ADJOINTNESS AND QUANTIFIERS

> *"... adjoints occur almost everywhere in many branches of Mathematics. ... a systematic use of all these adjunctions illuminates and clarifies these subjects."*
>
> Saunders Maclane

The isolation and explication of the notion of *adjointness* is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas. In this final chapter we shall look at the nature of this concept, and demonstrate its ubiquity with a range of illustrations that encompass almost all concepts that we have discussed. We shall then see how it underlies the proof of the Fundamental Theorem of Topoi, and finally examine its role in a particular analysis of quantifiers in a topos.

15.1. Adjunctions

The basic data for an *adjoint situation*, or *adjunction*, comprise two categories, $\mathscr{C}$ and $\mathscr{D}$, and functors F and G between them

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

in each direction, enabling an interchange of their objects and arrows. Given $\mathscr{C}$-object a and $\mathscr{D}$-object b we obtain

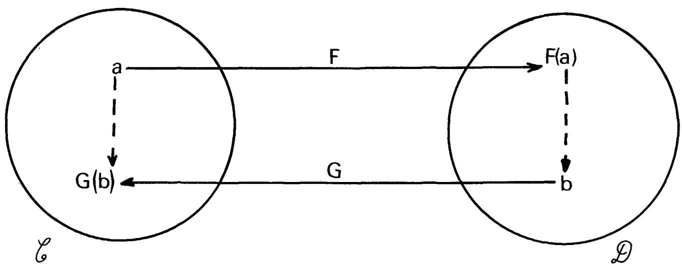

Fig. 15.1.

$G(b)$ in $\mathscr{C}$ and $F(a)$ in $\mathscr{D}$. Adjointness occurs when there is an exact correspondence of arrows between these objects in the directions indicated by the broken arrows in the picture, so that any passage from a to $G(b)$ in $\mathscr{C}$ is matched uniquely by a passage from $F(a)$ to b in $\mathscr{D}$. In other words we require for each a and b as shown, a bijection

(1) $$\theta_{ab}: \mathscr{D}(F(a), b) \cong \mathscr{C}(a, G(b))$$

between the set of $\mathscr{D}$-arrows of the form $F(a) \to b$ and the $\mathscr{C}$-arrows of the form $a \to G(b)$. Moreover the assignment of bijections θ_{ab} is to be "natural in a and b", which means that it preserves categorial structure as a and b vary. Specifically, the assignment to the pair $\langle a, b\rangle$ of the "hom-set" $\mathscr{D}(F(a), b)$ generates a functor from the product category $\mathscr{C}^{\mathrm{op}} \times \mathscr{D}$ to **Set** (why $\mathscr{C}^{\mathrm{op}}$ and not $\mathscr{C}$? Examine the details), while the assignment of $\mathscr{C}(a, G(b))$ establishes another such functor. We require that the θ_{ab}'s form the components of a natural transformation θ between these two functors.

When such a θ exists we call the triple $\langle F, G, \theta\rangle$ an *adjunction* from $\mathscr{C}$ to $\mathscr{D}$. F is then said to be *left adjoint* to G, denoted $F \dashv G$, while G is *right adjoint* to F, $G \vdash F$. The relationship between F and G given by θ as in (1) is presented schematically by

$$\frac{a \to G(b)}{F(a) \to b}$$

which displays the "left-right" distinction.

An adjoint situation is expressible in terms of the behaviour of special arrows associated with each object of $\mathscr{C}$ and $\mathscr{D}$:-

Let a be a particular $\mathscr{C}$-object, and put $b = F(a)$ in (1). Applying θ (i.e. the appropriate component) to the identity arrow on $F(a)$ we obtain the $\mathscr{C}$-arrow $\eta_a = \theta(1_{F(a)})$, to be called the *unit* of a. Then for any b in $\mathscr{D}$, we know that any $g: a \to G(b)$ corresponds to a unique $f: F(a) \to b$ under θ_{ab}. Using the naturality of θ in a and b we find in fact that η_a enjoys a certain co-universal property, namely that to any such g there is exactly one such f such that

