by $\mathcal{Q}$, which is generated by $\mathcal{Q}_1, \mathcal{Q}_2, \cdots$. For each $n = 1, 2, \cdots$, and each $\omega_{[n]} \in \Omega_{[n]}$, there is a unitary measure $u^{(n)}(\cdot\,|\omega_{[n]})$ on the subsets of $\mathcal{Q}$ in Ω; and for each $A \in \mathcal{Q}$, the function $u^{(n)}(A\,|\cdot)$ on $\Omega_{[n]}$ is measurable $\mathcal{Q}_{[n]}$. Also, there is a comprehensive measure, u, on $\mathcal{Q}$. These measures are characterized in terms of the measures of the preceding paragraph by the following relations:

$$u(A_1 \times \Omega_2 \times \Omega_3 \times \cdots) = u_1(A_1), \qquad A_1 \in \mathcal{Q}_1; \tag{3.1}$$

$$\begin{aligned} &u^{(n)}(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \times A_{n+1} \times \Omega_{n+2} \times \Omega_{n+3} \times \cdots \,|\, \omega_{[n]}) \\ &\quad = u_{n+1}(A_{n+1}\,|\,\omega_{[n]}), \qquad A_{n+1} \in \mathcal{Q}_{n+1}, \qquad n = 1, 2, \cdots; \end{aligned} \tag{3.2}$$

$$u^{(n)}(A\,|\,\omega_{[n]}) = \int_\Omega u^{(m)}(A\,|\,\omega_{[n]}, \cdot)\, du^{(n)}(\cdot\,|\,\omega_{[n]}), \qquad A \in \mathcal{Q}, \qquad m > n; \tag{3.3}$$

$$u(A) = \int_\Omega u^{(n)}(A\,|\,\cdot)\, du, \qquad A \in \mathcal{Q}, \qquad n = 1, 2, \cdots. \tag{3.4}$$

The mathematical structure of a personality strain is now completely laid out. In presenting it, we have stayed in the vocabulary of mathematics. We turn now to a discussion and renaming of the elements of the structure, to bring out their behavioral significance. The spaces $\Omega_1, \Omega_2, \cdots$ will be called the *basic* spaces of the strain. The sets in the σ-algebras $\mathcal{Q}_i$, $\mathcal{Q}_{[n]}$ and $\mathcal{Q}$ will be called *eventualities*. For each point ω_i in any Ω_i, the set consisting of this point, denoted by $\{\omega_i\}$, is therefore an eventuality; these will be called the *elementary* eventualities of the strain. The $\mathcal{Q}_i$, $i = 1, 2, \cdots$, will be called the *basic* σ-algebras (of eventualities) of the strain. The measures will be called *utilities*, or more precisely *utility functions*, and still more specifically *utility distributions*. The particular measures u_1 and $u_n(\cdot\,|\omega_{[n-1]})$, $n = 2, 3, \cdots$, will be called the *basic* utility distributions of the strain. The measure u will be called the *comprehensive* utility distribution. The distributions other than u_1 and u, which is to say the distributions $u^{(n)}(\cdot\,|\omega_{[n]})$, will be called *conditional* distributions. The number $u^{(n)}(A\,|\omega_{[n]})$ will be designated as the *conditional utility of* (or *for*) *the eventuality A relative to the eventuality* $\{\omega_{[n]}\}$.

An eventuality may be roughly described as a potential act. A particular eventuality may or may not actualize into an act. It is the burden of the law of behavior, given in the next section, to say what actualizations take place. Each basic space Ω_i, $i = 2, 3, \cdots$, is, of course, to be considered subsequent to its predecessor in time. We may examine the concept of an eventuality by looking at our example of the strain to play a game of ticktacktoe to lose. (The finite length of a ticktacktoe game does not imply that this strain is a finite one. That the player is playing to lose means that there is reference to such things as his intelligence, his disposition toward his opponent, etc.; and these are characters of a much more extensive personality strain than one that has reference merely to the moves of the game.) Among the basic spaces of this strain there will be one which has, among its elementary eventualities, the following nine: (1) that the man, on his second move, places his mark in the number one square, (2) that the man, on his second move, places his mark in the number two square, . . . , (9) that the man, on his second move, places his mark in the number nine square. Our reservations about describing an eventuality as a "potential act" can now be seen. One might object that two or three squares will have been filled before our man gets his second move, and therefore these squares, whichever they may turn out to be, do not offer "potential" acts. The word potential is thus not adequate to the description of eventualities. Nor would it be correct to suggest that eventualities are "potential acts as of such and such a time." In our conception, eventualities are simply there. That our man is going to play a legitimate game of ticktacktoe and avoid the already filled squares at his second move is to be found contained in the utilities and the law of behavior.

A few examples will illustrate the correspondence between the mathematical terminology for sets and the everyday language for eventualities. Consider the basic space which includes the nine elementary eventualities above. Take the two points corresponding to the second and eighth of these eventualities. The set consisting of these two points represents the eventuality: that the man, on his second move, places his mark in the number two square or in the number eight square. This set is also recognized as the *union* of the two sets consisting of the second point alone and the eighth point alone. We may then reasonably call this composite eventuality the union of the two elementary eventualities in question. Consider two sets of this basic space, one consisting of the second and eighth points, the other consisting of the third, eighth and ninth points. The *intersection* of these two sets is the set consisting of those points which are common to the two sets; that is, the set consisting of the eighth point alone. Correspondingly we may say that the eventuality: that the man . . . number eight square, is the intersection of the two eventualities (i) that the man . . . number two square or number eight square, and (ii) that the man . . . number three square or number eight square or number nine square. Intersection is thus the technical version of the conjunction *and.* The nontechnical expression of the fact concerning the eventualities (i) and (ii) is as follows: the eventuality that the man . . . number two square or number eight square *and* that he . . . number three square or number eight square or number nine square, is precisely the eventuality that he . . . number eight square. We shall leave off this discussion here, trusting that it constitutes a sufficient start for the nonmathematical reader.

We come now to a designation of fundamental importance. Each basic space Ω_i contains a particular point O_i. And these points have the following significance: if, for some n, the eventuality $\{O_n\}$ actualizes, then also $\{O_r\}$ actualizes for all $r = n + 1, n + 2, \cdots$. The force of this is that if some $\{O_n\}$ actualizes, then there is no further actualization of non-O eventualities. On this account we call the $\{O_n\}$ the (basic) *cessation eventualities* of the strain. To exemplify the role of the cessation eventualities, let us consider a very simple personality strain: to smoke. For each $n = 1, 2, \cdots$, the basic space may consist of just four points ω_{n1}, ω_{n2}, ω_{n3} and O_n, with $\{\omega_{n1}\}$ being (representing) the eventuality that the man will light a cigarette, $\{\omega_{n2}\}$ a pipe, and $\{\omega_{n3}\}$ a cigar. If this strain is the full smoking strain of the man, then actualization of, say, $\{O_{200,001}\}$ signifies that the man has stopped smoking for good after the 200,000th smoke. This may be due to death, or to swearing off smoking for one reason or another, or to taking up new residence in an area where smoking devices are not available, etc.; just what acts (of man-environment) may surround this cessation of smoking is a matter of information contained in a far more detailed strain, one in which the smoking strain itself is contained, in a certain precise sense (see section 7).

In introducing the cessation points we have made essential reference to the dynamics of the personality process, which is properly the domain of the law of behavior of the next section. Consistent with this law, the utilities associated with the cessation points have the following property:

$$(3.5)\quad \begin{aligned} &u_n(\{O_n\} \,|\omega_1, \omega_2, \cdots, \omega_{r-1}, O_r, \omega_{r+1}, \cdots, \omega_{n-1}) = 1\,, \\ &n = 2, 3, \cdots; \quad r < n\,; \quad \text{all} \quad \omega_1, \omega_2, \cdots, \omega_{r-1}, \omega_{r+1}, \cdots, \omega_{n-1}\,. \end{aligned}$$

We are now ready to go on to the statement of the law according to which eventualities actualize into acts. It may equally well be called the *law of behavior*, or *the law of evolution* (*development*) *of a strain*, or *the law of evolution* (*development*) *of a process*. The appearance of this law in our development is not unexpected as regards the concept of utility, but together with the cessation points, the law will be seen to represent our departure from all past thinking about the concept of probability.

