5. BOUNDS FOR THE DISTRIBUTION OF # W .

The principal result of this chapter is that

$$(5.1) \qquad P_p\{\#W(v) \geq n\}$$

decreases exponentially in n, provided certain crossing probabilities are sufficiently small. This is almost the only theorem which works for a general periodic percolation problem in any dimension. No axes of symmetry are required, nor does the graph have to be one of a matching pair. When Theorem 5.1 is restricted to one-parameter problems, then it shows that (5.1) decreases exponentially for $p < p_T$ and that in general $p_T = p_S$ (see (3.62)-(3.65) for definition). In Sect. 5.2 we discuss lower bounds for

$$(5.2) \qquad P_p\{\#W(v) = n\}$$

when p is so large that percolation occurs. In the one-parameter case this is the interval $p_H < p \leq 1$. It turns out that (5.2), and hence (5.1) does not decrease exponentially in this domain. We have no estimates for (5.1) for p-values at which

$$(5.3) \qquad E_p\{\#W(v)\} = \infty, \text{ but } \quad \theta(p,v) = P_p\{\#W(v) = \infty\} = 0 \ ,$$

except in the special cases of $\mathcal{G}_0$ and $\mathcal{G}_1$ (see Theorem 8.2). Of course if Theorem 3.1 and Cor. 3.1 apply then (5.3) can happen only on the critical surface, and one may conjecture that in general the set of p-values at which (5.3) holds has an empty interior. In one-parameter problems this amounts to the conjecture that $p_T = p_H$ in all periodic percolation problems. If one goes still further one might conjecture that (5.1) decreases only as a power of n whenever (5.3) holds. For bond- or site-percolation on $\mathbb{Z}^2$, Theorem 8.2 indeed gives a lower bound of the form $n^{-\gamma}$ for (5.1) at $p = p_H$.

In Sect. 5.3 we discuss a result of Russo (1981) which is more or less dual to Theorem 5.1. If in dimension two certain crossing probabilities are large enough, then percolation does occur. Sect. 5.2 and 5.3 are not needed for later chapters.

Throughout this chapter $\mathcal{G}$ will be a periodic graph imbedded in $\mathbb{R}^d$ which satisfies (2.2)-(2.5). We deal with a periodic[1] λ-parameter site problem and take $(\Omega, \mathcal{B}, P_p)$ as in (3.19)-(3.23). We also use the following special notation: For $\bar{n} = (n_1, \dots, n_d)$ we set

$$(5.4) \qquad T(\bar{n};i) = \{x = (x(1),\dots,x(d)) : 0 \le x(j) \le 3n_j, j \ne i, \ 0 \le x(i) \le n_i\}$$
$$= [0,3n_1] \times \dots \times [0,3n_{i-1}] \times [0,n_i] \times [0,3n_{i+1}] \times \dots \times [0,3n_d]$$

The block $T(\bar{n};i)$ is "short" in the i-th direction, as illustrated in Fig. 5.1 for $d = 2$ and $\bar{n} = (1,1)$.

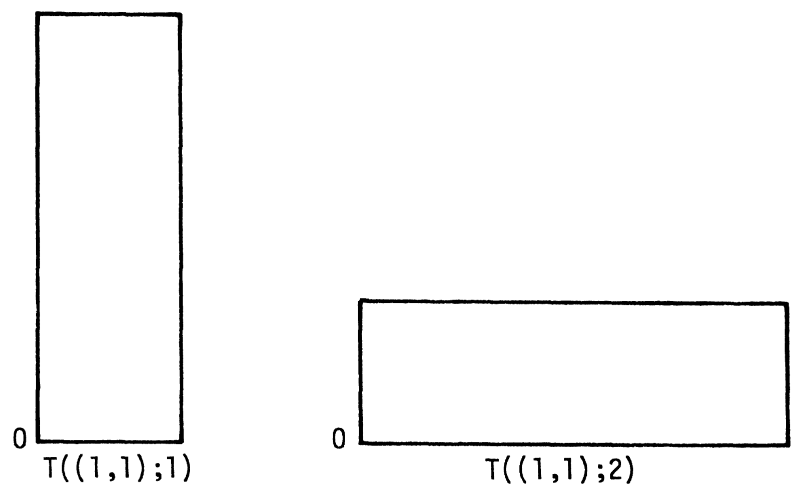

Figure 5.1

The corresponding crossing probabilities are defined as

$$(5.5) \qquad \tau(\bar{n};i,p) = \tau(\bar{n};i,p,\mathcal{G}) = \sigma((3n_1,\dots,3n_{i-1},n_i,3n_{i+1},\dots,3n_d);i,p,\mathcal{G})$$
$$= P_p\{\,\exists \text{ an occupied } i\text{-crossing on } \mathcal{G} \text{ of } T(\bar{n};i)\}$$

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[1] Actually one does not need periodicity for most results of this chapter, but it simplifies the formulation of the results.

and

(5.6) $\tau^*(\bar{n};i,p) = \tau(\bar{n};i,p,\mathcal{G}) = \tau(n;i,\bar{1}-p,\mathcal{G}^*) = \sigma((3n_1,\ldots,3n_{i-1},n_i,$ $3n_{i+1},\ldots,3n_d;i,\bar{1}-p,\ \mathcal{G}^*) = P_p\{\exists$ a vacant i-crossing on $\mathcal{G}^*$ of $T(\bar{n};i)\}$.

5.1 <u>Exponential fall off of</u> $P\{\#W \geq n\}$.

We need the following constants:

(5.7) μ = number of vertices $v = (v_1,\ldots,v_d)$ of $\mathcal{G}$ with $0 \leq v_i < 1,\ 1 \leq i \leq d$.

(5.8) Λ is any number such that $|v-w| \leq \Lambda$ for all adjacent pairs of vertices v,w of $\mathcal{G}$.

(5.9) $\kappa = \kappa(d) = \frac{1}{2d}(2e\ 5^d)^{-11^d}$.

Furthermore, z_0 is some fixed vertex of $\mathcal{G}$ and

$$W = W(z_0).$$

<u>Theorem 5.1. If for some</u> $\bar{N} = (N_1,\ldots,N_d)$ <u>with</u> $N_i \geq \Lambda,\ 1 \leq i \leq d$, <u>one has</u>

(5.10) $\tau(\bar{N};i,p) \leq \kappa,\ 1 \leq i \leq d,$

<u>then there exist constants</u> $0 < C_1,\ C_2 < \infty$ <u>such that</u>

(5.11) $P_p\{\#W \geq n\} \leq C_1 e^{-C_2 n},\ n \geq 0$.

(<u>The values of</u> C_1, C_2 <u>are given in (5.40)-(5.42)</u>). <u>If</u>

(5.12) $p(v) > 0$ for all v

<u>and</u>

(5.13) $E_p\{\#W\} < \infty$

<u>then</u>

(5.14) $\tau((n,\ldots,n);\ i,p) \to 0$ as $n \to \infty$

<u>and consequently (5.11) holds</u>.

Corollary 5.1. The set

$$(5.15)\qquad \{p \in \mathcal{P}_\lambda : p >> 0 \quad\text{and}\quad E_p\{\#W\} < \infty\}$$

is open in $\mathcal{P}_\lambda$.

Special Case. In the one-parameter problem with $p(v)$ independent of p the implication (5.10) $\Rightarrow$ (5.11) shows that if $p < p_S$, and hence (5.14), then (5.11) holds, and consequently $p < p_T$ (see (3.63),(3.65)). Conversely $p < p_T$ means (5.13) and this implies (5.14), i.e., $p < p_S$. Thus, in any periodic one-parameter percolation problem

$$(5.16)\qquad p_T = p_S \leq p_H \quad .$$

(The last inequality is just (3.64)). From the fact that $E_p\{\#W\} = \infty$ immediately to the right of p_T and Cor. 5.1 it further follows that

$$(5.17)\qquad E_{p_T}\{\#W\} = \infty$$

is any periodic one-parameter problem. ///

Kunz and Souillard (1978) already proved (5.11) when $p(v) < (z-1)^{-1}$ for all v, where z is as in (2.3). The present proof, which is taken from Kesten (1981) is a reduction to the case of small $p(v)$ by a block approach. The blocks $T(\bar{N};i)$ and suitable translates of them are viewed as the vertices of an auxiliary graph $\mathcal{L}$ with vertex set $\mathbb{Z}^d$. A vertex $\bar{\ell}$ of $\mathcal{L}$ is taken as occupied iff there is an occupied crossing of some associated block of $\mathcal{G}$, and this will have a small probability. Therefore, the distribution of the size of the occupied cluster of a vertex on $\mathcal{L}$ will have an exponentially bounded tail. This, in turn, will imply (5.11) via Lemma 5.2, which relates $\#W$ to the size of an occupied cluster on $\mathcal{L}$.

The proof will be broken down into a number of lemmas. As in Kunz and Souillard (1978) we bring in the numbers

$$(5.18)\qquad a(0,\ell) := \delta_{1,\ell} \quad ,$$

and for $n \geq 1$

$$(5.19)\qquad a(n,\ell) = a(n,\ell;z_0,\mathcal{G}) = \text{number of connected sets } C \text{ of vertices of } \mathcal{G}\text{, containing } z_0\text{, with } \#C = n,\ \#\partial C = \ell\ .$$

Here, analogously to (3.6), $\#C$ denotes the number of vertices in the set C, ∂C is the boundary of C on $\mathcal{G}$ as in Def. 2.8, and C is connected if any two vertices in C are connected by a path on $\mathcal{G}$ all of whose vertices belong to C.

Lemma 5.1. For any $0 \le p \le 1$, $q = 1-p$

$$(5.20) \qquad \sum_{n=0}^{\infty} \sum_{\ell \ge 0} a(n,\ell) p^n q^\ell = 1-\theta(p,z_0) \le 1 .$$

Consequently

$$(5.21) \qquad a(n,\ell) \le \left(\frac{n+\ell}{n}\right)^n \left(\frac{n+\ell}{\ell}\right)^\ell .$$

Also

$$(5.22) \qquad \sum_{\ell \ge 0} a(n,\ell) \le \{(z+1)^{z+1} z^{-z}\}^n$$

and for some universal constant $\varepsilon_0 > 0$ and $0 \le x \le \varepsilon_0$, $0 \le p \le 1$, $q = 1-p$ and $n \ge 1$

$$(5.23) \qquad \sum_{\substack{\ell \text{ with} \\ |p\ell - qn| \ge xnpq}} a(n,\ell) p^n q^\ell \le nz \exp(-\tfrac{1}{3} x^2 p^2 qn).$$

Proof: The relation (5.20) is well known, and is hardly more than the definition of the percolation probability. It is immediate from

$$(5.24) \qquad P_p\{\bar{W} = \bar{C}\} = p^n q^\ell$$

for any connected set C of vertices containing v_0 and with $\#C = n$, $\#\partial C = \ell$. In fact $\{W = C\}$ occurs iff all vertices of C are occupied, but all vertices adjacent to C but not in C are vacant. The left hand side of (5.20) is simply the sum of (5.24) over all possible finite C. (5.21) follows from (5.20) by taking $p = n/(n+\ell)$, $q = \ell/(n+\ell)$. For (5.22) observe that, by (2.3) and $\partial C \ne \phi$.

$$(5.25) \qquad 1 \le \#\partial C \le z.\#C \quad \text{when} \quad \#C \ge 1 ,$$

so that the sums in (5.20) and (5.22) can be restricted to $1 \le \ell \le n$. when $n \ge 1$. Thus, with $p = (z+1)^{-1}$, $q = z(z+1)^{-1}$ (5.20) yields for $n \ge 1$

$$\sum_{\ell=1}^{n} a(n,\ell) \leq (z+1)^n \left(\frac{z+1}{z}\right)^{zn} \sum_{\ell=1}^{n} a(n,\ell)p^n q^\ell \leq 1,$$

while for $n = 0$ (5.22) is true by definition of $a(0,\ell)$. Finally, by virtue of (5.25) and (5.21) the left hand side of (5.23) is bounded by

$$(5.26) \qquad \sum_{|p\ell-qn| \geq xnpq} a(n,\ell) \left\{\frac{(n+\ell)p}{n}\right\}^n \left\{\frac{(n+\ell)q}{\ell}\right\}^\ell \left(\frac{n}{n+\ell}\right)^n \left(\frac{\ell}{n+\ell}\right)^\ell$$

$$\leq nz \max \left\{\frac{(n+\ell)p}{n}\right\}^n \left\{\frac{(n+\ell)q}{\ell}\right\}^\ell ,$$

where the maximum in (5.26) is over all $1 \leq \ell \leq zn$ with $|p\ell - qn| \geq xnpq$. Now fix n and p and consider

$$(5.27) \qquad f(\ell) := n \log \frac{(n+\ell)p}{n} + \ell \log \frac{(n+\ell)q}{\ell}$$

as a function of a continuous variable ℓ on $(0,\infty)$. One easily sees that $f(.)$ is increasing if $(n+\ell)q/\ell \geq 1$ or $p\ell - qn \leq 0$, and decreasing for $p\ell - qn \geq 0$. Thus, the maximum of f over $|p\ell - qn| \geq xnpq$ is taken on when $p\ell - qn = xnpq$ or $p\ell - qn = -xnpq$. For such a choice

$$\ell = \frac{q}{p} n(1 \pm xp), \ \frac{(n+\ell)p}{n} = 1 \pm xpq, \ \frac{(n+\ell)q}{\ell} = \frac{1 \pm xpq}{1 \pm xp} .$$