(2)

$$\begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[dr, "g"'] & G(F(a)) \arrow[d, dashed, "G(f)"] & F(a) \arrow[d, dashed, "f"] \\ & G(b) & b \end{tikzcd}$$

commutes. Indeed $g = \theta_{ab}(f)$, and so

(3) $\theta_{ab}(f) = G(f) \circ \eta_a.$

Naturality of θ implies also that

$$\begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[d, "k"'] & G(F(a)) \arrow[d, "G(F(k))"] \\ a' \arrow[r, "\eta_{a'}"] & G(F(a')) \end{tikzcd}$$

commutes for all such $\mathscr{C}$-arrows k, and so the η_a's form the components of a natural transformation $\eta : 1_{\mathscr{C}} \dot{\to} G \circ F$, called the *unit* of the adjunction.

Dually, let b be a particular $\mathscr{D}$ object and put $a = G(b)$ in (1). If τ is the inverse to the natural isomorphism θ ($\tau_{ab} = \theta_{ab}^{-1}$), apply τ to the identity arrow on $G(b)$ to get the *co-unit* $\varepsilon_b = \tau(1_{G(b)})$ of b. ε_b has the universal property that to any $\mathscr{D}$-arrow $f : F(a) \to b$ there is exactly one $\mathscr{C}$-arrow $g : a \to G(b)$ such that

(4)

$$\begin{tikzcd} F(G(b)) \arrow[r, "\varepsilon_b"] & b & & G(b) \\ F(a) \arrow[u, dashed, "F(g)"] \arrow[ur, "f"'] & & & a \arrow[u, dashed, "g"'] \end{tikzcd}$$

commutes. Since $f = \tau_{ab}(g)$, we get

(5) $\tau_{ab}(g) = \varepsilon_b \circ F(g),$

while the ε_b's form the components of the natural transformation $\varepsilon : F \circ G \dot{\to} 1_{\mathscr{D}}$, the *co-unit* of the adjunction.

On the other hand, given natural transformations η and ε of this form, we could define natural transformations θ and τ by specifying their components by equations (3) and (5). If the universal properties of diagrams (2) and (4) hold, then θ_{ab} and τ_{ab} would be inverse to each other, hence each a bijection, giving θ as an adjunction from $\mathscr{C}$ to $\mathscr{D}$.

Thus, given F and G as above, the following are equivalent:

(a) F is left adjoint to G, $F \dashv G$

(b) G is right adjoint to F, $G \vdash F$

(c) there exists an adjunction $\langle F, G, \theta \rangle$ from $\mathscr{C}$ to $\mathscr{D}$

(d) there exist natural transformations $\eta : 1_{\mathscr{C}} \dot{\to} G \circ F$ and $\varepsilon : F \circ G \dot{\to} 1_{\mathscr{D}}$

whose components have the universal properties of diagrams (2) and (4) above.

Diagrams (2) and (4) are instances of a more general phenomenon. Suppose that $G:\mathscr{D}\to\mathscr{C}$ is a functor and a an object of $\mathscr{C}$. Then a pair $\langle b, \eta\rangle$ consisting of a $\mathscr{D}$-object b and a $\mathscr{C}$-arrow $\eta : a \to G(b)$ is called *free over a with respect to G* iff for any $\mathscr{C}$-arrow of the form $g : a \to G(c)$ there is exactly one $\mathscr{D}$-arrow $f : b \to c$ such that

(6)
$$
\begin{tikzcd}
a \arrow[r, "\eta"] \arrow[dr, "g"'] & G(b) \arrow[d, dashed, "G(f)"] \\
& G(c)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
b \arrow[d, dashed, "f"] \\
c
\end{tikzcd}
$$

commutes.

Such a pair $\langle b, \eta\rangle$ is also known as a *universal arrow* from a to G.

Thus, whenever $F \dashv G$, the pair $\langle F(a), \eta_a\rangle$ is free over a with respect to G.