4. The law of behavior

To enunciate this law we must first introduce some notation and terminology. To begin with, any sequence of eventualities $\{A_{[n]}, n = 1, 2, \cdots\}$, with $A_{[n]} \in \mathcal{Q}_{[n]}$ for each $n = 1, 2, \cdots$, will be called a *sequential tendency* in the strain in question. And any eventuality $A \in \mathcal{Q}$ will be called a *comprehensive* or *total tendency* in the strain. The reader should take careful note of the fact that nothing has been said here about behavior. There are dangers, at this stage, of confusion by virtue of classical connotations; we urge the reader to follow closely the sequence of definitions now being introduced. Tendencies can be nicely exemplified by referring to the smoking strain discussed in the preceding section. Consider, for example, the sequential tendency with

$$(4.1)\quad \begin{cases} A_{[1]} = \Omega_1\,, \\ A_{[2]} = \{\, (\omega_{11}, \omega_{22})\,, \quad (\omega_{11}, \omega_{23})\,, \quad (\omega_{12}, \omega_{21})\,, \quad (\omega_{13}, \omega_{21}) \,\}\,, \\ A_{[3]} = \begin{Bmatrix} (\omega_{11}, \omega_{22}, \omega_{31})\,, & (\omega_{11}, \omega_{23}, \omega_{31})\,, & (\omega_{12}, \omega_{21}, \omega_{32})\,, \\ (\omega_{12}, \omega_{21}, \omega_{33})\,, & (\omega_{13}, \omega_{21}, \omega_{32})\,, & (\omega_{13}, \omega_{21}, \omega_{33}) \end{Bmatrix}, \\ \qquad \vdots \end{cases}$$

In words, this is the tendency to smoke a cigarette every other time. To illustrate the total tendencies, consider all points ω, of Ω, of the form $(\omega_{1r_1}, \omega_{2r_2}, \cdots)$ where, for each $n = 1, 2, \cdots$, we have $r_n = 1, 2$ or 3. For such a point, and for each $k = 1, 2, \cdots$, let a_k^ω denote the number of the indices $r_1, r_2, \cdots, r_k$ which have the value 1; let b_k^ω denote the number of these indices which have the value 2; and c_k^ω the number of these indices which have the value 3. Of course, $c_k^\omega = k - a_k^\omega - b_k^\omega$. Now take A to be the set of those points ω, of the above form, for which

$$\lim_{k\to\infty} \frac{a_k^\omega}{k} = \frac{1}{2}, \tag{4.2}$$

$$\lim_{k\to\infty} \frac{b_k^\omega}{k} = \frac{1}{4}, \tag{4.3}$$

and

$$\lim_{k\to\infty} \frac{c_k^\omega}{k} = \frac{1}{4}. \tag{4.4}$$

Then A is a total tendency, describable in words as follows: it is the tendency to smoke a pipe and cigar equally frequently, but to smoke a cigarette twice as frequently as either of them.

For an eventuality $A_{[n]}$ we now introduce the symbol $D_{A_{[n]}}$ to denote another eventuality, related to $A_{[n]}$ in the following way: the points of $D_{A_{[n]}}$ are those of $A_{[n]}$ with nonempty terminal segments of components replaced by cessation points. Thus, for example, if

$$A_{[3]} = \{ (\omega_{11}, \omega_{21}, \omega_{33}), \quad (\omega_{14}, \omega_{22}, \omega_{36}) \}, \tag{4.5}$$

then

$$D_{A_{[3]}} = \left\{ \begin{matrix} (O_1, O_2, O_3), & (\omega_{11}, O_2, O_3), & (\omega_{11}, \omega_{21}, O_3), \\ (\omega_{14}, O_2, O_3), & (\omega_{14}, \omega_{22}, O_3) & \end{matrix} \right\}. \tag{4.6}$$

Similarly, if $A \in \mathcal{Q}$, D_A will denote the eventuality whose points are those of A with nonempty (always infinite in this case) terminal segments of components replaced by cessation points.

Tendencies of the forms $\{A_{[n]} \cup D_{A_{[n]}}, n = 1, 2, \cdots\}$ and $A \cup D_A$ will play a fundamental role. (The symbol $A \cup B$ denotes the eventuality which is the union of the two eventualities represented by A and B.) It will readily be seen—to illustrate the significance of the D's—that if $\{A_{[n]}, n = 1, 2, \cdots\}$ is the tendency to smoke a cigarette every other time, then $\{A_{[n]} \cup D_{A_{[n]}}, n = 1, 2, \cdots\}$ is (bringing our best command of language to the task) the tendency to smoke a cigarette every other time except insofar as smoking may cease altogether at some time. Similarly, if A is our above example of a total tendency in the smoking strain, then $A \cup D_A$ is the tendency to smoke a pipe and cigar equally frequently, but to smoke a cigarette twice as frequently as either of them, except insofar as smoking may cease altogether at some time.

We now make our next definition. Let $\{A_{[n]}, n = 1, 2, \cdots\}$ be a sequential tendency in a strain. We shall say, "the behavior actualizing from this strain *essentially has the tendency* $\{A_{[n]}, n = 1, 2, \cdots\}$" to mean (1) one of the elementary eventualities of each $\mathcal{Q}_n$ actualizes, and (2) for all n greater than some integer r, the eventualities $\{\omega_1\}, \{\omega_2\}, \cdots, \{\omega_n\}$ actualized up to the n-th stage are such that the corresponding point $(\omega_1, \omega_2, \cdots, \omega_n)$ belongs to the set $A_{[n]}$.

Similarly, let A be a total tendency of a strain. We say, "the behavior actualizing from the strain *has the tendency* A" to mean (1) one of the elementary eventualities of each $\mathcal{Q}_n$ actualizes, and (2) the eventualities $\{\omega_1\}, \{\omega_2\}, \cdots$ actualized are such that the corresponding point $(\omega_1, \omega_2, \cdots)$ belongs to the set A.

These last two definitions are designed for verbally relating behavior in a specific way to tendencies. And with their statement it is now clear what the law of behavior is to do: it will say exactly what tendencies the behavior has, or essentially has, as the case may be. To this end, we consider a particular strain, and define a particular type of sequential tendency:

Let the eventualities $A_{[n]} \in \mathcal{A}_{[n]}$, $n = 1, 2, \cdots$, be such that

$$\lim_{n \to \infty} u\,(A_{[n]} \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots) = 1\ , \tag{4.7}$$

and for each $n = 1, 2, \cdots$, if $\omega_{[n]} \in A_{[n]}$, then

$$\lim_{r \to \infty} u^{(n)}\,(A_{[r]} \times \Omega_{r+1} \times \Omega_{r+2} \times \cdots \mid \omega_{[n]}) = 1\ . \tag{4.8}$$

In this case we call the tendency $\{A_{[n]} \cup D_{A_{[n]}}, n = 1, 2, \cdots\}$ a *tendency of type* I.

And with this we may now state

THE LAW OF BEHAVIOR: *The behavior of a process essentially has every tendency of type* I *in the strain of the process.*

We may illustrate the application of this law by again referring to our simple example of the smoking strain. Let the basic utilities (which we have not specified up to now) of this strain be as follows:

$$u_1(\{\omega_{11}\}) = u_1(\{\omega_{12}\}) = u_1(\{\omega_{13}\}) = \tfrac{1}{3}\ , \quad u_1(\{O_1\}) = 0\ ; \tag{4.9}$$

$$u_2(\{\omega_{22}\} \mid \omega_{11}) = u_2(\{\omega_{23}\} \mid \omega_{11}) = \tfrac{1}{2}\ , \tag{4.10}$$

$$u_2(\{\omega_{22}, \omega_{23}, O_2\} \mid \omega_{1j}) = 0\ , \quad \text{for} \quad j = 2, 3\ ; \tag{4.11}$$

and in general,

$$u_n(\{\omega_{n2}\} \mid \omega_1, \omega_2, \cdots, \omega_{n-1}) = u_n(\{\omega_{n3}\} \mid \omega_1, \omega_2, \cdots, \omega_{n-1}) = \tfrac{1}{2} \quad \text{if} \quad \omega_{n-1} = \omega_{n-1,\,1}\ , \tag{4.12}$$

$$u_n(\{\omega_{n2}, \omega_{n3}, O_n\} \mid \omega_1, \omega_2, \cdots, \omega_{n-1}) = 0 \quad \text{if} \quad \omega_{n-1} = \omega_{n-1,\,2} \quad \text{or} \quad \omega_{n-1,\,3}\ . \tag{4.13}$$

This specification, together with that at the end of the preceding section, provides us with all the basic utilities (those not explicitly stated here being calculable from these by virtue of the properties of the utility distributions as unitary measures). We shall not here display the computations, but simply assert that if one calculates the pertinent conditional utilities of the eventualities in the already discussed tendency, $\{A_{[n]}, n = 1, 2, \cdots\}$, to smoke a cigarette every other time, then one finds that $\{A_{[n]} \cup D_{A_{[n]}}, n = 1, 2, \cdots\}$ is a tendency of type I. Hence, according to the law of behavior, the behavior issuing from this smoking strain has the latter tendency, which is to say that the person in question will smoke a cigarette every other time except insofar as his smoking may cease altogether at some time.