A simple expansion of the logarithms in (5.27) now shows that for small x and $p\ell - qn = \pm xnpq$

$$f(\ell) = -\frac{1}{2} n x^2 p^2 q \{1 + O(x)\} ,$$

with $|x^{-1} O(x)|$ bounded uniformly in n,ℓ,p. (5.23) follows. $\square$

We now define the auxiliary graph $\mathcal{L}$, and derive a relation between W and a certain occupied component on $\mathcal{L}$. The vertex set of $\mathcal{L}$ is $\mathbb{Z}^d$. The vertices $\bar{k} = (k(1),\ldots,k(d))$ and $\bar{\ell} = (\ell(1),\ldots,\ell(d))$ are adjacent on $\mathcal{L}$ iff

$$|k(i) - \ell(i)| \leq 2, \ 1 \leq i \leq d .$$

We associate with an occupancy configuration on $\mathcal{G}$ an occupancy configuration on $\mathcal{L}$ in the following manner: We take $\bar{k} \in \mathcal{L}$ occupied iff there exists an occupied path $r = (w_0,e_1,\ldots,e_\tau,w_\tau)$ on $\mathcal{G}$ whose initial point satisfies

(5.28) $$k(j)N_j \leq w_0(j) < (k(j)+1)N_j, \ 1 \leq j \leq d,$$

and whose final point w_τ satisfies

(5.29) $$w_\tau(i) \leq (k(i)-1)N_i \quad \text{or} \quad w_\tau(i) \geq (k(i)+2)N_i$$
$$\text{for some } 1 \leq i \leq d .$$

We shall now prove the estimate

(5.30) $$P_p \{\overline{k} \text{ is occupied}\} \leq 2 \sum_{i=1}^{d} \tau(\overline{N};i,p) ,$$

which is basic for our proof. To see (5.30) observe that if there exists an occupied path r for which (5.28) and (5.29) hold, then there is a smallest index b for which there exists an i such that e_b intersects one of the hyperplanes

$$H_i^- : \{x : x(i) = (k(i)-1)N_i\} \quad \text{or}$$
$$H_i^+ : \{x : x(i) = (k(i)+2)N_i\} .$$

e_b may intersect $H_i^- \cup H_i^+$ for several i. For each such i, let ζ_{bi} be the first intersection of w_b with $H_i^- \cup H_i^+$ and let i_0 be an index such that ζ_{bi_0} precedes all the other ζ_{bi} which exist. Then

(5.31) e_ℓ (including its endpoints $w_{\ell-1}$ and w_ℓ) lies strictly between H_j^- and H_j^+ for all $1 \leq \ell < b$ and $1 \leq j \leq d$; the same is true for the segment $[w_{b-1}, \zeta_{bi_0})$.

For the sake of argument assume e_b intersects $H_{i_0}^+$ so that $\zeta_{bi_0} \in H_{i_0}^+$. Then take a as the largest index less than b for which e_a intersects the hyperplane $x(i_0) = (k(i_0)+1)N_{i_0}$. Such an a exists by (5.28). Also take ζ_{ai_0} as the last intersection of w_a with the hyperplane $x(i_0) = (k(i_0)+1)N_{i_0}$. Then

(5.32) e_ℓ (including its endpoints $w_{\ell-1}$ and w_ℓ) lies strictly between the hyperplanes $x(i_0) = (k(i_0)+1)N_i$ and $H_{i_0}^+$ for all $a < \ell < b$; the same is true for the segment $(\zeta_{ai_0}, w_{a+1}]$.

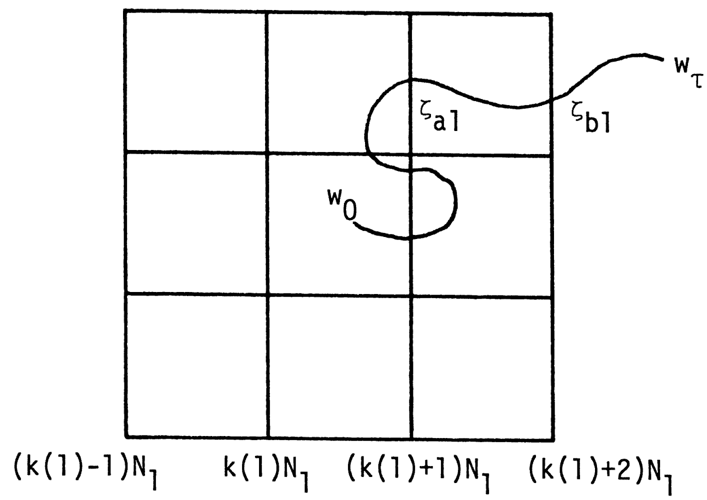

Figure 5.2

(5.31) and (5.32) say that $(w_a, e_{a+1}, \ldots, e_b, w_b)$ is an i_0-crossing of the block with sides

$$[(k(j) - 1)N_j,\ (k(j) + 2)N_j] \quad \text{for} \quad j \neq i$$

and

$$[(k(i_0) + 1)N_{i_0},\ (k(i_0) + 2)N_{i_0}] \quad \text{for} \quad j = i_0 \ .$$

This is precisely the block

$$(5.33) \qquad T(\bar{N}; i_0) + \sum_{j=1}^{d} (k(j) - 1)N_j\xi_j + 2k(i_0)N_{i_0}\xi_{i_0} \ ,$$

where, as before, ξ_j is the j-th coordinate vector in $\mathbb{R}^d$. Moreover, since r is occupied $(w_a, e_{a+1}, \ldots, e_b, w_b)$ is occupied. By periodicity, the probability that an occupied i_0-crossing of (5.33) exists is at most $\tau(\bar{N}; i_0, p)$. The same estimate holds when w_b intersects $H^-_{i_0}$ instead of $H^+_{i_0}$. (5.30) now follows by summing over all possible i_0.

We next define $\bar{\nu}$ by

$$(5.34) \qquad \nu_j N_j \le z_0(j) < (\nu_j + 1)N_j \ , \ 1 \le j \le d \ ,$$

where z_0 is the vertex which we singled out in $W = W(z_0)$. $\tilde{W}(\bar{\ell})$ will denote the occupied component of $\bar{\ell}$ on $\mathcal{L}$. Finally we remind the reader that $\bar{k}\,\mathcal{L}\,\bar{\ell}$ means that $\bar{k}$ and $\bar{\ell}$ are adjacent on $\mathcal{L}$.

Lemma 5.2. *Assume* W *contains a vertex* $w = (w(1),\dots,w(d))$ *with*

(5.35) $$k(j)N_j \le w(j) < (k(j)+1)N_j,\ 1 \le j \le d\ ,$$

for some $\bar{k}$ *with*

(5.36) $$|k(m) - \nu(m)| \ \ge\ 2 \ \text{ for some } \ 1 \le m \le d\ .$$

Then there exists an occupied path $(\bar{k}_0,\bar{e}_1,\dots,\bar{e}_\rho,\bar{k}_\rho)$ *on* $\mathcal{L}$ *with* $\bar{k}_0 = \bar{k}$ *and*

(5.37) $$|k_\rho(j) - \nu(j)| \ \le 3,\ 1 \le j \le d\ .$$

($\bar{e}_j$ *denotes an edge of* $\mathcal{L}$). *Furthermore*

(5.38) $$\max\ \#\tilde{W}(\ell) \ge \frac{\{\#W - \mu 4^d \prod_{j=1}^{d} N_j\}}{\mu 7^d \prod_{j=1}^{d} N_j}\ ,$$

where the max in (5.38) is over those $\bar{\ell}$ *with*

(5.39) $$|\ell(j) - \nu(j)| \ \le 3,\ 1 \le j \le d\ .$$

Proof: Assume $w \in W$ satisfies (5.35) and (5.36). Then there exists an occupied path $(w_0 = w, e_1,\dots,e_\tau, w_\tau = z_0)$ on $\mathcal{G}$ from w to z_0. By (5.36) and the definition of $\bar{\nu}$, $w_\tau = z_0$ satisfies (5.29). Since also w_0 satisfies (5.28) (see (5.35)) the vertex $\bar{k}$ of $\mathcal{L}$ is occupied, and there exists a smallest index b with $w_b(i) \le (k(i)-1)N_i$ or $w_b(i) \ge (k(i)+2)N_i$ for some i. We take $\bar{k}_0 = \bar{k}$ and $\bar{k}_1$ such that

$$k_1(j)N_j \le\ w_b(j)\ <\ (k_1(j)+1)N_j,\ 1 \le j \le d\ .$$

This $\bar{k}_1$ is uniquely determined, and by virtue of the minimality of b, and $N_j \ge \Lambda$,

$$|k_0(j) - k_1(j)|\ \le\ 2,\ 1 \le j \le d\ .$$

(Compare (5.31) and Fig. 5.2.) Thus $\bar{k}_0\,\mathcal{L}\,\bar{k}_1$.

We now repeat the procedure with w_b and $\bar{k}_1$ in the place of w_0 and $\bar{k}_0$. If the analogue of (5.36) still holds for $\bar{k}_1$, i.e., if

$|k_1(m) - \nu(m)| \geq 2$ for some m, then $\bar{k}_1$ is occupied and we find a neighbor $\bar{k}_2$ of $\bar{k}_1$ on $\mathcal{L}$, and so on. We continue this process as long as possible. It stops when we have obtained a sequence $\bar{k}_0, \ldots, \bar{k}_\rho$ of occupied points on $\mathcal{L}$ and a $\bar{k}_{\rho+1} \in \mathcal{L}$ such that

$$\bar{k}_t \; \mathcal{L} \; \bar{k}_{t+1}, \; 0 \leq t \leq \rho ,$$

while the analogue of (5.36) fails for $\bar{k}_{\rho+1}$, i.e.,

$$|k_{\rho+1}(j) - \nu(j)| \leq 1, \; 1 \leq j \leq d .$$

Since $\bar{k}_\rho \; \mathcal{L} \; \bar{k}_{\rho+1}$ this implies that (5.37) holds. Thus there exists an occupied path $(\bar{k}_0 = \bar{k}, \bar{e}_1, \ldots, \bar{e}_\rho, \bar{k}_\rho)$ as claimed in the first part of the lemma. (Note that we may have to apply the loop-removal procedure of Sect. 2.1 to make the path self-avoiding.)

The inequality (5.38) now follows easily from the first part of the lemma. Each vertex $w \in W$ with $|w(m) - z_0(m)| \geq 2N_m$ for some m satisfies (5.35) and (5.36) for some $\bar{k}$. There are at least

$$\#W - \mu \prod_{j=1}^{d} (4N_j).$$

such vertices w. Each such w leads to an occupied path of the above type on $\mathcal{L}$ starting at some $\bar{k}_0$ and ending at a $\bar{k}_\rho$ satisfying (5.37). A fixed $\bar{k}_0$ can arise as starting point for such a path only for a w with

$$k_0(j)N_j \leq w(j) < (k_0(j) + 1)N_j$$

(see (5.35)). Since there are at most $\mu \prod N_j$ such vertices w on $\mathcal{G}$, at least

(5.40) $$\{\mu \prod_{j=1}^{d} N_j\}^{-1} \{\#W - \mu \prod_{j=1}^{d} (4N_j)\}$$

distinct vertices $\bar{k}_0$ arise as the initial point of an occupied path on $\mathcal{L}$ which ends at some $\bar{k}_\rho$ satisfying (5.37). Since there are at most 7^d points $\bar{k}_\rho$ which satisfy (5.37), (5.38) now follows from (5.40). □

<u>Lemma 5.3.</u> <u>(5.10) implies (5.11) with</u>

(5.41) $$A = 7^{-d}(\mu N_1 \ldots, N_d)^{-1} ,$$

$$(5.42)\qquad C_1 = \left(\tfrac{7}{5}\right)^d e^{-1} \{2 \sum_{i=1}^{d} \tau(N;i,p)\}^{-11^{-d}}$$

$$[1 - e5^d \{2 \sum_{i=1}^{d} \tau(N;i,p)\}^{11^{-d}}]^{-1}$$

$$\leq \left(\tfrac{7}{5}\right)^d \{2 \sum_{i=1}^{d} \tau(N;i,p)\}^{-11^{-d}} \quad ,$$

$$(5.43)\qquad e^{-C_2} = (e\ 5^d)^A \{2 \sum_{i=1}^{d} \tau(\overline{N};i,p)\}^{A11^{-d}} \leq 2^{-A} \quad .$$

<u>Proof</u>: By (5.38)

$$(5.44)\qquad P_p\{\#W \geq n\} \leq \sum_{\substack{\overline{\ell} \text{ satisfying} \\ (5.39)}} P_p\{\#\tilde{W}(\overline{\ell}) \geq An - 1\} \ .$$

Set, for any $\overline{\ell} \in \mathcal{L}$

$\tilde{b}(m)$ = number of connected sets on $\mathcal{L}$ of m vertices and containing $\overline{\ell}$.

Note that $\tilde{b}(m)$ does not depend on $\overline{\ell}$ by the periodicity of $\mathcal{L}$. Recall also that at most 7^d points $\overline{\ell}$ satisfy (5.39). Therefore, (compare (3.15) and the proof of (5.20)) the right hand side of (5.44) is bounded by

$$(5.45)\qquad 7^d \sum_{m \geq An-1} \tilde{b}(m) \max_{\#\tilde{C} = m} P_p\ \{\text{all vertices in } \tilde{C} \text{ are occupied}\}$$

where $\tilde{C}$ in (5.45) runs over the connected sets of vertices of $\mathcal{L}$ with cardinality m. To estimate the probability appearing in (5.45) we observe that we are not dealing with a percolation problem on $\mathcal{L}$ because the occupancies of the vertices of $\mathcal{L}$ are not independent. However, the occupancy of a vertex $\overline{\ell}$ of $\mathcal{L}$ depends only on the occupancies of the vertices v of $\mathcal{G}$ with

$$(\ell(j)-2)N_j \leq (\ell(j)-1)N_j-\Lambda \leq v(j) \leq (\ell(j)+2)N_j+\Lambda \leq (\ell(j)+3)N_j, 1 \leq j \leq d.$$

Thus, if $\overline{\ell}_1,\dots,\overline{\ell}_t$ are vertices of $\mathcal{L}$ such that for each $r \neq s$ there exists an i with $|\ell_r(i) - \ell_s(i)| \geq 6$, then the occupancies of $\overline{\ell}_1,\dots,\overline{\ell}_t$ are independent (because they depend on disjoint sets of vertices of $\mathcal{G}$). Now given $\tilde{C}$ with $\#\tilde{C} = m$ we can choose $\overline{\ell}_1,\dots,\overline{\ell}_t \in \tilde{C}$ with the above property for some $t \geq 11^{-d}m$. With $\overline{\ell}_1,\dots,\overline{\ell}_t$ chosen in this way we have by virtue of (5.30)

$$P_p \{\text{all vertices in } \tilde{C} \text{ are occupied}\}$$
$$\leq P_p\{\overline{\ell}_1,\ldots,\overline{\ell}_t \text{ are occupied}\} \leq \{2 \sum_{i=1}^{d} \tau(\overline{N};p,i)\}^t .$$

Substitution of this estimate with $t = 11^{-d}m$ into (5.45) yields

$$(5.46) \quad P_p\{\#W \geq n\} \leq 7^d \sum_{m \geq An-1} \tilde{b}(m)\{2 \sum_{i=1}^{d} \tau(N;p,i)\}^{11^{-d}m} .$$

Finally, (5.22) applied to the graph $\mathcal{L}$ with $5^d - 1$ for z shows

$$\tilde{b}(m) \leq \{(z+1)(\frac{z+1}{z})^z\}^m \leq (e\ 5^d)^m .$$

This together with (5.46) implies (5.11) with the values (5.42) and (5.43) of C_1, C_2 . □

Lemma 5.4. (5.12) and (5.13) imply (5.14).