Dually, given a functor $F:\mathscr{C}\to\mathscr{D}$ and a $\mathscr{D}$-object b, a pair $\langle a, \varepsilon\rangle$, comprising a $\mathscr{C}$-object a and an arrow $\varepsilon : F(a)\to b$ is called *co-free over b with respect to F* if to each pair $\langle c, f\rangle$ comprising a $\mathscr{C}$-object c and an arrow $f : F(c)\to b$ there is a unique $g : c\to a$ in $\mathscr{C}$ such that

(7)
$$
\begin{tikzcd}
F(a) \arrow[r, "\varepsilon"] & b \\
F(c) \arrow[u, dashed, "F(g)"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \\
c \arrow[u, dashed, "g"']
\end{tikzcd}
$$

commutes. Such a pair is also called a universal arrow from F to b.

Exercise 1. Describe a right adjoint G to F in terms of pairs that are co-free over $\mathscr{D}$-objects with respect to F.

Exercise 2. Suppose that $\langle b, \eta\rangle$ is a universal arrow from a to $G:\mathscr{D}\to\mathscr{C}$. Show that the arrow $\eta : a\to G(b)$ is an initial object in the category $a \downarrow F$ whose objects are $\mathscr{C}$-arrows of the form $f : a\to G(c)$ and whose arrows are $\mathscr{D}$-arrows $g : c\to d$ such that

$$
\begin{tikzcd}
& a \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
G(c) \arrow[rr, "G(g)"] & & G(d)
\end{tikzcd}
$$

commutes.

EXERCISE 3. Dualise Exercise 2.

EXERCISE 4. Suppose that for every $\mathscr{C}$-object a, there is a universal arrow from a to $G:\mathscr{D}\to\mathscr{C}$. Construct a functor $F:\mathscr{C}\to\mathscr{D}$ such that $F\dashv G$.

EXERCISE 5. Dualise Exercise 4. □

The existence of an adjoint to a functor has important consequences for the properties of that functor. For example, if $F\dashv G$, then G preserves limits (i.e. maps the limit of a diagram in $\mathscr{D}$ to a limit for the G-image of that diagram in $\mathscr{C}$), while F preserves co-limits.

The details of this brief account of the theory of adjoints may be found in any standard text on category theory.

15.2. Some adjoint situations

Initial objects

Let $\mathscr{C}=\mathbf{1}$ be the category with one object, say 0, and G the unique functor $\mathscr{D}\to\mathbf{1}$. If $F:\mathbf{1}\to\mathscr{D}$ is left adjoint to G then for any b in $\mathscr{D}$,

$$\frac{0\to G(b)}{F(0)\to b}$$

since there is exactly one arrow $0\to G(b)$, there is exactly one arrow $F(0)\to b$. Hence $F(0)$ *is an initial object in* $\mathscr{D}$. The co-unit $\varepsilon_b : F(G(b))\to b$ is the unique arrow $F(0)\to b$.

EXERCISE 1. Show that $\mathscr{D}$ has a terminal object iff the functor $!:\mathscr{D}\to\mathbf{1}$ has a right adjoint. □

Products

Let $\Delta:\mathscr{C}\to\mathscr{C}\times\mathscr{C}$ be the diagonal functor taking a to $\langle a, a\rangle$ and $f:a\to b$ to $\langle f, f\rangle:\langle a, a\rangle\to\langle b, b\rangle$. Suppose Δ has a right adjoint $G:\mathscr{C}\times\mathscr{C}\to\mathscr{C}$.

Then we have

$$\frac{c\to G(x)}{\langle c, c\rangle\to x}$$

where c is in $\mathscr{C}$ and $x = \langle a, b\rangle$ is in $\mathscr{C}\times\mathscr{C}$. The co-unit $\varepsilon_x : \Delta(G(x)) \to \langle a, b\rangle$ is a pair of $\mathscr{C}$-arrows $p: G(x) \to a$ and $q: G(x) \to b$. Using the "co-freeness" property of ε_x, for any arrows $f: c \to a$, $g: c \to b$, there is a unique $h: c \to G(x)$ such that

$$\begin{tikzcd} \Delta(G(x)) \arrow[r, "\varepsilon_x"] & x \\ \Delta(c) \arrow[u, dashed, "\Delta(h)"] \arrow[ur, "\langle f, g\rangle"'] & \end{tikzcd} \qquad \begin{tikzcd} G(x) \\ c \arrow[u, dashed, "h"'] \end{tikzcd}$$

and hence

$$\begin{tikzcd} a & G(x) \arrow[l, "p"'] \arrow[r, "q"] & b \\ & c \arrow[ul, "f"] \arrow[u, dashed, "h"'] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. Thus $G(x)$ *is a product* $a \times b$ *of* a *and* b *with* ε_x *as the pair of associated projections*. We have the adjunction

$$\frac{c \to a \times b}{c \to a,\ c \to b}$$

The unit $\eta_c : c \to c \times c$ is the diagonal product arrow $\langle \mathbf{1}_c, \mathbf{1}_c\rangle$.