It is to be remarked that there is nothing absolute in the firstness of the first basic space in a strain. Strains are, in fact, generally[7] relative to past behavior. To see the meaning of this, let $\mathcal{S}$ be a strain whose elements are $\Omega_1, \Omega_2, \cdots, \mathcal{A}_1, \mathcal{A}_2, \cdots, u_1, u_2(\cdot \mid \cdot)$,

[7] This qualification is intended to be tentative. In the development of our theory up to this date, there is no indication that the qualification "always" is not the correct one. A decision between the two clearly rests on whether or not the theory is ever found to assert the existence of a very first, a primordial act.

$\cdots$, etc. Now suppose the first three elementary eventualities actualized are $\{\omega_1^\circ\}$, $\{\omega_2^\circ\}$ and $\{\omega_3^\circ\}$. Then $\mathcal{S}'$, defined by

$$\begin{aligned} \Omega_1' &= \Omega_4, & \mathcal{Q}_1' &= \mathcal{Q}_4 \\ \Omega_2' &= \Omega_5, & \mathcal{Q}_2' &= \mathcal{Q}_5 \\ &\vdots & &\vdots \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} u_1'(\cdot) &= u_4(\cdot \,|(\omega_1^\circ, \omega_2^\circ, \omega_3^\circ))\,, \\ u_2'(\cdot \,|\omega_1') &= u_5(\cdot \,|(\omega_1^\circ, \omega_2^\circ, \omega_3^\circ, \omega_4))\,, \qquad (\text{where } \omega_1' = \omega_4)\,, \end{aligned} \tag{4.15}$$

etc., *is also a strain*. And just as $\mathcal{S}'$ is related to $\mathcal{S}$ and its evolution, so is $\mathcal{S}$, in general, related to some other strain and its evolution. The reader will recognize that a question of consistency accompanies what we have just said. That is, the behavior actualizing from the strain $\mathcal{S}'$ is, *a fortiori*, behavior actualizing from the strain $\mathcal{S}$. Therefore, the law of behavior, applied to $\mathcal{S}$, will *a fortiori* be making assertions about the behavior actualizing from $\mathcal{S}'$. The question is: are these assertions over and above (and possibly in opposition to) the assertions of the law applied to $\mathcal{S}'$ itself? Or are they—as they should be if our theory is to hold together—no addition whatsoever to the assertions made by the law applied to $\mathcal{S}'$? The answer is the latter. It is a consequence of the mathematics of utility distributions that every tendency $\mathcal{T}$ of type I of the strain $\mathcal{S}$ has associated with it a tendency $\mathcal{T}'$ of type I of the strain $\mathcal{S}'$, such that the statements "the behavior out of $\mathcal{S}$ essentially has the tendency $\mathcal{T}$" and "the behavior out of $\mathcal{S}'$ essentially has the tendency $\mathcal{T}'$" are identical.

5. Alternative forms of the law of behavior

In the quest for new theory there is a most remarkable guiding principle. The principle is that the theory sought should be basically simple. What is remarkable about it is that investigators will disagree violently on what it means. We must confess that along with all others we too have a strong conviction about the principle. And we make explicit mention of it here at the beginning of this section, in particular, because we feel that the results to be described below serve it in a very positive way.

In addition to tendencies of type I, we define three other particular kinds of tendencies in a strain:

Let A be an eventuality in $\mathcal{Q}$. Let $A^{(n)}$ denote the projection of the set A into the space $\Omega_{[n]}$. [This mathematical designation means that $A^{(n)}$ is the set of all points $(\omega_1, \omega_2, \cdots, \omega_n)$ in $\Omega_{[n]}$ such that some point in A has these $\omega_1, \omega_2, \cdots, \omega_n$ for its first n respective components.] Let A be such that

$$u(A) = 1\,, \tag{5.1}$$

and for each $n = 1, 2, \cdots$, if $\omega_{[n]} \in A^{(n)}$, then

$$u^{(n)}(A \,|\omega_{[n]}) = 1\,. \tag{5.2}$$

In this case we call the total tendency $A \cup D_A$ a *tendency of type* II.

Let the eventualities $A_{[n]} \in \mathcal{Q}_{[n]}$, $n = 1, 2, \cdots$, be such that

$$\lim_{n\to\infty} u\,(A_{[n]} \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots) = 1\,. \tag{5.3}$$

Then we shall say that the sequential tendency $\{A_{[n]} \cup D_{A_{[n]}}, n = 1, 2, \cdots\}$ is a *tendency of type* III.

Finally, let the eventuality $A \in \mathcal{Q}$ be such that

$$u(A) = 1\,. \tag{5.4}$$

We say then that the total tendency $A \cup D_A$ is a *tendency of type* IV.

It is immediately clear that a tendency of type I is in particular a tendency of type III, and a tendency of type II is a tendency of type IV. But the general question that draws attention to itself is this: why, among these four types of tendencies, should it be the tendencies of type I that characterize the law of behavior; indeed, why should the law not assert that behavior has all these various tendencies, or equivalently (taking account of the above remark) that it essentially has all tendencies of type III and also has all tendencies of type IV? The complete answer to this question is supplied by the following:

THEOREM. *The following four assertions are all equivalent:*

(i) *the behavior essentially has every tendency of type* I,
(ii) *the behavior has every tendency of type* II,
(iii) *the behavior essentially has every tendency of type* III,
(iv) *the behavior has every tendency of type* IV.

The proof of this theorem is given in detail in [2], and will be published shortly. Its effect is to give us three alternative forms of the law of behavior:

THE LAW OF BEHAVIOR (first alternative form): *The behavior of a process has every tendency of type* II *in the strain of the process.*

THE LAW OF BEHAVIOR (second alternative form): *The behavior of a process essentially has every tendency of type* III *in the strain of the process.*

THE LAW OF BEHAVIOR (third alternative form): *The behavior of a process has every tendency of type* IV *in the strain of the process.*

As an example of the application of the last of these alternative forms, consider again the smoking strain with its utilities as specified in the preceding section. And let A be the total tendency we described there as an example; namely, to smoke a pipe and cigar equally frequently, but to smoke a cigarette twice as frequently as either of them. Calculation of the comprehensive utility of this tendency will show that it has the value 1. Hence $A \cup D_A$ is of type IV, and therefore, according to the third alternative form of the law of behavior, the behavior actualizing from this smoking strain will have the tendency $A \cup D_A$. That is, the person in question will smoke cigarettes, pipe and cigar with relative frequencies tending in the long run to the ratio 2:1:1, except insofar as his smoking may cease altogether at some time.

Notice that this fact about the person's smoking behavior, found with the aid of the third alternative form of the law, is in addition to the fact we discovered in the preceding section by applying the original form of the law; namely, that the person will smoke a cigarette every other time, except insofar as his smoking may cease altogether at some time.

6. Utility and probability

We have now reached a stage in our discussion of utility at which we can start to turn to the question of probability. We have detailed our conception of the individual as a complex of personality processes. These processes evolve according to the law of behavior; and the numerical-valued functions in the strains of the processes, to which the law has reference, are the utilities of the individual. In the Introduction we spoke of the fear that many people hold regarding mathematical description of human behavior, the fear that its progress would ultimately reveal our behavior to be a determined thing. What we have presented of our theory up to this point serves to dash this fear to the ground. The law of behavior does not in general make any assertions to the effect that specific elementary eventualities of specific basic spaces will actualize. It may do so, but then it is because the utilities have the particular values they do. If we take once again our numerical example of the smoking strain, we have seen two tendencies which the behavior out of this strain will have. Neither of them makes definite statements about specific smoking occasions. And in fact, the utilities of this strain—call it $\mathcal{S}$—are such that there is no definite assertion that can be made concerning any one smoking occasion. On the other hand, suppose the behavior issuing from this strain turns out with the individual smoking a cigar the first time. Let $\mathcal{S}'$ be the strain obtained from $\mathcal{S}$, relative to this first act, in the same way that the $\mathcal{S}'$ in section 4 was obtained from the strain $\mathcal{S}$ there, relative to the first three acts out of $\mathcal{S}$. Then the utilities of this present $\mathcal{S}'$ are such that definite, one-shot statements can be made; the law of behavior will assert, namely, that the individual will smoke a cigarette on the first, third, fifth, seventh, etc. occasions, and either a pipe or cigar on all other occasions—except insofar as his smoking may cease altogether at some time.