Proof: This lemma basically proves that the diameter of W has an exponentially decreasing distribution under (5.13). This fact was first proved by Hammersley (1957), Theorem 2. We make the following definition for positive integers m,M and u a vertex of $\mathcal{G}$

$$S_0 = S_0(u,M) = \{w \text{ a vertex of } \mathcal{G}: |w(j) - u(j)| \leq M,\ 1 \leq j \leq d\} ,$$

$$S_1 = S_0 \cup \partial S_0 = \{w \text{ a vertex of } \mathcal{G}: w \in S_0 \text{ or } w \text{ adjacent to a vertex in } S_0\},$$

$$(5.47) \quad A(u,m) = \{ \exists \text{ an occupied path on } \mathcal{G} \text{ from a neighbor of } u \text{ to a } w \text{ with } w(1) \geq m\} ,$$

$$g(u,w,M) = P_p\{ \exists \text{ occupied path } (w_0,e_1,\ldots,e_\rho,w_\rho) \text{ on } \mathcal{G} \text{ with } w_0 \mathcal{G} u,\ w_\rho \notin S_0(u,M) \text{ and one of the } w_i \text{ equal to } w\}.$$

We claim that if $u(1) < m - M$ then

$$(5.48) \quad P_p\{A(u,m)\} \leq \sum_{w \in S_1(u,M)} g(u,w,M) P_p\{A(w,m)\} .$$

To prove (5.48) assume that $A(u,m)$ occurs. Then there exists an occupied path $r = (v_0 = u,e_1,\ldots,e_\nu,v_\nu)$ on $\mathcal{G}$ with $v_0 = u$ and $v_\nu(1) \geq m > v_0(1) + M$. Therefore $v_\nu \in S_0$ and there exists a smallest index a, $1 \leq a \leq \nu$ with $v_a \notin S_0$. Now set

$$R = \{w \in S_1: \exists \text{ occupied path } (w_0, e_1, \dots, e_\rho, w_\rho) \text{ on } \mathcal{G} \text{ with } w_0 \mathcal{G} u,\ w_\rho \notin S_0 \text{ but } w_t \in S_0 \text{ for } t < \rho \text{ and one of the } w_i \text{ equal to } w\}.$$

R is the random set of vertices in S_1 through which there exists an occupied path from a neighbor of u to the complement of S_0, which except for its final point contains only vertices in S_0. By choice of a, $v_a \in R$. Let $b \geq a$ be the last index with $v_b \in R$. Now consider the occupied path $(v_{b+1}, e_{b+2}, \dots, e_\nu, v_\nu)$. All its vertices lie outside R, its initial point is adjacent to $v_b \in R$ and its final point v_ν satisfies $v_\nu(1) \geq m$. Thus $A(v_b, m)$ occurs. Summing over all possibilities for v_b and R gives the inequality

$$(5.49)\quad P_p\{A(u,m)\} \leq \sum_{w \in S_1} P_p\{w \in R \text{ and } \exists \text{ an occupied path } (w_0, f_1, \dots, f_\rho, w_\rho) \text{ on } \mathcal{G} \text{ with } w_0 \mathcal{G} w,\ w_\rho(1) \geq m \text{ and } w_i \notin R \text{ for } 0 \leq i \leq \rho\} = \sum_{w \in S_1} \sum_{\substack{C \subset S_1 \\ w \in C}} P_p\{R = C \text{ and } \exists \text{ an occupied path } (w_0, f_1, \dots, f_\rho, w_\rho) \text{ on } \mathcal{G} \text{ with } w_0 \mathcal{G} w,\ w_\rho(1) \geq m \text{ and } w_i \notin C \text{ for } 0 \leq i \leq \rho\}.$$

We now fix w and a subset C of S, containing w and estimate the last probability in (5.49). Observe that $R = C$ iff both the following two events occur:

$$\mathcal{C}_1 = \{\text{For every vertex } x \in C \text{ there exists an occupied path } (u_0, g_1, \dots, g_\tau, u_\tau) \text{ on } \mathcal{G} \text{ with } u_0 \mathcal{G} u,\ u_\tau \notin S_0, \text{ but } u_\tau \in S_0 \text{ for } t < \tau,\ u_i \in C \text{ for } 0 \leq i \leq \tau \text{ and } x \text{ equals one of the } u_i\},$$

$$\mathcal{C}_2 = \{\text{any path } (u_0, g_1, \dots, g_\tau, u_\tau) \text{ on } \mathcal{G} \text{ with } u_0 \mathcal{G} u,\ u_\tau \notin S_0, \text{ but } u_\tau \in S_0 \text{ for } t < \tau \text{ and not all } u_i \in C \text{ contains at least one vacant } u_j \notin C\}.$$

All vertices on the paths $(u_0, g_1, \dots, u_\tau)$ in the description of $\mathcal{C}_1$ must belong to C, because whenever such a path satisfies $u_0 \mathcal{G} u$, $u_\tau \notin S_0$

$u_\tau \in S_0$ for $t < \tau$ and all u_i occupied, then all its u_i automatically belong to R. Not all sets $C \subset S_1$ are such that $\mathcal{C}_1$ can occur; e.g., C can only have components which contain a neighbor of u. But in any case $I_{\mathcal{C}_1}(\omega)$ is a function of the occupancies of the vertices in C only. If $\mathcal{C}_1$ can occur, then

$$\mathcal{C}_1 = \{\text{all vertices of } C \text{ are occupied}\} .$$

Also $I_{\mathcal{C}_2}(\omega)$ is a function of the $\omega(y)$ with $y \notin C$ and it is a decreasing function. On the other hand

$$\text{(5.50)} \qquad \begin{aligned} J(\omega) = J(w,\omega) := I[\ \exists \text{ an occupied path } (w_0,f_1,\dots,f_\rho,w_\rho) \\ \text{on } \mathcal{G} \text{ with } w_0 \mathcal{G} w,\ w_\rho(1) \geq m \text{ and } w_i \notin C \text{ for } 0 \leq i \leq \rho\} \end{aligned}$$

is an increasing function of the occupancies of the $\omega(v)$, $y \notin C$. By the independence of the $\omega(y)$ with $y \in C$ and with $y \notin C$ the last probability in (5.49) can be written as

$$E_p\{I_{\mathcal{C}_1} I_{\mathcal{C}_2}\ J\} = E_p\{I_{\mathcal{C}_1}\}\ E_p\{I_{\mathcal{C}_2}\ J\} .$$

Next it follows immediately from the FKG inequality (apply Prop. 4.1 to $I_{\mathcal{C}_2}$ and $1-J$ for instance) that

$$E_p\{I_{\mathcal{C}_2} J\} \leq E_p\{I_{\mathcal{C}_2}\}\ E_p\{J\} .$$

Substituting this into (5.49) and using the independence of $\mathcal{C}_1$ and $\mathcal{C}_2$ once more, as well as the simple inequality $J(w,\omega) \leq I[A(w,m)]$ we obtain

$$\begin{aligned} P_p\{A(u,m)\} &\leq \sum_{w \in S_1} \sum_{\substack{C \subset S_1 \\ w \in C}} E_p\{I_{\mathcal{C}_1}\}\ E_p\{I_{\mathcal{C}_2}\}\ E_p\{J\} \\ &= \sum_{w \in S_1} \sum_{\substack{C \subset S_1 \\ w \in C}} E_p\{I_{\mathcal{C}_1} I_{\mathcal{C}_2}\} E_p\{J\} \\ &\leq \sum_{w \in S_1} \sum_{\substack{C \subset S_1 \\ w \in C}} P_p\{R = C\}\ P_p\{A(w,m)\} \\ &= \sum_{w \in S_1} P_p\{w \in R\}\ P_p\{A(w,m)\} \leq \sum_{w \in S_1} g(u,w,M)\ P_p\{A(w,m)\}. \end{aligned}$$

This proves (5.48). We next show that we can choose M such that

$$(5.51) \qquad \sum_{w \in S_1(u,M)} g(u,w,M) \le \frac{3}{4} \text{ for all } u \in \mathcal{G} .$$

This is easy, because any path from a neighbor of u to the complement of $S_0(u,M)$ has diameter $\ge M-\Lambda$ and therefore contains at least M/Λ vertices. Consequently

$$g(u,w,M) \le P_p\{w \in W(x) \text{ and } \#W(x) \ge M/\Lambda \text{ for some neighbor } x \text{ of } u\}$$

and, by virtue of (4.8)

$$(5.52) \qquad \sum_{w \in S_1(u,M)} g(u,w,M) \le \sum_{\substack{x \text{ such that} \\ x \mathcal{G} u}} E_p\{\#W(x);\ \#W(x) \ge \frac{M}{\Lambda}\}$$

$$\le \sum_{x \mathcal{G} u} [P_p\{x \text{ and } u \text{ are occupied}\}]^{-1} E_p\{\#W(u);\ \#W(u) \ge \frac{M}{\Lambda}\} .$$

Under (5.12) and (5.13) the right hand side of (5.52) tends to zero as $M \to \infty$ when $u = z_0$. But, by the Application in Sect. 4.1 (5.12) and (5.13) imply

$$E_p\ \{\#W(u)\} < \infty \text{ for all } u \in \mathcal{G}$$

Consequently the right and left hand side of (5.52) tend to zero as $M \to \infty$ for any vertex u. In particular

$$(5.53) \qquad \lim_{M \to \infty} \sum_{w \in S_1(u,M)} g(u,w,M) = 0$$

uniformly for the finitely many u in $[0,1)^d$. By periodicity,

$$\sum_{w \in S_1(u,M)} g(u,w,M)$$

is unchaged if u is replaced by $u + \sum k_j \xi_j$, so that (5.53) holds uniformly in u.

The above shows that (5.51) holds for large enough M. Pick such an M. Then, it follows from (5.48) and the fact that

$$w(1) \le u(1) + M + \Lambda \quad \text{for} \quad w \in S_1(u,M) ,$$

that for $r < m-M$

$$\sup_{u(1) \le r} P_p\{A(u,m)\}$$

$$\le \sum_{w \in S_1(u,M)} g(u,w,M) \sup_{u(1) < r+M+\Lambda} P_p\{A(u,m)\}$$

$$\le \frac{3}{4} \sup_{u(1) \le r+M+\Lambda} P_p\{A(u,m)\} .$$

It follows immediately that

$$\text{(5.54)} \qquad \sup_{u(1) \le 0} P_p\{A(u,m)\} \le \left(\frac{3}{4}\right)^{\lfloor (m-M)/(M+\Lambda) \rfloor} .$$

(5.54) says that the probability that $W(u)$ extends m units in the 1-direction decreases exponentially in m. $\tau((n,\dots,n);1,p)$ is the probability that there exists an occupied 1-crossing $(v_0,e_1,\dots,e_\nu,v_\nu)$ on $\mathcal{G}$ of $T((n,\dots,n);1)$. By Def. 3.1 (cf. (3.30) and (3.31)) such a crossing must satisfy $v_\nu(1) - v_0(1) \ge n-2\Lambda$ and the initial point v_0 has to lie in

$$[-\Lambda,\Lambda] \times [-\Lambda,3n+\Lambda]\times\dots\times[-\Lambda,3n+\Lambda]$$

(see (3.29), (3.30), (5.8)). By periodicity and (5.7) there are at most

$$\mu(2\Lambda+1)\ (3n+2\Lambda+1)^{d-1}$$

such vertices v_0. Therefore, by periodicity

$$\text{(5.55)} \quad \tau((n,\dots,n);1,p) \le \mu(2\Lambda+1)(3n+2\Lambda+1)^{d-1} \sup_{u(1) \le 0} P_p\{A(u,n-3\Lambda)\},$$

so that $\tau((n,\dots,n);1,p)$ tends to zero exponentialy as $n \to \infty$, by virtue of (5.54). The same holds for $\tau((n,\dots,n);i,p)$ for any $1 \le i \le d$. This proves the lemma. □

<u>Theorem 5.1</u> is now just a combination of Lemmas 5.3 and 5.4.