But in general there will be no one-shot (or two-shot, etc.) statements at all concerning the behavior out of a strain $\mathcal{S}$; and likewise, no such statements pertaining to any $\mathcal{S}'$ obtained from $\mathcal{S}$, relative to an initial sequence of acts out of $\mathcal{S}$, of whatever length.

The crucial fact is this, that the conception of the individual's behavior being motivated by the organization of eventualities and their utilities, in accordance with the law we have stated—this conception, by its formal breadth, pays the fullest respect to the free will that is quite obviously there in human beings. If the theoretical structure that we are building is one day denied by a piece of knowledge that explicitly contradicts it, then we shall be obliged to admit our mistake. But lacking any such knowledge at this date, we firmly believe that the full extent of human freedom of choice is captured in the personality strain as subject to the law of behavior.

Anticipating our discussion of probability, the reader has perhaps been most immediately struck by the formal similarity between a personality process and what in the natural sciences we refer to as a *stochastic process*. Except for the cessation points and their special role, the two appear to be formally the same. If the reader noticed this likeness, and began by assuming that we were merely bent upon translating the terminology of stochastic processes, he found himself constrained to negate this assumption when he came to the law of behavior. For, at that point he recognized the situation to be this: The unitary measures of a stochastic process are called *probability distributions*. They are the mathematical precisement of the *uncertainty* attending the acts of the system in question (a die, a volume of gas, a group of voters, etc.). And the concept of uncertainty, or equivalently, the concept of probability, is still today without a generally accepted explanation of its precise relation to the system. In contrast to this, we began by renouncing all reliance on an *a priori* givenness of the human being—the system of our considerations; instead, eventualities, utilities and acts are the basic elements for us. These elements are composed in the structures that we have called personality processes, and the individual *is* a complex of such processes. The law of behavior rounds out the theoretical construction, describing the actualization of eventualities into acts. For us there is no question of how the unitary measures are related to the system; we have said how the system is related to the measures.

What we have just displayed is *not* a contrast between personality processes and stochastic processes. It is the contrast between our conception of a personality process and current, inconclusive conceptions of a stochastic process. We are therefore not in the position of having to accept a fundamental distinction between these two kinds of processes. We are, to the contrary, now ready to state the opposite as our position: *personality processes and stochastic processes are one and the same kind of thing.* By this identification, the concept of probability receives full explication within the context of reality.

Consider any system whose behavior we are wont to describe in terms of a stochastic process; for brevity, let us refer to the system as M. With our example of the psychologist, at the end of section 2, we indicated that M is sometimes a human being. Let us develop our contention first for this case. To begin with, we affirm the unmitigated objectivity of the uncertainty attending the acts of the individual M. An interested party, for example our psychologist, may have M in his thoughts, and may say "I am uncertain what M is going to do in this next situation." The uncertainty alluded to in this phraseology is a strictly objective thing, and is part of the fabric that is M. To draw from this verbal formation the idea that the uncertainty inheres in the psychologist is to confuse the latter's perception of an objective entity with the entity itself. The statement that he "has M in his thoughts" already comprehends the statement that he "is uncertain (or certain) what M is going to do"; the latter statement is part of the detail of the former. (This does not mean that the psychologist necessarily perceives all the probabilities exactly correctly, any more than he need perceive the presence of a scar on the right upper arm of the fully clothed M.) Thus understood, the uncertainty appears immediately as an attribute deriving its entire substance from the freedom of choice of M. This insight has come about at the level of intuition. At that level we could also pose the reverse question: does M's freedom of choice have its entire substance in the uncertainty attending his acts (and therefore, in the light of the first insight, are this freedom of choice and this uncertainty coextensive, one and the same thing in essence)? But at the intuitive level one tends to feel that human free will is a larger thing than the uncertainty attending human acts; at this level one would demur giving an affirmative answer to the question. However, we are not confined to the level of intuition; our development has brought everything down to the level of mathematical precision. We have already made our stand explicit: human self-determination is *completely* expressed in the personality strains under the law of behavior. Couple with this the fact that, granting as an oversight the customary neglect to mention the cessation eventualities explicitly in discussions of stochastic processes, the mathematical form of a personality strain has no characteristics which are not present also in the corresponding structure in a stochastic process. And add finally the allowance that also the law of behavior has gone unmentioned with the notion of a stochastic process as it has come down to us. Then the conclusion falls out: there is nothing in the character of the freedom of choice that is not also there in the character of the uncertainty. There is nothing in its structure to which one can point and say "therefore this personality process is not a stochastic process." If at first one had thought that the cessation eventualities constituted such a distinguishing mark, one soon saw, on pondering for example the eventualities of M's death, that these eventualities are undeniably of account also in the uncertainty context. And if one thereupon tried the premise that the law of behavior was the distinguishing feature, then one found oneself with a notion of stochastic process having a big hole in it.

Speaking in behalf of the individual himself, we say he has certain utilities which motivate his acts; speaking for the rest of the world, we say there are certain probabilities that underlie his acts. We have thereby said the same thing in two ways.

Turn now to the case of M not a human being. In particular, let us have in mind such systems as have never been widely considered to warrant our imputing a free will to them. For example, an elementary physical particle, an aggregate of such particles; the winds, ocean currents, et cetera, which are the substance of the weather; a rose bush; and so on. To understand the behavior of such systems, investigators (physicists, meteorologists, geneticists, and so on) have brought the notion of stochastic process to bear. But, as we have just pointed out, the notion has lacked complete explication. The cessation eventualities have never been explicitly recognized[8] as always pertinent in the strain[9] of the process. And, going much deeper, the nature of the bond between eventualities and acts has never been satisfactorily understood; which is to say, in other words, that no past attempts at the resolution of the meaning of probability have been universally convincing.

Again as before, not much argument is needed to get the idea of the constant pertinence of cessation eventualities. Physical particles are not the indestructible kernels of matter of ancient atomistic conceptions. An atom has no materialistic identity which it preserves against all circumstances. It may lose some of its electrons in collisions with other particles. One of these electrons may couple with a positron, the two then being annihilated to produce two gamma rays, and so on until eventually there are in many places little increments of heat energy, whereas in the beginning there was an atom with all its electrons. Obviously many instances of cessation are contained in this description. Think of the process in which the eventualities are that various weather conditions will be recorded by the inhabitants of a small island. It is clearly pertinent that a volcanic eruption may produce a sizeable volcanic mountain where before there were peaceful fields, thus changing the island's topography appreciably; or, that the island might one day be blown out of the ocean. And in the geneticist's process describing the production of roses by the bush in his experimental plot, if it already takes account of the eventualities of mutations, does it take also into consideration the possible eventualities of thievery and destruction by pests or fire?

The explicit recognition of the cessation eventualities in our understanding of a stochastic process stands in direct opposition to the premise of the only other objectivistic approach to probability that has any force today. This latter is the frequency conception of von Mises (see [20] and [21]). We shall be critical of his ideas, but we desire to express here our own humble tribute to them. They have carried the burden of keeping alive the exacting prescripts of scientific inquiry in the search for understanding of probability. In

[8] However, for something close to them in spirit, cf. Loève's random variables in the wide sense, and the associated notion of nonobservability. See, for example, [14].

[9] We are herewith beginning to refer to the *stochastic strain* of a stochastic process; by this we mean, of course, the structure that we call a personality strain when we call the process a personality process. The terminology of stochastic processes is not completely set at this date. For example, most mathematicians, in speaking of a stochastic process, actually have reference, more specifically, only to the strain of a process. It is in the strain of a process that all the mathematical questions lie, and beyond which their professional interest does not go.

some quarters their reining influence has not been particularly needed; but in others, the regression to pre-Copernican anthropocentricity has been appalling, and there the frequency conception has given the main body of scientific workers something to hold on to.

The premise of the frequency theory to which we have reference is that of indefinite replicability of identical situations. There is no definition whatsoever of "identical situations" which will render the assertion of indefinite replicability correct. One may attempt to achieve such circumstances for the tossing of a die; and for the first thousand tosses one might be able to detect nothing which would dispose one to say that the situations were not identical. But all experience teaches us that eventually something to the contrary is to be expected. The die may crack, or at least wear down appreciably; or the tosser may have to depart on more urgent business; the die may be irretrievably lost. For a photon that has given up its energy to heat, there is no reconstituting that same photon to have it go through its paces again. One is tempted to suspect that the historical assumption of indefinite replicability represents one of the more subtle cases of anthropocentricity insinuating itself into our science. Let us explain. Reality being as it is, we have found it within human capability to reproduce "like situations" of many sorts. Of none of these have we ever produced more than a finite number, of course. But in the case of a large class of them, we have been rewarded with "like consequences" following out of the like situations. The physical sciences have come to their present fruition on this basis of the "verified experiment." With such a tremendous body of knowledge thus attributable to the human power to replicate like situations, of what extremes is that power then not capable? It is an easy jump from this careless attitude to the indiscriminate presumption of indefinite replicability, with the human being as either active agent or passive observer.

The reader may be starting this paragraph with raised eyebrows, wondering if it was our intention to slander the idea of experimental verification. Needless to say, it was not. What the reader has rightly detected is the implication that a much larger view of scientific experiment is pressing its way into our awareness. We can understand this by drawing on two famous experiments: Galileo's dropping of objects from the Tower of Pisa (1590) and the electron diffraction experiments of Thomson and Kikuchi (1928). In the former, the concern is with satisfactorily cognizable and controllable systems, for example, stones. Unargued replications of the initial disposition of the systems are at our beck and call: we can carry the stones repeatedly to the same height and drop them. On doing so, and noting the times of descent on each repetition, we find these times to be sensibly the same; thus, the likeness of consequences. In contrast to this, we find ourselves much less firmly in command in the diffraction experiments. We must recognize many component systems there: the several electrons in the beam. And to begin with, we cannot recapture these same electrons for a repetition of the experiment. But furthermore, it is not a detailed before-after correlation for individual electrons that we can attempt to verify with replications of this experiment; it is merely an over-all diffraction pattern that is thus verified as consequence. The contrast between these two experiments may be sharply summed up in the following language: in the former, we make each system do what we wish, and we observe the individual outcomes; in the latter, we set a stage for the systems, let them proceed, and observe the aggregate outcome. The new understanding of scientific experiment is precisely to this effect, that we have no choice but to allow, to a greater or less extent (as the case may be), the systems concerned to exercise their several and group individualities. It is no arbitrary use of language to say that electrons (for example) have a vital aspect, a freedom of choice, that we are obliged to respect.

Incidentally, it is fitting to note here that the plaint often heard from economists, that theirs is a field without the benefit of experimentation, is incorrect. The true version of their disadvantage is that the individual and group idiosyncrasies of the systems they study, namely human beings, are so manifold and diversified as to make the detection of regularities extremely difficult. They suffer from an extremity of degree, not a distinction of kind.

Two more points are to be made concerning von Mises' frequency conception, and we shall make them briefly. The first is that the ideological clash between the *Regellosigkeits-axiom* and the mathematical sense of "existence" is a very disturbing one. One is perhaps not bothered by this if one feels that mathematics and objective reality are two separate things in their essence. We, however, who feel strongly that there is no such cleavage in the nature of things, find this clash intolerable. And to our way of thinking, it represents a second distortion necessitated by the previous commitment to the first distortion of indefinite replicability—just as, in constructing an affirmative sentence, one must somewhere introduce a second negative if a first one has already been used. The second point is that the limiting frequencies, which are von Mises' probabilities, are evaluations resulting from the counting of acts conceived of as already there: in short, the theory is a *deterministic* one. The reader will note this sharp contrast between von Mises' theory and our own: in the former, probabilities are "after the fact"; in our theory, they are "before the fact."

The difficulties in the frequency theory, that we have discussed, are the roots of its insufficiency. A thoroughgoing objectivism must take account of cessations, must probe the question of why mathematics is so "useful" in understanding reality, and must carry us on firm ground beyond the old determinism. All of this is accomplished with the recognition of the cessation eventualities in a stochastic process and of the pertinence of the law of behavior. If one would hold out now by taking the single exception that one cannot see admitting a free will to, for example, electrons, then, in our opinion, one is permitting oneself to be lost in the web of words. There is always the privilege of using the word probability rather than the word utility. But above all, one must understand the error of falling prey to the fear of relinquishing his dignity as a human being. There is no such implication. The stochastic process which is a human being is infinitely times infinitely richer than that which is an electron, infinitely times richer, even, than that which is a member of any other living species.

We have attempted, above, to exhibit the identity of personality processes and stochastic processes. We have endeavored to do this the more convincingly by arguing successively in both directions. The possibility of doing this was there by virtue of the unsatisfactory state of affairs in current understanding of probability; our conclusion is a new conception of probability and the identity of free will and uncertainty. We shall go on in the next section to describe the over-all picture of reality that is implied by this conclusion. But first we wish to remark briefly on other approaches that have been taken in the past to the question of probability.

There are two main positions apart from the objectivistic one. These are the *subjectivistic* and the *necessaristic* positions. Subjective views hold that probability is a personal degree of belief on the part of an individual concerning anticipated events. Necessary views maintain that probability is the degree of confirmation attaching to anticipated events relative to actual events. This brief twofold classification is necessarily rough, and does not begin to reflect all the shades of interpretation that the many investigators have sought to make. It is, nevertheless, an adequate classification for our purposes. Actually, holders of necessary views adhere to an objectivistic position, in that the interested individual is in no way an agent for their probabilities. And indeed, a necessarist might consider the probabilities of our theory to be degrees of confirmation of eventualities relative to other eventualities. But what is crucial in our conception is first that there is no confusion as between "anticipated events" and "actual events," our probabilities are "before the fact" in a clear-cut way; and secondly, if they are viewed as degrees of confirmation, they are not left up at that intuitive level—there is a very specific law of behavior to assert the consequences of these degrees of confirmation. The most recent exponent of the necessary point of view is Carnap [4]. But the very fact that he has found it indispensable to posit a long-run frequency conception of probability to stand alongside his necessary conception—as two distinct, cooperating conceptions—this, in our view, points up the weakness in the necessaristic position. To explain: Granting that our probabilities are interpretable as degrees of confirmation (and this consolidation of ideas seems indicated), the law of behavior in our theory then goes on to imply *deductively* all the characteristics of the ensuing behavior; in particular, any characteristics in the form of long-run frequencies are so deduced by the law of behavior from the probabilities (recall the example of the smoking strain and the tendency to smoke cigarettes, pipe, cigar with frequencies in the ratio 2:1:1). In contrast to this, the necessarist has no such law. But Carnap in particular, sensing this inadequacy, attempts to fill the gap with a second concomitant conception of probability, the frequency conception; and this attempt amounts to trying to introduce the consequences of the law on an entirely separate basis. This comparative picture of Carnap's theory and our theory reflects our opinion that the general outline of the necessaristic intuitions is in the main correct. However, it indicates also why the necessaristic position fails as an *approach* to the crucial, fundamental revisions of ideas that are called for.

Earlier expositions of the notion of degree of confirmation are due to Keynes [13] and Jeffreys [12]; but they do not seem to have severed this concept from the subjectivist viewpoint as well as Carnap has.

Like von Mises, Reichenbach [17] defends the frequency interpretation of probability. However, his position does not appear to resolve any of the objections we have pointed out to the von Mises conception; rather, it is to be described as a withdrawal to safer ground. For example, the clash between the notions of a mathematical infinite sequence and of an infinite sequence of acts—this he avoids by introducing the uncomfortably vague idea of a "*practical limit* for sequences that, in dimensions accessible to human observation, converge sufficiently and remain within the interval of convergence." The hierarchy of levels of probability that Reichenbach feels compelled to construct is illustrative of his compounding of notions. This kind of attack we cannot reconcile with our trust in the simplicity of reality.

Finally, we must mention the major advocates of the subjectivistic conception of probability; these are Ramsey [16], de Finetti [8] and Savage [19]. Toward the subjectivistic viewpoint we frankly admit our attitude to be one of strong emotion. This viewpoint finds probability to be a personal degree of belief, seated entirely within the observing, anticipating human being. Thus, all the intricate mathematical detail of the calculus of probability is a built-in machinery in the mind of the individual. But what, then, of those individuals who believe a particular die is fair, but who cannot, even with all the advantages of time and pencil and paper, produce the correct value of the probability that in six presumed independent tosses the die will turn up exactly four even faces? For them, subjectivism must have a designation; and indeed it does. It varies from author to author; some examples are: "inconsistent," "incoherent," "irrational." Here, if you please, all wrapped up in a nice, neat bundle, sealed by the holy sanction of mathematics, is the inferior race! What cruelty will men unwittingly perpetrate in the name of science?

7. The structure of reality and the discrete character of behavior

We have now accomplished the major conceptual step of our theory, the identification of personality processes and stochastic processes. Henceforward we shall tend to use the terminology "stochastic process" uniformly unless special occasion warrants our use of "personality process."