<u>Proof of Cor. 5.1</u>: Assume

$$E_{p_0}\{\#W\} < \infty \quad \text{and} \quad p_0 >> 0$$

for some $p_0 \in \mathcal{P}_\lambda$. By (5.14) we can then find an $n \geq \Lambda$ such that $\tau((n,\ldots,n);i,p_0) < \kappa$ for all $1 \leq i \leq d$. Since $\tau((n,\ldots,n);i,p)$ is a continuous function of p for fixed n - it only involves the occupancies of a finite number of vertices - it follows that

$$\tau((n,\ldots,n);i,p) < \kappa \text{ , } 1 \leq i \leq d \text{ ,}$$

holds for p in some neighborhood of p_0. For any p in this neighborhood (5.11) holds, and consequently also (5.13). $\square$

5.2. Estimates above the percolation threshold.

Let $\mathcal{G}$ be a periodic graph imbedded in $\mathbb{R}^d$ and P_p a periodic probability measure. Assume that p is such that percolation occurs, i.e., that

$$\theta(p,v_0) > 0 \quad \text{for some} \quad v_0 \in \mathcal{G} \text{ .} \tag{5.56}$$

Aizenman, Delyon and Souillard, (1980) proved that in this case (5.2) does not decrease exponentially. In fact they showed that

$$P_p\{\#W=n\} \geq C_3\{p \wedge (1-p)\}^{C_4\Theta^{-2}n^{\frac{d-1}{d}}} \tag{5.57}$$

for all n, where

$$p \wedge (1-p) = \min_{v \in [0,1)^d} \{P\{v \text{ is occupied}\} \wedge P\{v \text{ is vacant}\}\}, \tag{5.58}$$

$$\Theta = \sum_{v \in [0,1)^d} \theta(p,v). \tag{5.59}$$

C_3 is a constant depending only on Θ and d, and C_4 is a constant depending only on $\mathcal{G}$ and d. Aizenman et al. (1980), Remark 2.2, pointed out that (5.57) does not give the right behavior near the critical surface, i.e., when Θ becomes small. Indeed one expects Θ to tend to zero as p approaches the critical surface, and for $\Theta \to 0$ the exponent in the right hand side of (5.57) blows up. On the other hand, on the basis of Theorem 8.2 (dealing with one-parameter problems on $\mathcal{G}_0$ and $\mathcal{G}_1$) we expect (5.2) to decrease only polynomially in n when p is on the critical surface. Theorem 5.2 gives a lower bound for (5.2) with an exponent containing Θ to a positive power. Even though this improvement meets the above objection, we have to pay a price. Our estimate is not valid for all n, and we do not have much control over the domain of n-values for which the estimate holds. It should also be said that Aizenman et al. prove their estimate (5.57) in much more general models than our independent site-percolation models. To avoid uninteresting combinatorial complications we restrict ourselves in Theorem 5.2 to site-percolation on $\mathbb{Z}^d$. The proof should, however, go through for most periodic percolation problems.

Theorem 5.2. *Let* P *be a probability measure on the occupancy configurations on* $\mathbb{Z}^d$ *which satisfies*

$$(5.60)\qquad P\{v \text{ is occupied}\} = P\{v+k_0\xi_i \text{ is occupied}\}$$

for some integer k_0 *and* $1 \le i \le d$[1]. *Let*

$$(5.61)\qquad \pi := \min_v \{P\{v \text{ is occupied}\} \wedge P\{v \text{ is vacant}\}\} > 0$$

and

$$(5.62)\qquad \Theta := \sum_{v \in [0,k_0)^d} P\{\#W(v) = \infty\} > 0 .$$

Then for $d \ge 3$ *there exists a* $C_3 = C_3(d)$, *depending on* d *only, and an* N_0, *such that for* $n \ge N_0$ *and all* $w \in \mathbb{Z}^d$ *one has*

$$(5.63)\qquad P\{\#W(w) = n\} \ge \pi^{C_3 k_0^{2d-1}\, \Theta^{1/d}\, n^{(d-1)/d}} .$$

The estimate (5.63) remains valid for $d = 2$ *if (5.61) holds and (5.62) is strengthened*[2] *to*

$$(5.64)\qquad E\{\#W^*(v)\} < \infty \quad \text{for some } v,$$

where $W^*(v)$ *is the vacant component of* v *on* $(\mathbb{Z}^2)^* = \mathcal{G}_0^*$ (*see Ex. 2.2(i) for* $\mathcal{G}_0^*$).

---

1) As usual ξ_i is the i-th coordinate vector. For simplicity of notation we required (5.60) instead of our usual periodicity condition (3.18) which corresponds to $k_0 = 1$. To obtain (3.18) one has to replace $\mathbb{Z}^d$ by k_0^{-1} times $\mathbb{Z}^d$.

2) (5.64) and (5.61) imply (5.62) by Lemma 7.3.

Remark.

(i) By Theorem 3.2 (see also Application 3.4(iv)) (5.64) and hence (5.63), hold as soon as (5.61) and (5.62) hold, provided the probability measure P has enough symmetry properties. In particular (5.63) holds for the two-parameter site-percolation problem on $\mathbb{Z}^2$ of Application 3.4(iv) anywhere in the restriction of the percolative region to the interior of $\mathcal{P}_2$, i.e., whenever the parameters $p(1)$, $p(2)$ satisfy $0 < p(1) < 1$, $p(1) + p(2) > 1$. ///

Kunz and Souillard (1978) also prove for $\max_v P\{v \text{ is vacant}\}$ sufficiently small that there exists a constant D for which

$$P\{\#W(w) = n\} \leq \exp{-Dn^{(d-1)/d}} .$$

To give a proof of this estimate for general d would require too much topological groundwork. We shall therefore only prove this result for $d = 2$ and $\mathcal{G}$ one of a matching pair.

Theorem 5.3. Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair of periodic graphs in $\mathbb{R}^2$. Denote by P_p a λ-parameter periodic probability measure defined by means of a periodic partition $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ of the vertices of $\mathcal{G}$ as in (3.17)-(3.23). Assume that $p_0 \in \mathcal{P}_\lambda$ satisfies

(5.65) $$0 \ll p_0 \ll 1 \quad\text{and}\quad E_{p_0}\{\#W^*(z_0)\} < \infty ,$$

where $W^*(z_0)$ is the vacant cluster of z_0 on $\mathcal{G}^*$. Then there exist constants $0 < D_i = D_i(p_0,\mathcal{G}) < \infty$ such that

(5.66) $$P_p\{n \leq \#W(z_0) < \infty\} \leq D_1 e^{-D_2 n^{1/2}} \quad\text{for all}$$

$$p = (p(1),\dots,p(\lambda)) \in \mathcal{P}_\lambda \quad\text{with}\quad p(i) \geq p_0(i),\ 1 \leq i \leq \lambda .$$

Remarks.

(ii) In particular (5.66) holds for any $(\mathcal{G},\mathcal{G}^*)$ to which Cor. 3.1 applies if we take $p \in \mathcal{P}_\lambda^+$, $0 \ll p \ll 1$. I.e., (5.66) holds in the whole percolative region of $(0,1)^\lambda$ (cf. (3.51)). In some two-dimensional examples (such as the two-parameter site-percolation problem on $\mathbb{Z}^2$ of Application 3.4 (iv)) both (5.66) and (5.63) hold, when percolation occurs. For such examples one obtains in the percolative region

$$0 < \liminf -\frac{1}{\sqrt{n}} \log P_p\{\#W(z_0) = n\}$$
$$\leq \limsup -\frac{1}{\sqrt{n}} \log P_p\{\#W(z_0) = n\} < \infty .$$

(iii) Russo (1978) uses estimates of the form (5.66) in one-parameter problems to show that for various graphs $\mathcal{G}$, which are one of a pair of matching periodic graphs, the functions

$$p \to \theta(p,z_0) \quad \text{and} \quad p \to E_p\{\#W(z_0);\ \#W(z_0) < \infty\}$$

are infinitely often differentiable on $(p_H,(\mathcal{G}),1]$. The same argument works for $p \to E_p\{\pi(\#W(z_0));\ \#W(z_0) < \infty\}$ for any polynomial π.

(iv) Delyon (1980) shows that for most periodic graphs $\mathcal{G}$ the $a(n,\ell)$ of (5.19) satisfy

$$\lim_{\substack{n \to \infty \\ \frac{\ell}{n} \to \gamma}} \{a(n,\ell)\}^{\frac{1}{n}} = (1+\gamma)^{1+\gamma}\ \gamma^{-\gamma} \tag{5.67}$$

whenever

$$\gamma < \frac{1 - p_H(\mathcal{G})}{p_H(\mathcal{G})} .$$

The remarkable part of this result is that the limit in (5.67) is independent of $\mathcal{G}$; only the range of γ's for which the limit relation (5.67) holds depends on $\mathcal{G}$. One only needs some aperiodicity assumptions on the relation between $\#C$ and $\#\partial C$ for connected sets C of vertices on $\mathcal{G}$ to obtain (5.67). The proof rests on subadditivity arguments such as in Lemma 5.9 below, an estimate like (5.23) and the fact that $P_p\{\#W(z_0) = n\}$ does not decrease exponentially for $p > p_H(\mathcal{G})$. ///

We turn to the proof of Theorem 5.2. Until further notice we deal with the set up of Theorem 5.2 and all its hypotheses are in force. As in Aizenman et al. (1980) the main estimate will be obtained by connecting a number of vertices inside a large cube by occupied paths, and making several vertices in the boundary of the cube vacant. The latter change disconnects a cluster inside the cube from the outside; this allows us to control (from above) the size of a cluster which we constructed inside the cube. Nevertheless the size of this cluster is not fixed, and this method only yields a lower bound for $P\{\#W = n\}$ along a subsequence of n's. The general n is then handled by Lemma 5.9, which shows how lower bounds for various n's can be combined.

C_i, K_i will denote various constants; the C_i depend on d only, while the K_i depend the probability distribution P as well. It is understood that $0 < C_i, K_i < \infty$. In addition we shall use the following sets and events:

$$S(v,M) = [v(1) - M, v(1) + M] \times \dots \times [v(d) - M, v(d) + M]$$

(a cube of size 2M centered at $v = (v(1),\dots,v(d))$,

$\Delta S(v,M) = Fr(S(v,M))$ = topological boundary of $S(v,M)$.

$B(v,M) = \{\exists$ an occupied path on $\mathbb{Z}^d$ inside $S(v,M)$ which connects v with a point in $\Delta S(v,M)\}$,

$B_k(v,M,j,\pm) = \{$at least k vertices on the face $[v(1)-M, v(1)+M] \times \dots \times [v(j-1)-M, v(j-1)+M] \times \{v(j) \pm M\} \times \dots \times [v(d)-M, v(d)+M]$ of $S(v,M)$ are connected by an occupied path on $\mathbb{Z}^d$ inside $S(v,M)$ to $v\}$.

Finally

$$\theta(v) = P\{\#W(v) = \infty\} .$$

<u>Lemma 5.5. There exist constants</u> M_0 <u>and</u> K_i <u>such that for each set</u> A <u>of vertices of</u> $\mathbb{Z}^d$

$$P\{B(v,M_0) \text{ occurs for more than } 2\sum_{w \in A} \theta(w) \text{ vertices } v \text{ in } A\} \le K_1 \exp - K_2(\#A). \tag{5.68}$$

<u>In addition, for each</u> k <u>there exists an</u> M_k <u>such that for all</u> $v \in \mathbb{Z}^d$ <u>and</u> $M \ge M_k$

$$P\{B_k(v,M,j,\varepsilon)\} \ge \frac{\theta(v)}{4d} \tag{5.69}$$

for some j, ε , which may depend on v,k,M.

<u>Proof:</u> First note that by (4.8) (with $n = \infty$) we have for any two vertices v_1 and v_2 of $\mathbb{Z}^d$ in $[0,k_0)^d$

$$\theta(v_1) \geq P\{\exists \text{ occupied path from } v_1 \text{ to } v_2\}\, \theta(v_2)$$
$$\geq \pi^{C_4 k_0} \quad \theta(v_2).$$

If we write K_3 for $\pi^{C_4 k_0}$, then this can be written as

$$(5.70) \qquad \theta(v_1) \geq K_3 \theta(v_2),$$

By virtue of the periodicity assumption (5.60), $\theta(.)$ is periodic with periods $k_0 \xi_i$, $1 \leq i \leq d$, and hence (5.70) holds for any pair of vertices v_1, v_2. Moreover (5.70) implies for any vertex w

$$(5.71) \qquad \theta(w) \geq \frac{K_3}{k_0^d} \sum_{v \in [0,k_0)^d} \theta(v) = \frac{K_3}{k_0^d} \quad \Theta > 0$$

(see (5.62)). Next observe that the events $B(v,M)$ decrease to $\{\#W(v) = \infty\}$ as $M \uparrow \infty$. Consequently we can find an M_0 such that

$$(5.72) \qquad P\,\{B(v,M_0)\} \leq \frac{3}{2}\, \theta(v) \;,$$

and by the periodicity assumption (5.60) we can choose M_0 independent of v. Now if A is any set of vertices of $\mathbb{Z}^d$ we can write A as a union of at most $(2M_0+1)^d$ disjoint sets A_i such that for each pair of vertices v and w in a single A_i one has $|v(j)-w(j)| > 2M_0$ for some $1 \leq j \leq d$. For any such pair of vertices v and w $S(v,M_0)$ and $S(w,M_0)$ are disjoint. Consequently the events $\{B(v,M_0): v \in A_i\}$ are independent for fixed i. It follows from standard exponential bounds for independent bounded variables (see Renyi (1970), Ch. VII.4 or Freedman (1973, Theorem (4)) that for all $\lambda \geq 0$

$$(5.73) \qquad P\{B(v,M_0) \text{ occurs for more than } 2\sum_{w \in A} \theta(w) \text{ vertices in } A\}$$
$$\leq \sum_i P\{B(v,M_0) \text{ occurs for more than } \frac{3}{2}\sum_{w \in A_i} \theta(w) + \frac{1}{2}(2M_0+1)^{-d} \sum_{w \in A} \theta(w)$$
$$\text{vertices in } A_i\}$$

$$\leq \sum_i \exp\left(-\frac{\lambda}{2(2M_0+1)^d}\sum_{w\in A}\theta(w) - \frac{3\lambda}{2}\sum_{w\in A_i}\theta(w)\right)$$

$$\cdot \prod_{w\in A_i}\{1 + P\{B(w,M_0)\}\,(e^\lambda - 1)\}$$

$$\leq \sum_i \exp\{-\frac{\lambda}{2(2M_0+1)^d}\sum_{w\in A}\theta(w) + \sum_{w\in A_i}\theta(w)(\tfrac{3}{2}(e^\lambda - 1) - \tfrac{3}{2}\lambda)\}$$

(use (5.72)). Now pick $\lambda > 0$ such that

$$\frac{3}{2}(e^\lambda - 1) - \frac{3}{2}\lambda < \frac{\lambda}{4(2M_0+1)^d} \; .$$

For such a λ the last member of (5.73) is at most

$$\sum_i \exp - \frac{\lambda}{4(2M_0+1)^d}\sum_{w\in A}\theta(w) \; .$$

This, together with (5.71) implies (5.68).