In the course of our development we have endeavored to so express ourselves that a certain significant fact would appear the more evident. This fact is that the three concepts of *eventuality*, *probability* and *act* express the entire substance of reality, in the view of our theory. This is not to say that we have *demonstrated* this to be the case; the demonstration will consist in continued, positive substantiation of the theory through verification of its implications. What we are asserting is that no other concept has had to be set down beside these three, and on a par with them, for the development thus far, nor will this be necessary for the further development we shall present in this article. We are asserting, furthermore, our considered conjecture that no other concept will ever have to be brought to stand beside them on their level, for the understanding of reality. And we are asserting, finally, our definite feeling that eventualities, probabilities and acts are very likely the primitive real entities, the ultimate objective elements of reality.

If this is true, then how—it may be asked, to begin with—do we explain our assertion, referred to several times above, that the individual is a *complex* of stochastic (that is, personality) processes? That is, is it not necessary to appeal to some altogether different kind of structural element to account for the binding together of these processes into the individual? The answer is *no*. And the key to this answer is a very familiar mathematical notion, that of a *marginal distribution*. In detail, the answer is this: the individual is one large stochastic process, S, and the processes to which we have had reference as making up the individual are precisely all the various marginal processes of S. That is, if S_1 is one of the particular processes (for example, the smoking process that we used many times as an illustration), and $\mathcal{S}$, $\mathcal{S}_1$ denote the strains of S and S_1, respectively, then the eventualities of $\mathcal{S}_1$ are projections (in the set-theoretical sense) of the eventualities of $\mathcal{S}$, and the probability distributions of $\mathcal{S}_1$ are the corresponding marginal distributions of the probability distributions of $\mathcal{S}$. And the acts of S_1 are, correspondingly, components of the acts of S (for example, the act of "smoking a cigarette" is a component of the act of "smoking a cigarette while reading the morning newspaper and drinking a cup of coffee which he raises to his lips with his right hand, etc., etc."). To illustrate these ideas with a detailed mathematical example would require more space than we may allow ourselves here. The essential point is, however, clear: the complex of processes making up the individual is itself one large process. Thus, no new elements appear for the plastering together of the several processes; these processes are interrelated already in an exceedingly rich way as marginals of a parent process, which is the complete individual.

Another question presents itself now. What is the nature of the interrelation of individual and environment? Does *this* perhaps call for some new structural elements? Again the answer is *no*. With the universal identification of utility and probability clearly in our view, we see immediately that the environment of the individual, like the individual himself, is a stochastic process. And what is the relation between these two processes? Again as before, they are two marginals of a parent process which is the complete system of individual-environment.

The over-all picture now emerges: all reality is one grand stochastic process,[10] *and any system is a marginal process of this universal process.*

Several items crowd for discussion after this comprehensive statement. First, what does it mean to say that a system is a stochastic process? We have proceeded to this in steps which began with saying that an individual is a stochastic process. What is the sense of such statements? How, in these terms, does a wooden table, for example, get its solidness, its there-ness that our senses teach us? The answer is that the table consists in the acts in which it has participated, in the eventualities in which it figures, and in the probabilities of those eventualities. Its there-ness of the past is not a thing in itself; it is the sum total of our having eaten at it for the last five years, of the baby having scratched its surface with the bread knife, of its veneer having begun to peel at one corner, etc., etc. Its future there-ness lies in that it may have its legs gnawed at by the new puppy in the house, that we may eat dinner at it for the next ten years, that we may sell it next week, that in five years it may be in a fire and some of its combustion gases be inhaled by the dog and remain for some time as irritant foreign matter in the dog's lungs, etc., etc.—and, along with these eventualities, their probabilities. What must be grasped when we see the table, feel it with our hands, stub our toe against its hard, very-definitely-there leg, and so on, is that always it is an act that is involved, that in reality there is no isolation of that table to prove its in-itself-ness apart from the other systems that interact with it. And in our anticipatory thoughts of the table, it is always eventualities and probabilities with which we deal; there is not a comprehension of table-in-itself. Such in-itselfness is pure illusion, abstracted from experience and a subsequent science which have understandably but mistakenly conceived human sense-perception to hold a privileged conceptual position.

Indeed, in our view of reality, a flower wafting in the breeze which blows across an uninhabited (by human beings) island perceives that breeze in the same conceptual sense that a man, standing on a dock, visually and auditively perceives the boat moving in. Likewise does a bridge perceive the traffic passing over it, a book perceive the reader perusing it; the earth perceive the sun, a physical particle perceive other physical particles, our white blood cells perceive a locus of infection, an automatic telephone exchange perceive our dialing of a number, and so on. Such perceptions, and eventualities of such perceptions, are among the acts and eventualities that comprise the mentioned systems—and comprise, more particularly, those marginal processes of these systems that

[10] Pending the full resolution of the remaining fundamental questions in the theory, it must be allowed that this universal process (and consequently some of its marginals) may be a process in a sense extended beyond what we have been working with. For example, the process may not have a first basic space; this is clearly related to the question, pointed up in a previous footnote, of existence of a primordial act. Secondly, there is the question of continuous stochastic processes. This question is related to two others whose answers are not achieved with this article: the question of the nature of (space-) time in the context of our theory, and the question of the place of continuity in the context.

we call "seeing," "hearing," "feeling," etc. (this list comes to an end too soon for lack of names for the perceptory functions of flowers, bridges, books, etc.).

One of the immediate consequences of our conception of reality is the total elimination, from the conceptual picture, of the agency of *chance*. This notion is rooted in the times when theoretical understanding and determinism were one. Where investigators were driven to account for the acts of a system (for example, the outcomes of the toss of a die) only in an aggregate fashion by means of probabilities, not being able to penetrate to a satisfactory cause-effect accounting, there the perverse agency of chance was presumed to be at work. In classical statistical mechanics, the concern being with large collections of similar systems, the probabilities were interpretable as a device specific to the handling of such assemblages; and physics, thus free to continue its assumption of an underlying determinism for the individual systems, had no particular concern with the question of the palatability of the idea of chance at work in the world. Quite the reverse is true today in physics, by virtue of the problem of interpretation that quantum theory brought with it. Most physicists today consider that the ψ-function represents a description of an individual system, whatever be the ultimate significance of the associated probability assertions. Strongly in opposition to that stand is Professor Einstein (see [9], also the book [7] of de Broglie); and one of his principal underlying motivations appears to be an abhorrence for the idea of a demon of chance operating to confound men. For Einstein there is no chance (at least in the area of behavior that he would define as the domain of physics); in his view, probability continues to be a tool for the description of ensembles of like systems, and along with that he maintains a firm faith in an underlying determinism. In our view of reality there is likewise no agency of chance, but this, in our case, is for the reason that probabilities and eventualities (together with acts) are the very stuff of the existence of systems. (And this position is of universal scope.) This is what the present theory proposes as alternative to determinism. But neither do we see it to be the fact that the quantum-theoretical stochastic processes arising out of the ψ-functions necessarily describe an individual particle in every case. It is rather to be guessed, at this point, that in general these processes engender the evolution of aggregate random variables over the pertinent basic spaces of the *collection* of particles present.

There are two remarks that we can usefully make here in connection with the article [11] of Feynman. In that article the point of view taken is, in its own words, ". . . that in nature the laws of combining probabilities [are] *not* those of the classical probability theory of Laplace." This statement can be read as intending to imply that the probability of quantum theory is something of a different kind from the probability occurring in other scientific fields. Such is certainly not the case in our viewpoint. It would seem, rather, that the present mathematical machinery of quantum theory is but a special technique for dealing with the particular stochastic processes which are the domain of atomic phenomena. Feynman illustrates the superposition principle with the example of an electron beam directed against a screen with two parallel slits, and a detector behind that. After discussing the pertinent aspects of this experimental setup, he states the physicist's view very nicely as follows: "To avoid the logical inconsistencies into which it is so easy to stumble, the physicist takes the following view. When no attempt is made to determine through which hole the electron passes one cannot say it must pass through one hole or the other. Only in a situation where an apparatus is operating to determine which hole the electron goes through is it permissible to say that it passes through one or the other hole, but if you are not looking you cannot say that it either goes one way or the other! Such is the logical tightrope on which Nature demands that we walk if we wish to describe her." The point we now wish to make is that this is no tightrope at all from the point of view of our theory. The holes in question are not things-in-themselves; they are structured complexes of acts, eventualities and probabilities (that is, they are stochastic processes). In particular, they are, in part, what has happened and what may happen when a beam of electrons is directed at them. If some of their acts, in conjunction with electrons, detecting devices and human observers, are in accord with our usual conception of the hole-in-itself, all well and good. If others of their acts are not, there is no occasion for perplexity, but simply occasion for reconsidering our further guesses about eventualities and their probabilities.