Now for the proof of (5.69). Let $\mathfrak{F}(M) = \mathfrak{F}(v,M)$ be the σ-field generated by $\{\omega(w): w \in S(v,M)\}$. By the martingale convergence theorem (see Breiman (1968), Cor. 5.22)

$$P\{\#W(v) = \infty | \mathfrak{F}(v,M)\} \to I[\#W(v) = \infty] \; (M \to \infty)$$

with probability one. As pointed out above

(5.74) $\qquad I[B(v,M] \downarrow I[\#W(v) = \infty] \; (M \to \infty),$

so that

(5.75) $\qquad P\{\#W(v) = \infty | \mathfrak{F}(M)\} - I[B(v,M)] \to 0 \quad (M \to \infty)$

with probability one. Now define

$W_M(v)$ = collection of edges and vertices of $\mathbb{Z}^d$ which are connected to v by an occupied path in $S(v,M)$

and

$\Gamma_M = \Gamma_M(v)$ = number of vertices in $\Delta S(v,M)$ which are connected by an occupied path in $S(v,M)$ to v.

Γ_M is just the number of vertices of $W_M(v)$ in $\Delta S(v,M)$. $\#W(v)$ will be finite if all neighbors outside $B(v,M)$ of the $\Gamma_M(v)$ vertices of $W_M(v) \cap \Delta S(v,M)$ are vacant. Indeed, if this occurs no occupied path starting at v can leave $S(v,M)$. Since any vertex has 2d neighbors it follows that

$$(5.76) \qquad P\{\#W(v) = \infty | \mathfrak{F}(M)\} \leq 1 - \pi^{2d\Gamma_M}$$

(see (5.61) for π). (5.61) and (5.74) - (5.76) imply that for each fixed k

$$P\{B(v,M) \text{ occurs, but } \Gamma_M \leq 2dk\} \to 0 \ (M \to \infty),$$

and hence

$$P\{B(v,M) \text{ occurs and } \Gamma_M > 2dk\} \to \theta(v)$$

But, by the definition of Γ_M and $B_k(.)$

$$\{B(v,M) \text{ and } \Gamma_M > 2dk\} \subset \bigcup_{\substack{j=1,\dots,d \\ \varepsilon = \pm}} B_k(v,M,j,\varepsilon).$$

Consequently, for each k there exists an M_k such that

$$\sum_{\substack{j=1,\dots,d \\ \varepsilon = \pm}} P\{B_k(v,M,j,\varepsilon) \geq \frac{1}{2} \theta(v)$$

for all $M \geq M_k$. Again by the periodicity assumption (5.60) we can choose M independent of v. (5.69) is now immediate. □

Without loss of generality we shall assume that the origin has been chosen such that

$$(5.77) \qquad \theta(0) = \max_v \theta(v)$$

and consequently (see (5.62))

$$(5.78) \qquad 0 \leq \Theta \leq k_0^d \, \theta(0) .$$

We next define for $v \in S(0,M)$

$W_M(v,0)$ = collection of edges and vertices of $\mathbb{Z}^d$ which are connected to v by an occupied path in $S(0,M)$, and for $m \geq M_k$ choose a $j=j(k,m)$ and an $\varepsilon = \varepsilon(k,m)$ such that

(5.79) $$P\{B_k(0,m,j,\varepsilon)\} \geq \frac{\theta(0)}{4d}.$$

Note that $W_M(v,0) \subset S(0,M)$. A face of $S(0,M)$ is a set of the form

$$\{x \in S(0,M): x(j) = \varepsilon M\}, \quad \varepsilon = +1 \text{ or } -1$$

Any face of $S(0,M)$ is contained in $\Delta S(0,M)$, and in fact, $\Delta S(0,M)$ is the union of all faces of $S(0,M)$.

Lemma 5.6. Assume the origin is chosen such that (5.77) holds. Then there exists a constant $C_5 > 0$ and for all $k \geq 1$ an $\tilde{M}_k$ such that for $M \geq \tilde{M}_k$

(5.80) $$P\{\exists \text{ set of vertices } D \text{ in some face of } S(0,M) \text{ with } \#D \leq 3^d k^{-1}\theta(0)M^{d-1} \text{ and } C_5\theta(0)(M/k_0)^d \leq \#\Big(\bigcup_{w\varepsilon D} W_M(w,0)\Big) \leq 3^{d+1}\theta(0)M^d \text{ and } \#\{\bigcup_{w\varepsilon D} W_M(w,0) \cap \Delta S(0,M)\} \leq 4d\theta(0)(3M)^{d-1}\}$$
$$\geq \frac{1}{2} C_5(3k_0)^{-d}\ \theta(0).$$

Proof: Fix k and let $M \geq 4M_k + 8M_0 + 20k_0$. For $m \geq M_k$ we take $j(k,m)$ and $\varepsilon(k,m)$ such that (5.79) holds. For some j_0, ε_0 there exist at least

$$\frac{1}{2d}\left\lfloor\frac{M}{4k_0}\right\rfloor \geq \frac{M}{10d\ k_0}$$

integers m satisfying

(5.81) $$M_k+2M_0 \leq m \leq \frac{M}{2},\ k_0 \text{ divides } M-m \text{ and } j(k,m) = j_0, \varepsilon(k,m) = \varepsilon_0.$$

Without loss of generality we assume that $j_0 = 1$, $\varepsilon_0 = -$. For the corresponding m we then have $j(k,m), \varepsilon(k,m) = (j_0,\varepsilon_0) = (1,-)$ and by (5.79) and the periodicity assumption (5.60)

(5.82) $$P\{B_k(v,m,1,-)\} \geq \frac{\theta(0)}{4d}$$

for each vertex $v = (v(1),\ldots,v(d))$ with $v(i)$ divisible by k_0 for each i. We put

$$F_M = \{-M\} \times [-M,M]\times\ldots\times[-M,M]$$

F_M is the "left face" of $S(0,M)$. Now let v be such that

(5.83) $\quad v(1) = -M + m$ for some m which satisfies (5.81) and $|v(i)| \le \frac{M}{2}$, $v(i)$ divisible by k_0 for $i=2,\ldots,d$.

If $B_k(v,M - |v(1)|, 1,-)$ occurs for such a v, then v is connected inside $S(v,M-|v(1)|)$ to at least k vertices in

$$\{v(1) - M + |v(1)|\} \times [v(2) - M + |v(1)|, v(2) + M -|v(1)|] \times\ldots\times [v(d) - M + |v(1)|, v(d) + M -|v(1)|] \subset F_M$$

Moreover, $S(v,M - |v(1)|) \subset S(0,M)$. Thus, if we define

$$\Gamma_M(v,0) := \text{number of vertices of } W_M(v,0) \text{ in } F_M = \#(W_M(v,0) \cap F_M),$$

then for a v satisfying (5.83) $B_k(v,M - |v(1)|,1,-)$ implies $\Gamma_M(v,0) \ge k$. In addition for any v satisfying (5.83), k_0 divides $v(i)$, $1 \le i \le d$, (use (5.81) for $i=1$) and by (5.82)

$$P\{B_k(v,M-|v(1)|, 1,-) = P\{B_k(v,m,1,-) \ge \frac{\theta(0)}{4d}$$

It follows that for $M \ge 4M_k + 8M_0 + 20k_0$

(5.84) $\quad E$ {number of v in $S(0,M)$ which satisfy (5.83) with $\Gamma_M(v,0) \ge k\} \ge \frac{\theta(0)}{4d}$ {number of v satisfying (5.83)}

$$\ge 2C_5 \quad (0)\ (M/k_0)^d$$

for some C_5. Since the total number of v in $S(0,M)$ is $(2M+1)^d$, the left hand side of (5.84) is at most

$$P\{(\text{number of } v \in S(0,M) \text{ which satisfy (5.83) and with } \Gamma_M(v,0) \ge k) \text{ is at least } C_5\theta(0)(M/k_0)^d\}\ (2M+1)^d + C_5\ \theta(0)(M/k_0)^d .$$

It follows from this and (5.84) that

(5.85) $\quad P$\{there are at least $C_5\theta(0)(M/k_0)^d$ vertices v in $S(0,M)$ which satisfy (5.83) and with $\Gamma_M(v,0) \ge k\}$

$$\ge C_5(3k_0)^{-d}\ \theta(0) .$$

Now consider the collection of w in F_M which belong to some $W_M(v,0)$ for a v satisfying (5.83) and $\Gamma_M(v,0) \geq k$. Call two vertices w_1 and w_2 of this kind equivalent if they belong to the same $W_M(v,0)$ with v satisfying (5.83) and $\Gamma_M(v,0) \geq k$. From each equivalence class pick one representative and denote by D the collection of representatives chosen in this way. Note that $D \subset F_M$ and that by definition each equivalence class contains at least k elements. Consequently

$$\#D \leq k^{-1} \text{ (number of vertices in } F_M \text{ which belong to some } W_M(v,0) \text{ with } v \text{ satisfying (5.83)).}$$

Now if $w \in W_M(v,0)$, then $v \in W_M(w,0)$, and if $w(1) = -M$, $v(1) \geq -M + 2M_0$ then $B(w_0,2M_0)$ must occur. Consequently, by (5.68) and (5.77).

$$\#D \leq k^{-1} \text{ (number of } w \in F_M \text{ for which } B(w,2M_0) \text{ occurs)} \leq 2\frac{\theta(0)}{k} \#F_M = 2\frac{\theta(0)}{k} (2M+1)^{d-1} \tag{5.86}$$

outside a set of probability at most

$$K_1 \exp - K_2(2M+1)^{d-1} \quad .$$

Also, by our choice of D

$$\bigcup_{w \in D} W_M(w,0) = \cup W_M(v,0) , \tag{5.87}$$

where the union in the right hand side is over all v which satisfy (5.83) and have $\Gamma_M(v,0) \geq k$. If the event in braces in (5.85) occurs then this union contains at least $C_5\theta(0)(M/k_0)^d$ vertices. On the other hand, the union in (5.87) is contained in the set

$$\{u \in S(0,M) : B(u,M_0) \text{ occurs}\} \quad .$$

To see this note that if $u \in W_M(w,0)$ for some $w \in D$, then $W_M(u,0) = W_M(w,0) = W_M(v,0)$ for some v satisfying $|v(1)-w(1)| \geq 2M_0$ (by (5.81)) and hence $|u(1)-v(1)| \geq M_0$ or $|u(1)-w(1)| \geq M_0$. In any case, such a u is connected by an occupied path to $\Delta S(u,M_0)$ and $B(u,M_0)$ occurs. The number of vertices in the union in (5.87) is therefore for large M at most

$$(5.88) \qquad \#\{u \in S(0,M) : B(u,M_0) \text{ occurs}\}$$
$$\le 2\theta(0)\ \#S(0,M) \le 3^{d+1}\ \theta(0)M^d ,$$

outside a set of probability at most

$$K_1 \exp - K_2(2M+1)^d .$$

(again by (5.68) and (5.77)). For the same reasons

$$(5.89) \qquad \#\{(\bigcup_{w \in D} W(w,0)) \cap \Delta S(0,M)\} \le 2\theta(0)\ \#\Delta S(0,M)$$
$$\le 4d\theta(0)\ (3M)^{d-1}$$

outside a set of probability at most

$$K_1 \exp - K_2 2d\ (2M+1)^{d-1} .$$

Thus, if the event in braces in (5.85) occurs, and if the estimates (5.86), (5.88) and (5.89) are valid, then the event in braces in (5.80) also occurs. In view of (5.85) and the above estimates this shows that the left hand side of (5.80) is at least

$$C_5(3k_0)^{-d}\,\theta(0) - 2K_1 \exp - K_2(2M+1)^{d-1}$$
$$- K_1 \exp - K_2(2M+1)^d ,$$

from which (5.80) follows for large M. □

<u>Lemma 5.7</u>. <u>Assume (5.77). For</u> $d \ge 3$ <u>there exist constants</u> C_6, C_7 <u>and</u> $\tilde{M}$ <u>such that for all</u> $M \ge \tilde{M}$ <u>the interval</u>

$$(5.90) \qquad [C_5 k_0^{-d}\,\theta(0)M^d ,\ 3^{d+2}\,\theta(0)M^d]$$

<u>contains an integer</u> m <u>with</u>

$$(5.91) \qquad P\{\#W(v) = m\} \ge \pi^{C_6\theta(0)M^{d-1}}$$
$$\ge \pi^{C_7 k_0^{d-1}(\theta(0))^{1/d}\, m^{(d-1)/d}}$$

for some $v \in [0,k_0)^d$.