The above discussion serves to show how the problems of interpretation in quantum theory find resolution in our theory. Whether this happy circumstance will continue or not, as our theory develops, is a question whose answer cannot be given now, of course. However, we can present here an immediate result which, in our opinion, is some of the strongest evidence that quantum theory is comprehended by the present theory. Consider any particular basic space, say Ω_n, of a stochastic process. Let u be the comprehensive probability distribution of the process. For any eventuality $A_n \in \mathcal{A}_n$ we shall write simply $u(A_n)$ to denote the number $u(\Omega_1 \times \cdots \times A_n \times \Omega_{n+1} \times \cdots)$. Now, it is a well-known mathematical argument, using the additivity of u and the finiteness of the number $u(\Omega_n) = 1$, that establishes the following fact: there is an at most denumerably infinite collection of elementary eventualities of Ω_n for each of which the comprehensive probability is positive (that is, not zero). Now suppose $\{\omega_n\}$ is an elementary eventuality with $u(\{\omega_n\}) = 0$. Then for the eventuality $\Omega_n - \{\omega_n\}$ we have $u(\Omega_n - \{\omega_n\}) = 1$. Hence, by the third alternative form of the law of behavior, there will actualize an elementary eventuality included in $\Omega_n - \{\omega_n\}$. That is, $\{\omega_n\}$ will *not* actualize. Thus, an elementary eventuality of comprehensive probability zero will not actualize. On the other hand, there are at most denumerably many elementary eventualities of positive comprehensive probability. Since $u(\Omega_n) = 1$, one of these latter will actualize (by the third alternative form of the law) unless cessation takes place or has taken place. And so our conclusion: *except insofar as cessation takes place, there is an at most denumerable subcollection of elementary eventualities in each basic space, one of which will actualize.* Thus, for example, if the basic space is the positive half-line, representing the observation of energy of a particle, and the probability distribution of energies is depicted in the usual way by a plot of the distribution function over the half-line, then only those energies are observable at which there is a jump in the distribution function. And, for another example, a man about to make a transaction at his bank will deposit or withdraw a whole number of currency units.

Our theory hence implies a universal discreteness of all behavior. The question arises how this result is to be reconciled with current successful use of continuous distribution functions in fields other than physics. A complete answer to this question must await the full resolution of basic problems to which we have already referred above. We must come to know exactly the position of continuity in the context of the theory. But in any case, it seems clear at this stage that the law of behavior admits partial application in conjunction with the use of approximate, continuous distributions.

The question of causality is promised a full answer with the present theory. We have already indicated this by our statement to the effect that the theory replaces determinism altogether. The central point is this: anticipating a complete explication of the notions of space-time and continuity within the theory, all so-called causal laws (of physics, etc.) will appear as merely compact descriptions of stochastic processes. Our conviction that this is true stems from the following fact in the theory: if the elementary eventualities $\{\omega_1\}, \{\omega_2\}, \cdots, \{\omega_{n+1}\}$ are such that $u_{n+1}(\{\omega_{n+1}\} | \omega_1, \omega_2, \cdots, \omega_n) = 1$, and if the eventualities $\{\omega_1\}, \{\omega_2\}, \cdots, \{\omega_n\}$ actualize, then the eventuality $\{\omega_{n+1}\}$ will actualize if not cessation. That is, the probabilities can be such that specific eventualities are pinpointed for actualization. In such a case the behavior that will actualize can be simply described without explicit reference to the probabilities and the law of behavior, and this simpler description is what has been habitually called a causal law.

Finally, let us point out the place of characteristic constants. We have, with the preceding paragraph, clarified the position which the present theory must, with its anticipated complete development, hold with respect to past theories: they will appear as descriptions of particular stochastic processes. With this, all characteristic constants of systems will then appear as parameters of their probability distributions; for example, mass, charge, capacity, viscosity, hardness, tensile strength, coefficient of friction, reaction time, recovery time, blood pressure, intelligence quotient, etc., etc.

8. Induction and statistics

These two activities represent the human endeavor to predict and to influence the future. Even today, owing to the unsatisfactory state of affairs surrounding the concept of probability, one finds in most quarters a delimited conception of induction, or inductive inference, or inductive behavior. Attitudes reveal a confinement of the pertinence of induction to the area of behavior calling for explicitly recognizable probabilistic treatment, or better, a denial of the pertinence of induction in the realm of so-called causal law. Yet, as Reichenbach [17] points out, the universal pertinence of induction for all our theoretical activity was already clearly stated by Hume. Designing a bridge according to the principles of classical mechanics is as much an inductive inference as assigning the hypergeometric distribution to the number of nontoxic pills in a fixed-size sample from a batch of pills under test. Building the bridge and trusting one's safety to it is as much inductive behavior as putting the batch of pills on the market (or not putting it on the market).

How an author will ascribe domains to induction and statistics depends on his theoretical conception of reality. In the case of the theory we have presented, the combined domain of induction-statistics is more suitably described with two other terminologies; and the universality of the theory itself bespeaks the universal pertinence of both induction and statistics. There is first the *guessing* at stochastic strains; this means the guessing at eventualities and their probabilities. When such guessing is not ventured to the point of complete specification of all the probability distributions, then the burden falls to statistical investigation to present a comparative study of all the specific strains in the class of strains singled out by the guessing as far as it goes. It is important to understand that the strains we speak of here are to be considered as including the eventualities having to do with the human observer's behavior toward the system with which that observer is concerned; it is a class of guesses about the full observer-observed system that is treated under statistical investigation. Thus, the not completely specific guess may submit a family of hypergeometric distributions relating to batches of pills and a certain family of probabilities relating to an observer's behavior in respect to the sampling of those batches and the putting of those batches on the market. Statistical investigation will show the implications of the law of behavior for each of the specific strains comprehended by the guess, and the statistician will explain these implications to the observer. On the basis of the observer's expressed intentions regarding the batches of pills, the statistician will indicate that certain of the strains, which have particular specific probabilities relating to the observer's behavior, are more meritorious than others. If, relative to the past which includes the very act of the statistician making his recommendation to the observer, the strain in question is indeed one of the meritorious ones, then that strain will evolve and in so doing will bear out the statistician's recommendation and consequently also the observer's intentions.

If, in this example, an electrically charged body is substituted for the batches of pills, and the observer's intention is to ascertain the mass and charge of the body, then one again has an instance of the operation of statistical methodology. In this case, when classical physical theory is considered to provide a good guess, the statistician's deliberations will show the usual exact determinability of the mass and charge in finitely many steps. This is because the probability distributions in the strains of which classical theory is a description assign probability 1 to specific eventualities (otherwise expressed, their distribution functions are constant except at one point where the jump has the value one. See the discussion of "causal" law at the end of the preceding section).

9. Concerning mathematics

We have, we hope, made it abundantly clear, concerning our theory, that the development in the preceding pages is not the ultimate form of its development. This is suggested by at least three observations. First, why should the cessation eventualities appear so obtrusively in the formalism of the theory? One tends to feel that the established success of applications of mathematical probability on an intuitive basis, without explicit recognition of the cessation eventualities, must mean that these eventualities are to have their place in the formalism in a much more subtle guise. Secondly, it is evident that nowhere in the substantive discussions of the theory have we had occasion to recognize the *points* of the basic spaces as such; that is, as things of an effectively different kind from the single-point sets, which, along with all other sets, represent eventualities. Seeing this, there emerges a very definite feeling that the formal representatives of eventualities, that is, sets, must be found to hold a more fundamental position than points; and, in fact, the idea occurs quite readily that, in general, the points of the formalism must be in the nature of eventualities which, for the particular process, are not considered in further refinement—that is, as we have called them, the *elementary* eventualities of the particular process. Thirdly, one cannot but turn a sharp notice to the extreme simplicity of the third alternate form of the law of behavior. One's intuition fairly insists that in the ultimate presentation of the theory it is this form of the law that will appear as fundamental.

These are three of the pivots on which further insight into the heart of the theory turns. They have led to some interesting considerations, which, however, are not yet fully integrated. One, in particular, we shall mention here in order to further clarify the subsequent discussion. It is that the cessation eventualities are, in fact, already represented by the null sets of the respective σ-algebras. This conjecture arose in pursuit of that principle of simplicity in theory which we spoke of in section 5. But it turns out to have much more to recommend it. For, together with the assertion of the theory that cessation eventualities are always there, it implies that the intersection of any two eventualities is again an eventuality; that is, it is not the case that some pair of eventualities has an intersection which is "just nothing" in any sense. This is immediately recognized as posing a direct opposition to the *tertium non datur* or "law of the excluded middle." And this position finds support for example in the work of Reichenbach [18], who reasons himself to a similar position in his study of quantum mechanics.

It may be felt that the *tertium non datur* to which we have just referred is, in some sense, an empirical one, and therefore distinct from the principle of the same name in logic. What we are now about to suggest will be seen to entail a denial that there is any logico-mathematical versus empirical distinction at all.