Proof: We take

$$k = \{\theta(0)\}^{-1/(d-2)}.$$

Fix an ω for which the event in braces in (5.80) occurs. Note that this event depends only on the occupancies in $S(0,M)$, and is therefore independent of all $\omega(u)$ with $u \notin S(0,M)$. We want to show that the $W_M(w,0)$ with $w \in D$ can be connected by paths on $\mathbb{Z}^d$ which (apart from their endpoints in D) lie outside $S(0,M)$ and which contain at most $C_8\theta(0)M^{d-1}$ vertices. To do this fix D such that it satisfies the requirements in (5.80). For the sake of argument assume again that D lies in the face $F_M = \{-M\} \times [-M,M]\times\ldots\times[-M,M]$. Take $D' = D - 2\xi_1$. D' is the translate by $(-2,0,\ldots,0)$ of D so that $\#D' = \#D$. Also each vertex v in D can be connected to $v - 2\xi_1 \in D'$ via a straight line segment of length two containing only the vertex $v-\xi_1$ outside $D \cup D'$. Moreover

$$D' \subset F'_M := F_M - 2\xi_1$$

and F'_M lies outside $S(0,M)$. The paths connecting the vertices in D will consist of all the line segments from $v \in D$ to $v-2\xi_1 \in D'$ plus a number of paths in F'_M connecting all vertices of D'. Hence, they will indeed contain only vertices outside $S(0,M)$ plus endpoints in D, as desired. To construct paths in F'_M connecting all vertices of D' consider the collection of vertices $w = (w(1),\ldots,w(d)) \in F'_M$ of the following form:

(5.92) $\quad w(1) = -M-2$, $w(r)$ is a multiple of ρ for $2 \le r \le d$, $r \ne s$, and $-M \le w(i) \le M$, $2 \le i \le d$,

where

$$\rho = \left\lceil \left\{ \frac{k}{\theta(0)} \right\}^{1/(d-1)} \right\rceil \sim \{\theta(0)\}^{-1/(d-2)}$$

and s is anyone of the indices $2,\ldots,d$. There are at most $(d-1) \times \rho^{2-d}(2M+\rho)^{d-1}$ such vertices. When $d \ge 3$ all the vertices satisfying (5.92) are connected by line segments in F'_M , containing only vertices of the form (5.92). Also, each vertex $v \in F'_M$ can be connected to one of these vertices by a straight line segment in F'_M containing at most

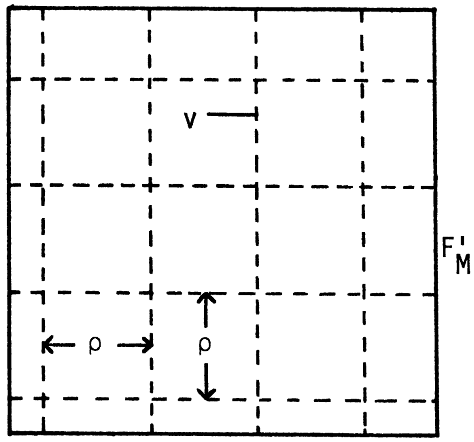

Figure 5.3 The vertices satisfying (5.92) are connected by the dashed lines in F'_M. These lines are distance ρ apart. The solid line from v shows how to connect v to this system of lines.

$(d-2)\rho$ vertices for $d \geq 3$. (see Fig. 5.3). Choose such a segment for each vertex $v \in D'$. Let E be the set of all vertices which satisfy (5.92), the vertices on the segments in F'_M which connect vertice of D' to one of the vertices which satisfy (5.92), as well as the vertices in $D-\xi_1$. It follows from the above that any pair of vertices of D can be connected by a path on $\mathbb{Z}^d$ which, apart from its endpoints in D, contains only vertices from E. By construction E lies outside $S(0,M)$ and for $M \geq \rho$

$$\#E \leq (d-1)\rho^{2-d}(2M+\rho)^{d-1} + ((d-2)\rho+1)\#D \leq C_8\theta(0)M^{d-1} \quad .$$

(the last inequality follows from the upper bound on #D in (5.80)). Moreover, if all vertices in E are occupied, then

$$(5.93) \qquad \bigcup_{w\in D} W_M(w,0) \cup E$$

forms a connected occupied set. When the event in braces in (5.80) occurs and M is sufficiently large, then the number of vertices in the set (5.93) lies in the interval (5.90). Thus E has all the desired properties for connecting the $W_M(w,0)$ with $w \in D$. In addition - because E is disjoint from $S(0,M)$ - the conditional probability that all vertices in E are occupied, given any information about the occupancies in $S(0,M)$, is at least $\pi^{\#E}$. This almost proves our lemma. We merely have to make sure that (5.93) is a maximal occupied component when all of E is occupied, i.e., that it is not part of a bigger occupied component (whose cardinality may lie outside the interval (5.90)). We claim that (5.93) will indeed be a maximal occupied component if all vertices in the following set G are vacant.

$$G := \{\text{all vertices outside } S(0,M) \cup E \text{ which are adjacent to a vertex in } \bigcup_{w \in D} W_M(w,0) \cup E\}.$$

To see this, note that all vertices inside $S(0,M)$ adjacent to the set (5.93), but not belonging to (5.93) itself, are already vacant. This is so because the $W_M(w,0)$ are already maximal occupied components inside $S(0,M)$, so that their neighbors in $S(0,M)$ are vacant. The only vertices of

$$S(0,M) \cap \partial\Big(\bigcup_{w \in D} W_M(w,0) \cup E\Big)$$

which might be occupied would have to lie in $S(0,M) \cap \partial E$. But by our choice of E no such vertices exist, because $S(0,M) \cap \partial E = D$, and this is part of the set (5.93). This proves that if all vertices in G are vacant, then all vertices in

$$\partial\Big(\bigcup_{w \in D} W_M(w,0) \cup E\Big)$$

are vacant. It therefore justifies our claim and thereby shows

$$\begin{aligned}
(5.94) \qquad & P\{\exists \text{ a maximal occupied cluster in } [-M-2,M] \times [-M,M]^{d-1} \\
& \qquad \text{with cardinality in the interval (5.90)}\} \\
& \geq P\{\text{the event in braces in (5.80) occurs, all vertices in } \\
& \qquad E \text{ are occupied and all vertices in } G \text{ are vacant}\} \\
& \geq E\{\pi^{\#E + \#G};\ \text{the event in braces in (5.80) occurs}\}.
\end{aligned}$$

It remains to estimate $\#G$. But by definition

$$G \subset \partial E \cup \partial\Big(\bigcup_{w \in D} W_M(w,0) \cap \Delta S(0,M)\Big),$$

so that

$$\#G \leq 2d\,\{\#E + 4d\theta(0)(3M)^{d-1}\}$$

by the bound on $\#\{\bigcup_{w\in D} W_M(w,0) \cap \Delta S(0,M)\}$ in (5.80). Therefore the last member of (5.94) is at least

$$E\ \{\exp(\log \pi)\ \{(2d+1)\#E + 8d^2\theta(0)(3M)^{d-1}\ ;$$

$$\text{the event in braces in (5.80) occurs}\}$$

$$\geq \exp\{C_9\theta(0)M^{d-1} \log \pi\}.\ P\{\text{the event in braces in (5.80) occurs}$$

$$\geq \exp\{C_9\theta(0)M^{d-1} \log \pi\}\ .\ \tfrac{1}{2}C_5(3k_0)^{-d}\ \theta(0).$$

The lemma follows easily from this lower bound for the first member of (5.94), because any maximal occupied cluster which lies entirely in $[-M-2,M] \times [-M,M]^{d-1}$ equals $W(v)$ for one of the $(2M+3)(2M+1)^{d-1}$ vertices v in this box. In addition the interval (5.90) contains at most $3^{d+2}\theta(0)M^d$ integers, so that

$$\max P\{\#W(v) = m\} \geq (2M+3)^{-d}\ (3^{d+2}\theta(0)M^d)^{-1}$$

$$\tfrac{1}{2}C_5(3k_0)^{-d}\theta(0)\ \exp\{C_9\theta(0)M^{d-1} \log \pi\}\ .$$

where the max is over all v, and over all m in the interval (5.90). We may restrict v to $[0,k_0)^d$ by the periodicity assumption (5.60). This gives the first inequality in (5.91) for large M (since $\pi<1$). The second inequality follows from the fact that m lies in the interval (5.90). □

The proof of the preceding lemma breaks down for $d=2$, because the collection of vertices satisfying (5.92) is no longer connected; it consists merely of the vertices $(-M-2, \ell\rho)$, $|\ell| \leq M/\rho$. Nevertheless the conclusion of Lemma 5.7 remains valid.

Lemma 5.8. Assume (5.77). If $d=2$ and the conditions (5.60), (5.61) and (5.64) hold, then there exists an $\tilde{M}$ such that for each $M \geq \tilde{M}$ the interval

$$[\ \tfrac{1}{2}C_5\ k_0^{-2}\ \theta(0)M^2,\ 3^4\ \theta(0)M^2\ ] \tag{5.95}$$

contains an integer m for which (5.91) with $d=2$ holds.

Proof: We do not give a detailed proof for $d=2$ here. We shall rely in part on a simple result from Ch. 7. This result shows that for $d=2$ there exists with high probability an occupied path in $S(0,M)$

which connects most occupied clusters in $S(0,M)$. We therefore automatically obtain a large cluster $W_M(v,0)$ in $S(0,M)$, even without the use of any such set as E outside $S(0,M)$, as in the preceding lemma. Specifically we shall prove

$$(5.96)\qquad P\{\ \exists \text{ a vertex } v \text{ in } S(0,M) \text{ for which } \#W_M(v,0) \text{ lies in the interval (5.95) and } \#W_M(v,0) \cap \Delta S(0,M) \le 20\,\theta(0)M\} \ge 2^{-6} C_5(3k_0)^{-2}\ \theta(0)\ .$$

This estimate will take the place of the construction of E in Lemma 5.7. For any ω for which the event in braces in (5.96) occurs choose a v in $S(0,M)$ for which $\#W_M(v,0)$ lies in the interval (5.95) and define G as

$$G := \{\text{all vertices outside } S(0,M) \text{ which are adjacent to } W_M(v,0)\}\ .$$

From here on the proof is practically the same as in Lemma 5.7. When all vertices in G are vacant, then $W_M(v,0)$ is a maximal occupied component, i.e., it equals $W(v)$. Thus

$$P\{\ \exists \text{ a vertex } v \text{ in } S(0,M) \text{ with } \#W(v) \text{ in the interval (5.95)}\} \ge E\{\pi^{\#G}\ ;\ \text{event in braces in (5.96) occurs}\}\ ,$$

and we can estimate this as before. We shall therefore restrict ourselves to proving (5.96) and leave further details to the reader.

The proof of (5.96) relies on Lemma 5.6 which does hold for $d=2$. It is a trivial consequence of (5.80) for $d=2$, that for some face F_M of $S(0,M)$

$$(5.97)\qquad P\{\ \#\bigcup_{w\in F_M} W_M(w,0) \ \ge\ C_5\theta(0)(M/k_0)^2\ \} \ \ge\ \frac{1}{8}C_5(3k_0)^{-2}\ \theta(0)\ .$$

Again without loss of generality we assume that (5.97) holds with $F_M = \{-M\} \times [-M,M]$, the left side of the square $S(0,M)$. Now note that $\mathbb{Z}^2$ is just the graph $\mathcal{G}_0$ of Ex. 2.1. (i). Under the periodicity assumption (5.60), (5.61) and the extra hypothesis (5.64) for $d=2$ we can apply (7.15) as well as Theorem 5.1. (7.15) together with periodicity gives

$$(5.98)\qquad \begin{aligned} &P\{\ \exists \text{ occupied vertical crossing on } \mathcal{G}_0 \text{ of} \\ &[-M,-M+C\log M]\times[-M,M]\} \geq \sigma((C\log M-k_0,2M+k_0);\,2,p,\mathcal{G}_0) \\ &\geq 1-P\{\ \exists \text{ vacant horizontal crossing on } \mathcal{G}_0^* \text{ of} \\ &[0,C\log M-k_0-\Lambda]\times[0,2M+k_0+\Lambda]\} \end{aligned}$$

for some constant Λ (which depends on $\mathcal{G}_0 = \mathbb{Z}^2$ only). Moreover, as at the end of the proof of Lemma 5.4, a horizontal crossing on $\mathcal{G}_0^*$ of $[0,C\log M-k_0-\Lambda]\times[0,2M+k_0+\Lambda]$ has to contain one of the vertices $v=(0,\ell)$, $0\leq\ell\leq 2M+k_0+\Lambda$. If such a v is part of a vacant horizontal crossing of $[0,C\log M-k_0-\Lambda]\times[0,2M+k_0+\Lambda]$, then its vacant component an $\mathcal{G}_0^*$, $W^*(v)$, must contain at least $C\log M-k_0-\Lambda$ vertices. Thus, the right hand side of (5.98) equals at least

$$1-\sum_{\ell=0}^{2M+k_0+\Lambda} P\{\#W^*((0,\ell)) \geq C\log M-k_0-\Lambda\}\ .$$

Now Theorem 5.1 applied to $\mathcal{G}_0^*$ shows that (by virtue of (5.61) and (5.64)).

$$P\{\#W^*(0,\ell) \geq C\log M-k_0-\Lambda\} \leq K_4 M^{-K_5 C}$$

uniformly in ℓ, for some constants K_4,K_5 which depend only on the probability measure P. Thus for $C > K_5^{-1}$ the right hand side of (5.98) is greater than $\frac{1}{2}$ eventually. From the FKG inequality, Proposition 4.1, and (5.97) we now obtain

$$(5.99)\qquad \begin{aligned} &P\{\#\bigcup_{w\varepsilon F_M} W_M(w,0) \geq C_5\,\theta(0)(M/k_0)^2 \text{ and there exists an} \\ &\text{occupied vertical crossing of } [-M,-M+C\log M]\times[-M,M]\} \\ &\geq \frac{1}{2}P\{\bigcup_{w\varepsilon F_M} W_M(w,0) \geq C_5\theta(0)(M/k_0)^2\} \\ &\geq \frac{1}{16}C_5\,(3k_0)^{-2}\,\theta(0)\ . \end{aligned}$$

Now assume that there exists an occupied vertical crossing r of $[-M,-M+C\log M]\times[-M,M]$. Then any occupied cluster $W_M(v,0)$ in $S(0,M)$

which is connected inside $S(0,M)$ to w in $F_M = \{-M\} \times [-M,M]$, and which contains a vertex in $[-M + C \log M, M] \times [-M,M]$ must intersect r. Hence r will be part of any such $W_M(v,0)$. In particular all $W_M(v,0)$ with the above property are connected via r and form a single cluster. On the other hand, any cluster $W_M(v,0)$ which contains a w in F_M but does not intersect r must be contained in $[-M,-M + C \log M) \times [-M,M]$. Hence, all such clusters contain together at most $(C \log M + 1)(2M + 1)$ vertices. Thus, if the event in braces in the first member of (5.99) occurs, then $S(0,M)$ contains a single $W_M(v,0)$ of at least

$$\#\Big(\bigcup_{w \varepsilon F_M} W_M(w,0) - (C \log M + 1)(2M + 1)$$

$$\geq C_5\, \theta(0)(M/k_0)^2 - (C \log M + 1)(2M + 1)$$

vertices. For sufficiently large M this number exceeds $\frac{1}{2} C_5 k_0^{-2}\, \theta(0) M^2$ so that

$$\text{(5.100)} \qquad P\{\exists\ v \varepsilon S(0,M) \text{ with } \#W_M(v,0) \geq \tfrac{1}{2} C_5 k_0^{-1}\, \theta(0) M^2\}$$

$$\geq \frac{1}{16} C_5 (3k_0)^{-2}\ \theta(0)$$

for large M. This gives us a $W_M(v,0)$ (with a certain probability) whose cardinality equals at least the lower bound of (5.95). To make sure that $\#W_M(v,0)$ actually falls in the interval (5.95) we once more appeal to (5.68). For each w in $W_M(v,0)$ with $|w(i)-v(i)| > M_0$ for $i = 1$ or 2, $B(w,M_0)$ occurs. Indeed any such w is connected to $v \notin S(w,M_0)$. Therefore, if $\#W_M(v,0) > 3^4\theta(0)M^2$, then $B(w,M_0)$ occurs for more than $3^4\, \theta(0)M^2 - (2M_0+1)^2 \geq 2\theta(0)(2M+1)^2 = 2\theta(0)\ \#S(0,M)$ vertices in $S(0,M)$. Thus, by (5.68), for large enough M

$$P\{\#W_M(v,0) > 3^4\theta(0)M^2 \text{ for some } v \varepsilon S(0,M)\}$$

$$\leq K_1 \exp - K_2(2M+1)^2 \leq 2^{-6} C_5 (3k_0)^{-2}\ \theta(0)\ .$$

By the same argument one has for large M

$$P\{\#(W_M(v,0) \cap \Delta S(0,M)) > 20\theta(0)M \quad \text{for some} \quad v \in S(0,M)\}$$
$$\leq P\{B(w,M_0) \text{ occurs for more than } 20\theta(0)M - (2M_0+1)^2$$
$$\geq 2\theta(0)\ \#\Delta S(0,M) \text{ vertices in } \Delta S(0,M)\}$$
$$\leq K_1 \exp - 8K_2 M \leq 2^{-6}\ C_5(3k_0)^{-2}\ \theta(0)\ .$$

These estimates together with (5.100) prove (5.96). □

From here on the proof closely follows Aizenman et al. (1980).

Lemma 5.9. For any vertices v_1, v_2 and integers $n_1, n_2 \geq 1$ one has on $\mathbb{Z}^d$

$$(5.101) \qquad P\{\#W(v_1) = n_1 + n_2\} \geq \pi^{2dn_2}\ P\{\#W(v_1) = n_1\}$$

and, for some constant $C_{10} = C_{10}(d)$

$$(5.102) \qquad P\{W(v_1) = n_1 + n_2 + (d+1)k_0\}$$
$$\geq \frac{1}{n_2(d+1)k_0}\ \pi^{C_{10}k_0}\ P\{W(v_1) = n_1\}\ P\{W(v_2) = n_2\}\ .$$

Proof: Let G_1 be a connected set of vertices of $\mathbb{Z}^d$ containing v_1 with $\#G_1 = n_1$. Let $w_1 = (w_1(1), w_1(2),\dots,w_1(d))$ be a vertex in G_1 with maximal first coordinate, i.e., $w_1(1) \geq w(1)$ for all $w \in G_1$. (w_1 is a "right most" point in G_1). Then form a connected set G of vertices by adding to G_1 the n_2 vertices

$w_1 + j\xi_1 = (w_1(1) + j, w_1(2),\dots,w_1(d))$ for $j=1,\dots,n_2$. Then $\#G = n_1 + n_2$. As in (5.24)

$$P\{W(v_1) = G_1\} = \prod_{w \in G_1} P\{w \text{ is occupied}\}$$
$$\cdot \prod_{w \in \partial G_1} P\,\{u \text{ is vacant}\}$$

and similarly for G. Since G consists of G_1 plus n_2 vertices, and ∂G consists of $\partial G_1 \setminus \{w_1 + \xi_1\}$ plus at most $(2d-1)n_2$ points it follows that

$$(5.103) \qquad P\{W(v_1) = G\} \geq \pi^{2dn_2}\ P\{W(v_1) = G_1\}\ .$$

Finally,

$$(5.104) \qquad P\{\#W(v_1) = n_1\} = \sum_{\#G_1 = n_1} P\{W(v_1) = G_1\}$$

where the sum runs over all connected G_1 with $\#G_1 = n_1$ and containing v_1. Since distinct G_1's lead to distinct G's in the above construction we find

$$P\{\#W(v_1) = n_1 + n_2\} \geq \sum_{\#G_1 = n_1} P\{W(v_1) = G\}$$
$$\geq \pi^{2dn_2} \sum_{\#G_1 = n_1} P\{W(v_1) = G_1\} \quad .$$

This proves (5.101).

To prove (5.102) we also bring in a connected set of vertices G_2 which contains v_2 and with $\#G_2 = n_2$. We take $w_2 = (w_2(1), \dots, w_2(d))$ as a "left most" point of G_2, i.e., one with $w_2(1) \leq w(1)$ for all $w \in G_2$. We shall now form a connected set G with $\#G = n_1 + n_2 + (d+1)k_0$ by connecting G_1 and a translate G_2' of G_2. Let m_i be the unique integer for which

$$(m_i - 1)k_0 \leq w_1(i) - w_2(i) < m_i k_0, \; 1 \leq i \leq d.$$

For G_2' we take $G_2 + \sum_1^d m_i k_0 \xi_i + k_0 \xi_1$. Let $w_2' = w_2 + \sum_1^d m_i k_0 \xi_i + k_0 \xi_1$. Then

$$w_1(1) + k_0 < w_2'(1) \leq w_1(1) + 2k_0 \;,$$

$$w_1(i) < w_2'(i) \leq w_1(i) + k_0, \; 2 \leq i \leq d,$$

and we can therefore connect w_1 to w_2' by a path r of at most $k_0(d+1)$ vertices, all of which lie in the strip $\{w_1(1) < x(1) < w_2'(1)\}$. By the periodicity assumption (5.60)

$$P\{W(w_2) = G_2\} = P\{\text{all vertices in } G_2 \text{ are occupied and all vertices in } \partial G_2 \text{ are vacant}\} = P\{\text{all vertices in } G_2' \text{ are occupied and all vertices in } \partial G_2' \text{ are vacant}\}$$

Very much as in (5.103) one obtains from this

$$(5.105) \qquad P\{W(v_1) = G_1 \cup r \cup G_2'\} \geq \pi^{2d(d+1)k_0} P\{W(v_1) = G_1\} \; P\{W(v_2) = G_2\} \; .$$

We now sum this over all connected sets G_1, G_2 with $\#G_i = n_i$ and containing v_i, $i = 1,2$. By (5.104) the sum of the right hand side will be

$$(5.106) \qquad \pi^{2d(d+1)k_0} P\{\#W(v_1) = n_1\} \; P\{\#W(v_2) = n_2\} \; .$$

As G_1, G_2 run over these sets, $G_1 \cup r \cup G_2'$ will run over certain connected sets of vertices G, containing v_1 and with $n_1 + n_2 < \#G \leq n_1 + n_2 + (d+1)k_0$. It is nevertheless not true that the sum of the left hand side is at most $P\{n_1 + n_2 < \#W(v_1) \leq n_1 + n_2 + (d+1)k_0$, because any given G may arise from many pairs G_1, G_2. It is, however, not hard to derive an upper bound for the number of pairs which can give rise to the same G. In fact G_1 is uniquely recoverable from G. One merely has to find the smallest integer m such that G has exactly n_1 vertices in the half space $\{x : x(1) \leq m\}$. G_1 is then the piece of G in this half space. This is so because r and G_2' lie in $\{x : x(1) > w_1(1)\}$ in our construction. In the same way one can recover G_2' as the piece of G in $\{x : x(1) \geq m'\}$ where m' is the maximal integer for which the above halfspace contains n_2 vertices of G. Finally $r = G \setminus G_1 \cup G_2'$. When G_2' is known there are at most n_2 possible choices for G_2, since G_2 is obtained from G_2' by a translation which takes one of the n_2 vertices of G_2' to v_2. From this it follows that the sum of the left hand side of (5.105) over G_1 and G_2 is at most

$$n_2 P\{n_1 + n_2 < \#W(v_1) \leq n_1 + n_2 + (d+1)k_0\} \; .$$

Together with (5.106) this proves that there exists an m in $(n_1 + n_2, n_1 + n_2 + (d+1)k_0]$ with

$$P\{\#W(v_1) = m\} \geq \frac{1}{n_2(d+1)k_0} \pi^{2d(d+1)k_0} P\{\#W(v_1) = n_1\} \, P\{\#W(v_2) = n_2\} \; .$$

An application of (5.101) with n_1 replaced by m and n_2 by

$n_1 + n_2 + (d+1)k_0 - m$ now yields (5.102). $\square$

Proof of Theorem 5.2. Given an integer $n_1 \geq (d+1)k_0 + 3^{d+2}\,\theta(0)\tilde{M}^d$ we find the largest integer M with

$$3^{d+2}\,\theta(0)M^d + (d+1)k_0 \leq n_1 ,$$

and then find an m_1 in the interval

$$[\tfrac{1}{2} C_5 k_0^{-d}\ \theta(0)M^d,\ 3^{d+2}\ \theta(0)M^d] \tag{5.107}$$

for which (5.91) holds. Such an m_1 exists by virtue of Lemma 5.7 and 5.8 since $M \geq \tilde{M}$. Next we find the maximal integer s_1 for which $s_1(m_1 + (d+1)k_0) \leq n_1$. Since

$$3^{d+2}\ \theta(0)(M+1)^d + (d+1)k_0 > n_1 ,$$

and m_1 lies in the interval (5.107), it follows that for n_1 greater than some $n_0 = n_0(\theta(0), d, \tilde{M})$

$$m_1 \geq \frac{1}{4} C_5 k_0^{-d}\ 3^{-d-2}\ n_1 . \tag{5.108}$$

Consequently for $n_1 \geq n_0$ and $C_{11} = 4C_5^{-1}\ 3^{d+2}$

$$1 \leq s_1 \leq C_{11}\ k_0^{-d} \tag{5.109}$$

and

$$n_2 := n_1 - s_1(m_1 + (d+1)k_0) \leq \frac{1}{2} n_1 . \tag{5.110}$$

By repeating the above procedure for n_2 instead of n_1 and so on we can represent n_1 as

$$n_1 = \sum s_i(m_i + (d+1)k_0) + t$$

with integers m_i satisfying (5.91) and (5.108), s_i satisfying (5.109), and $t < n_0$. Repeated application of Lemma 5.9 and (5.91) now shows that for some $v \in [0,k_0)^d$

$$P\{\#W(v) = n_1\} \geq \pi^{2dn_0 + C_{10}k_0 \sum s_i}\ \Pi(m_i(d+1)k_0)^{-s_i}\ P\{\#W(v_i) = m_i\}^{s_i}$$

$$\geq \exp\{2dn_0 \log \pi + (C_{10}k_0 \log \pi - \log k_0(d+1)) \sum s_i$$
$$- \sum s_i \log m_i + C_7(\log \pi)k_0^{d-1}(\theta(0))^{1/d} \sum s_i m_i^{(d-1)/d}\} .$$

It is clear that we can fix n_0 so large that the inequalities $m_i \geq C_{11}^{-1} k_0^{-d} n_i \geq C_{11}^{-1} k_0^{-d} n_0$ and $s_i \leq C_{11} k_0^d$ (cf. (5.108) and (5.109)) imply that the exponent in the right hand side is at least

$$2C_7(\log \pi)k_0^{d-1} (\theta(0))^{1/d} \sum s_i m_i^{(d-1)/d} + 2dn_0 \log \pi$$

$$\geq 2C_7 C_{11} k_0^{2d-1} \log \pi (\theta(0))^{1/d} \sum m_i^{(d-1)/d} + 2dn_0 \log \pi .$$

It is also easy to show from $m_i \leq n_i$ and $n_{i+1} \leq \frac{1}{2} n_i$ (cf. (5.110)) that

$$\sum_{i \geq 1} m_i^{(d-1)/d} \leq \sum_{i \geq 1} n_i^{(d-1)/d} \leq C_{12}\, n_1^{(d-1)/d} .$$

Since $\Theta \geq \theta(0)$, it follows that (5.63) holds for $n \geq N_0$ with a suitable choice of N_0, $C_3 = 3C_7 C_{11} C_{12}$ and w equal to some $w_n \in [0,k_0)^d$. To obtain (5.63) for all w we use Lemma 5.9 once more. By (5.102) with $v_1 = w$, $v_2 = w_{n-1-(d+1)k_0}$

$$P\{\#W(w) = n\} \geq \frac{1}{n(d+1)k_0} \pi^{C_{10}k_0}$$

$$P\{W(w) = 1\} \; P\{W(w_{n-1-(d+1)k_0}) = n-1 - (d+1)k_0\} .$$

Since

$$P\{W(w) = 1\} = P\{w \text{ is occupied and all its neighbors are vacant}\} \geq \pi^{2d+1} ,$$

and (5.63) holds for n replaced by $n-1 - (d+1)k_0$ and $w = w_{n-1-(d+1)k_0}$, we obtain (5.63), in general, at the expense of increasing C_3 and N_0 slightly. $\square$