Although the attitude prevails that mathematics (inclusive of logic) is a body of knowledge not rooted by its nature in the empirical, the fact remains that the fundamental articles of mathematics come to us by an intuition in which the empirical is undeniable. We are referring to *sets of points*, and to the intuition of a set of points as a *collection, or aggregate, or class, of things that we designate as points.* It is this picture of a "collection" of "things" that underlies all deliberations concerning sets, up to the present time. The picture leads to no difficulties where finite sets are concerned. But taken at face value for infinite sets, it produces, for example, the well known Russell's paradox (see [6]). For an excellent critical discussion of the various formulations put forward with intention to resolve all such difficulties, we refer the reader to the book [23] of Weyl. It is to be pointed out here that one of these formulations, the *intuitionist* approach of L. E. J. Brouwer, denies the law of the excluded middle in the domain of infinite sets. And it comes by doing so as a result of proceeding in the same general spirit as that which guides our own venturing; namely, to keep one's feet always planted firmly on the empirical ground. For Brouwer, infinite sets are not collections-in-being, but collections-in-growth.

The point of view that we wish to set forth is not in the nature of an independent whim. It is intimately tied to the development of our theory. To see this, consider some of the steps through which the development has carried us. We set down the mathematical description of a stochastic process. This involved the procedure of appealing to *spaces of points* and to the algebras of *sets of points* in these spaces. These sets represented what, in the usual terminology, would have been called "events." The "algebra of sets" was the customary mathematical translation of the "logic of events"—or, rather, would have been if we had spoken of "events." But, from the very beginning, we did not speak of "events." And our persistence in this soon showed itself to be a purposeful intention to split the classical notion of "event," in which there is no distinction as between actual and potential, into two notions: *act* and *eventuality*. Ultimately this intention was realized in a concrete way when we laid down the proposal that acts and eventualities are primitive real entities. Thus we arrived at the representative identification of "sets of points" with eventualities. This identification, we now assert, groans with imbalance. On the one hand, a "set of points" is a conceptually composite affair, a "collection" of "things"; and on the other hand, an eventuality is a primitive entity, having no deeper conceptual analysis. A mathematical description of reality should reflect faithfully the organization of real concepts; there are no sufficient grounds for believing it must be otherwise.

The suggestion to be made is now clearly motivated. It is that the "sets" of mathematics may be of at least two distinct varieties: those that represent eventualities and those that represent aggregates of systems. Concerning the first of these, it remains to be seen with the complete clarification of the foundations of the present theory what their algebra is like. Concerning the second, their algebra and its closure seem evident along the following lines: an aggregate of systems is, just as a system, a stochastic process. A *union* of such aggregates is the smallest parent process; an *intersection* is the largest common marginal process.

We content ourselves with this brief indication. It makes our point, namely, that the resolution of questions at the foundations of mathematics may very well be found only in a unified mathematical-empirical approach. And if we are correct in this, then the rules of logic will be found in the category of natural laws.

10. Concerning language

One of the most striking implications of the theory we have expounded is that the structure of language is oblique to the structure of reality. It would perhaps be more correct to refer to this circumstance rather as a striking *corollary* to the revision of the classical conceptual picture. For, our science has undoubtedly been strongly influenced by the manner in which we must communicate concerning phenomena. In any case, the point is this: single words (man, book, walking, playing) describe stochastic processes, while acts and eventualities, which are the primitive building blocks of processes, require extensive verbal constructions for their description (Mr. Smith had steak for dinner last night; that I shall soon drop this ball and that it will hit the ground in what I shall read from my watch to be one and two-thirds seconds). We do not intend to pursue here a detailed study of this situation, but only to make a few observations relating to it.

It is an experience met with repeatedly by everyone that our attempts to describe people and other things (systems) in a few words prove inadequate, and recourse must be had to describing how they behaved and how (in our estimation) they will behave. For example, if we say to a friend that our automobile is reliable, or that Mr. Jones is a good man, our friend may find himself unsatisfactorily informed. A further attempt in the same direction may still not fill the bill; for example: Mr. Jones is a friendly and generous man. Being steadily pressed by our friend, we will yield to speaking as follows: "Well, for example, when our lawn mower broke last week right in the middle of the job, Jones noticed it and immediately offered us the use of his mower. And whenever we need some help on a heavy job around the house, we can always count on his assistance with a smile and good spirits. If anyone at all needed help, he would probably leave anything he was doing to lend a hand." Now, yielding to this manner of conveying the information about Mr. Jones seems to be a less worthy procedure. One tends to feel that resorting to such long, detailed descriptions means that one is removing oneself from the mode of essential description of the system (man, good, friendly, generous, . . .). But the fact of the matter is that just the reverse is true. What has been conveyed in the quotes above are acts and eventualities and some rough estimates of probabilities. And these are the primitive entities, the building blocks which make up the process that is Mr. Jones. That Mr. Jones is a man, that he is good, friendly, generous, etc., these are over-all qualities of the process. And these qualities, lacking exact definitions in terms of eventualities, acts and probabilities, do not, with their specification, always take the place of getting down to detailed specification of acts, eventualities and probabilities themselves. (See again our discussion of the chess player in section 3.) It is a deceptive circumstance that getting down to a discussion of the basic elements of the structure of a system involves expanding to the full use of all the syntactical structure of language.

The structure of language *is* in conformity with the *classical* conceptual picture of reality. And it is by this marriage that scientists have prevailed upon humanity at large to see itself—tied as it is to discussing reality exclusively in full verbal expressions—as beneath the dignity of science, which, in its old framework, has expressed itself in single nouns and verbs with a sprinkling of adjectives. We submit that that science, with its laconic verbalizations, has been speaking of the intricate, composite qualities of stochastic processes, while from the beginning the poets and novelists and the common man have been discussing the basic elements of reality.

REFERENCES

[1] K. J. Arrow, "Alternative approaches to the theory of choice in risk-taking situations," *Econometrica*, Vol. 19 (1951), pp. 404–437.

[2] E. W. Barankin, "Quantitative theory of human behavior, I: the single individual," RAND Corporation Research Memorandum, RM-900 (1952).

[3] ———, "Foundations of a theory of behavior" (abstract), *Econometrica*, Vol. 21 (1953), pp. 474–475

[4] R. Carnap, *Logical Foundations of Probability*, Chicago, University of Chicago Press, 1950.

[5] H. Chernoff, "Rational selection of decision functions," *Econometrica*, Vol. 22 (1954), pp. 422–443.

[6] R. Courant and H. Robbins, *What Is Mathematics?* New York, Oxford University Press, 1951.

[7] L. de Broglie, *The Revolution in Physics*, New York, The Noonday Press, 1953.

[8] B. de Finetti, "La prévision: ses lois logiques, ses sources subjectives," *Ann. de l'Inst. Henri Poincaré*, Vol. 7 (1937), pp. 1–68.

[9] A. Einstein, *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp, Evanston, Ill., The Library of Living Philosophers, 1949.

[10] G. C. Evans, *Mathematical Introduction to Economics*, New York, McGraw-Hill Book Co., 1930.

[11] R. P. Feynman, "The concept of probability in quantum mechanics," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1951, pp. 533–541.

[12] H. Jeffreys, *Theory of Probability*, Oxford, The Clarendon Press, 1948.

[13] J. M. Keynes, *A Treatise on Probability*, London, Macmillan and Co., 1952.

[14] M. Loève, "Remarques sur les ensembles de lois," *C. R. Acad. Sci. Paris*, Vol. 222 (1946), pp. 628–630.

[15] J. Marshak, "Rational behavior, uncertain prospects, and measurable utility," *Econometrica*, Vol. 18 (1950), pp. 111–141.

[16] F. P. Ramsey, *The Foundations of Mathematics*, edited by R. B. Braithwaite, New York, The Humanities Press, 1950.

[17] H. Reichenbach, *The Theory of Probability*, Berkeley and Los Angeles, University of California Press, 1949.

[18] ———, *Philosophic Foundations of Quantum Mechanics*, Berkeley and Los Angeles, University of California Press, 1944.

[19] L. J. Savage, *The Foundations of Statistics*, New York, John Wiley and Sons, 1954.

[20] R. von Mises, *Wahrscheinlichkeitsrechnung*, New York, Mary S. Rosenberg, 1945.

[21] ———, *Probability, Statistics and Truth*, New York, Macmillan Co., 1939.

[22] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton, Princeton University Press, 1947.

[23] H. Weyl, *Philosophy of Mathematics and Natural Science*, Princeton, Princeton University Press, 1949.

[24] N. Wiener, *Cybernetics*, New York, John Wiley and Sons, 1949.