<u>Proof of Theorem 5.3</u>: By Cor. 2.2, if $0 < \#W(v_0) < \infty$, then there exists a vacant circuit J on $\mathcal{G}^*$ surrounding $W(v_0)$. If $\#W(v_0) \geq n$, then $W(v_0)$ contains some vertex v_1 with

$$|v_1(i) - v_0(i)| \geq \frac{1}{2} (\sqrt{n/\mu} - 1) \quad \text{for } i = 1 \text{ or } 2,$$

where μ is as in (5.7). Therefore, the diameter of J - which has

v_0 and v_1 in its interior - is at least $\frac{1}{2}(\sqrt{n/\mu}-1)$. Let the diameter of J be $L \geq (\sqrt{n/\mu}-1)$. Since J surrounds v_0 it intersects the horizontal half line $[0,\infty) \times \{v_0(2)\}$, and if Λ is some constant which exceeds the diameter of each edge of $\mathcal{G}^*$, then J contains a vertex v^* of $\mathcal{G}^*$ in the strip

$$S = [-\Lambda,\infty) \times [v_0(2)-\Lambda,\ v_0(2) + \Lambda].$$

Moreover, $|v^*(1)-v_0(1)| \leq L$, since v_0 lies in the interior of J. Also, J must contain at least

$$\frac{L}{\Lambda} \geq \frac{1}{\Lambda} \max\{|v^*(1)-v_0(1)|,\ \tfrac{1}{2}(\sqrt{n/\mu}-1)\}$$

vertices of $\mathcal{G}^*$ and all of those belong to the vacant cluster of v^* on $\mathcal{G}^*$, $W^*(v^*)$. Thus,

$$\text{(5.111)} \qquad P_p\{n \leq \#W(v_0) < \infty\}$$
$$\leq \sum_{v^* \varepsilon S} P_p\{\#W^*(v^*) \geq \frac{1}{\Lambda} \max\{|v^*(1)-v_0(1)|,\ \tfrac{1}{2}(\sqrt{n/\mu}-1)\}\} .$$

By virtue of Lemma 4.1 the right hand side of (5.111) can only be increased if we replace p by p_0 with $p(i) \geq p_0(i)$, $1 \leq i \leq d$. Moreover, by Theorem 5.1 (applied to $\mathcal{G}^*$) (5.65) implies that

$$\text{(5.112)} \qquad P_{p_0}\{\#W^*(v^*) \geq m\} \leq C_1 e^{-C_2 m} , \quad m \geq 0,$$

and by the periodicity this estimate is uniform in v^*. (5.66) is immediate from (5.111), (5.112), and (5.7). $\square$

5.3. Large crosssing probabilities imply that percolation occurs.

Even though this section does not deal with the distribution of $\#W$ we include it here, since the proof of next theorem, due to Russo (1981), is in a sense dual to that of Theorem 5.1. The argument works for any graph $\mathcal{G}$ imbedded in $\mathbb{R}^2$ which satisfies the following condition.

Condition C. If e' and e'' are edges of $\mathcal{G}$ with endpoints v',w' and v'',w'', respectively, and e' intersects e'' in a point which is not an endpoint of both e' and e'', then there exists an edge of $\mathcal{G}$ from v' or w' to v'' or w''. ///

Condition C holds for instance if $\mathcal{G}$ is one of a matching pair $(\mathcal{G},\mathcal{G}^*)$ based on some $(\mathcal{M},\mathcal{F})$. In such a graph two edges e', e" can intersect in a point which is not an endpoint of both, only if e' and e" both belong to the closure of the same face $F \in \mathcal{F}$ (cf. Comment 2.2. (vii)). This F is close-packed in $\mathcal{G}$ and the endpoints v', w', v", w" of e' and e" must lie on the perimeter of F, and there exist an edge of $\mathcal{G}$ between any pair of these vertices. This argument also shows that even the graphs discussed in the Remark in Sect. 2.3 satisfy Condition C.

The reader should note that in the next theorem (5.113) and (5.114) are conditions on the crossing probabilities in the "long direction" of the blocks, while (5.10) is for crossing probabilities in the "short direction".

<u>Theorem 5.4</u>. (Russo 1981) <u>Let $\mathcal{G}$ be a periodic graph imbedded in $\mathbb{R}^2$ which satisfies (2.2)-(2.5) and Condition C. Let P_p be a λ-parameter periodic probability measure and let Λ satisfy (5.8). If for some integers</u> $N_1, N_2 > 2\Lambda$

$$(5.113) \qquad \sigma((3N_1,N_2);\ 1,p,\mathcal{G}) > 1-7^{-81}$$

<u>as well as</u>

$$(5.114) \qquad \sigma((N_1,3N_2);\ 2,p,\mathcal{G}) > 1-7^{-81}$$

<u>then</u>

$$(5.115) \qquad \theta(p,v) > 0 \quad \text{for some } v \in \mathcal{G}.$$

<u>Remark</u>.

Russo (1981), Prop. 1 uses Theorem 5.4 to show that for periodic site-percolation problems on graphs $\mathcal{G}$ in $\mathbb{R}^2$ which satisfy conditions somewhat stronger than those of Theorem 3.2 no percolation can occur on the critical surface. In other words $\theta(p_0,v) = 0$ for the p_0 defined in Theorem 3.2. In particular $\theta(p_H,v) = 0$ in one-parameter problems of this kind. This is of course also a consequence of Theorem 3.2 (see (3.43)). Actually using Theorem 6.1 and a refinement of Russo's argument one can prove this result under more general conditions. Specifically the following left-continuity property holds: <u>Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair of periodic graphs imbedded in $\mathbb{R}^2$ and $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ a periodic partition of the vertices of $\mathcal{G}$ such that one of the coordinate axes is an axis of symmetry for $\mathcal{G},\mathcal{G}^*$ and the partition $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$. Let P_p be as in (3.20)-(3.23). If $p_0 \in \mathcal{P}_\lambda$ is such that $p_0 \gg 0$ and $\theta(p,v) = 0$ for all</u>

$p << p_0$, $p \in \mathcal{P}_\lambda$, _then_ $\theta(p_0,v) = 0$.

There also is a continuity result if $\theta(p_0,v) > 0$, which can be derived from Theorem 12.1. _Under the above conditions_ $p \to \theta(p,v)$ _is continuous at all points_ $p >> 0$ _for which_ $\theta(p,v) > 0$.

We do not proof either of these results.

Finally, it is worth pointing out that Russo (1978) proved that $\theta(.,v)$ is always right continuous for any graph in any dimension. I.e., if $p(i) \downarrow p_0(i)$, $1 \le i \le d$, then $\theta(p,v) \downarrow \theta(p_0,v)$. This is so because $\theta(.,v)$ is the decreasing limit of the sequence of continuous increasing fuctions $p \to P_p\{v$ is connected by an occupied path to some point outside $S(v,M)\}$. (See the lines before Lemma 5.5 for $S(v,M)$.)

Proof of Theorem 5.4. As in Theorem 5.1 we use an auxiliarly graph and set up a correspondence between vertices of this graph and blocks of $\mathcal{G}$. This time the auxiliarly graph is the simple quadratic lattice $\mathcal{G}_0$ of Ex. 2.1(i). For each occupancy configurations ω on $\mathcal{G}$ we construct an occupancy configuration on $\mathcal{G}_0$ as follows. If (i_1,i_2) is a vertex of $\mathcal{G}_0$ with $i_1 + i_2$ even, then we take (i_1,i_2) occupied iff there exists an occupied horizontal crossing on $\mathcal{G}$ of

$$[i_1N_1,(i_1+3)N_1] \times [i_2 N_2,(i_2+1)N_2] . \tag{5.116}$$

If (j_1,j_2) is a vertex of $\mathcal{G}_0$ with $j_1 + j_2$ odd we take (j_1,j_2) occupied iff there exists an occupied vertical crossing on $\mathcal{G}$ of

$$[(j_1+1)N_1,\ (j_1+2)N_1] \times [(j_1-1)N_2,\ (j_2+2)N_2] . \tag{5.117}$$

We claim that if (i_1,i_2) with $i_1 + i_2$ even and (j_1,j_2) with $j_1 + j_2$ odd are two adjacent vertices of $\mathcal{G}_0$ which are both occupied, then there exists an occupied horizontal crossing $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ of (5.116) and an occupied vertical crossing $s = (w_0,f_1,\dots,f_\rho,w_\rho)$ of (5.117), and any such pairs of crossings must intersect. We check this for the case $j_1 = i_1+1$, $j_2 = i_2$; the other cases are similar. Since (i_1,i_2) is occupied, there exists an occupied horizontal crossing r of (5.116) on $\mathcal{G}$. By Def. 3.1, if $r = (v_0,e_1,\dots,e_\nu,v_\nu)$, then the curve made up from $e_1,\dots,e_\nu$ contains a continuous path in $[i,N_1,(i_1+3)N_1] \times [i_2 N_2,(i_2+1)N_2]$ which connects the left and right edges of this rectangle. Similarly, there exists an occupied vertical crossing $s = (w_0,f_1,\dots,f_\rho,w_\rho)$ of (5.117), and s contains a continuous curve in $[(j_1+1)N_1,\ (j_1+2)N_1] \times [(j_2-1)N_2,\ (j_2+2)N_2]$ which connects the top and bottom edges of this rectangle. Since

$j_1 = i_1+1$, $j_2 = i_2$, the latter rectangle equals

$$(5.118) \qquad [(i_1+2)N_1, (i_1+3)N_1] \times [(i_2-1)N_2, (i_2+2)N_2] .$$

It is now evident from the relative location of the rectangles (5.116) and (5.118) that r and s intersect.

This proves the claim. By condition C it follows that either r and s intersect in a vertex of $\mathcal{G}$, common to both, or there exists vertices v of r and w of s which are adjacent to each other on $\mathcal{G}$. In either case all the vertices of r and s (which are all occupied) belong to the same occupied component on $\mathcal{G}$. Thus, if (i_1,i_2) and (j_1,j_2) are occupied neighbors on $\mathcal{G}_0$, then necessarily one of them has an even sum of its coordinates, and one an odd sum, and the corresponding blocks on $\mathcal{G}$ contain crossings which belong to the same occupied component on $\mathcal{G}$. Therefore, if $\mathcal{G}_0$ contains an infinite occupied cluster, then so does $\mathcal{G}$. To complete the proof it therefore suffices to show

$$(5.119) \qquad P_p\{\mathcal{G}_0 \text{ contains an occupied cluster}\} > 0,$$

since this will imply (5.115). (5.119) is proved by the standard Peierls argument. Let W_0 be the occupied cluster of (0,0) on $\mathcal{G}_0$. By Cor. 2.2., $0 < \#W_0 < \infty$ happens only if there exists a vacant circuit J surrounding (0,0) on $\mathcal{G}_0^*$. $\mathcal{G}_0^*$ is described in Ex. 2.2(i). Every vertex has eight neighbors on $\mathcal{G}_0^*$. The number of self-avoiding paths starting at the origin and containing n vertices is therefore at most 8.7^{n-2}. The number of circuits of n vertices containing the origin in its interior is therefore at most $8n.7^{n-2}$ (since any such circuit must contain one of the points $(i,0)$, $1 \le i \le n$, as in the argument preceding (5.111)). On the other hand, the probability that any vertex of $\mathcal{G}_0^*$ is vacant is strictly less than 7^{-81}, by virtue of (5.113) and (5.114). Not all vertices of $\mathcal{G}_0^*$ are independent, but if $\overline{\ell}_1,\dots,\overline{\ell}_t$ are vertices of $\mathcal{G}_0^*$ (and hence of $\mathcal{G}_0$) such that for each $1 \le r, s \le t$, $r \ne s$, there is an $i = 1,2$, with $|\ell_r(i)-\ell_s(i)| \ge 5$ ($\overline{\ell}_r = (\ell_r(1), \ell_r(2))$ then the occupancies of $\overline{\ell}_1,\dots,\overline{\ell}_t$ are independent, because they depend on disjoint sets of vertices of $\mathcal{G}$. Any circuit on $\mathcal{G}_0^*$ of n vertices contains at least $n/81$ such independent vertices, and hence the probability that a given circuit on $\mathcal{G}_0^*$ of n vertices is vacant is at most

$$(7^{-81} - \eta)^{n/81}$$

for some $\eta > 0$. It follows that for a suitably large N

(5.120) P_p {there does not exist a vacant circuit on $\mathcal{G}_0^*$ surrounding $(0,0)$ and containing at least N vertices}

$$\geq 1 - \sum_{n \geq N} 8n\, 7^{n-2} (7^{-81} - \eta)^{n/81} \geq \frac{1}{2}.$$

Now the event that there does not exist a vacant circuit of a certain type on $\mathcal{G}_0^*$ is an increasing event for the percolation on $\mathcal{G}$. Thus, by the FKG inequality

(5.121) P_p {the origin of $\mathcal{G}_0$ is occupied and there does not exist any vacant circuit on $\mathcal{G}_0^*$ surrounding $(0,0)$

$\geq P_p$ {the origin of $\mathcal{G}_0$ is occupied and there does not exist any vacant circuit on $\mathcal{G}_0^*$ surrounding $(0,0)$ and containing less than N vertices} $\times$ {the left hand side of (5.120)}

$$\geq \prod_{\substack{|i_1| \leq N \\ |i_2| \leq N}} P_p\{\text{the vertex } (i_1,i_2) \text{ of } \mathcal{G}_0^* \text{ is occupied}\} \cdot \frac{1}{2} > 0.$$

As we saw above, the event in braces in the first member of (5.121) implies $\#W_0 = \infty$, so that we proved

$$P_p\{\#W_0 = \infty\} > 0$$

which in turn implies (5.119). $\square$