EXERCISE 2. Show that $\mathscr{C}$ has co-products iff $\Delta : \mathscr{C} \to \mathscr{C}\times\mathscr{C}$ has a left adjoint. □

It can be shown that the limit and co-limit of any type of diagram in a category $\mathscr{C}$ arise, when they exist, from right and left adjoints of a "diagonal" functor $\mathscr{C} \to \mathscr{C}^J$, where J is a canonical category having the "shape" of that diagram (for products, J is the discrete category $\{0, 1\}$). The unit for the left adjoint is the universal co-cone, the co-unit for the right adjoint is the universal cone.

Topology and algebra

There are many significant constructions that arise as adjoints to forgetful functors. The forgetful functor $U: \mathbf{Grp} \to \mathbf{Set}$ from groups to sets has as left adjoint the functor assigning to each set the free group generated by that set (here "free" has precisely the above meaning associated with units of an adjunction).

The construction of the field of quotients of an integral domain gives a functor left adjoint to the forgetful functor from the category of fields to the category of integral domains.

The specification of the discrete topology on a set gives a left adjoint to $U:\mathbf{Top} \to \mathbf{Set}$, while the indiscrete topology provides a right adjoint to U.

The completion of a metric space provides a left adjoint to the forgetful functor from complete metric spaces to metric spaces.

The reader will find many more examples of adjoints from topology and algebra in Maclane [71] and Herrlich and Strecker [73].

Exponentiation

If $\mathscr{C}$ has exponentials, then there is (§3.16) a bijection

$$\mathscr{C}(c \times a, b) \cong \mathscr{C}(c, b^a)$$

for all objects a, b, c, indicating the presence of an adjunction.

Let $F:\mathscr{C} \to \mathscr{C}$ be the right product functor $- \times a$ of §9.1 taking any c to $c \times a$. Then F has as right adjoint the functor $(\ \ )^a : \mathscr{C} \to \mathscr{C}$ taking any b to b^a and any arrow $f: c \to b$ to $f^a : c^a \to b^a$, which is the exponential *adjoint* to the composite $f \circ ev' : c^a \times a \to c \to b$, i.e. the unique arrow for which

$$\begin{tikzcd}
b^a \times a \arrow[r, "ev"] & b \\
c^a \times a \arrow[u, dashed, "f^a \times 1_a"] \arrow[ur, "f \circ ev'"'] &
\end{tikzcd}$$

commutes.

The co-unit $\varepsilon_b : F(b^a) \to b$ is precisely the evaluation arrow $ev: b^a \times a \to b$, and its "co-freeness" property yields the axiom of exponentials given in §3.16.

The adjoint situation is

$$\frac{c \to b^a}{c \times a \to b}.$$

Thus *$\mathscr{C}$ has exponentials iff the functor $- \times a$ has a right adjoint for each $\mathscr{C}$-object a.*

Relative pseudo-complements

This is a special case of exponentials (cf. §8.3). In any r.p.c. lattice the condition

$$c \sqcap a \sqsubseteq b \quad \text{iff} \quad c \sqsubseteq a \Rightarrow b$$

yields the adjunction

$$\frac{c \to (a \Rightarrow b)}{c \sqcap a \to b}$$

A lattice is r.p.c. iff the functor -$\sqcap$ a *taking* c *to* $c \sqcap a$ *has a right adjoint for each* a.

Natural numbers objects (cf. Lawvere [69])

A $\mathscr{C}$ arrow f is *endo* (from "endomorphism") iff $\operatorname{dom} f = \operatorname{cod} f$, i.e. f has the form $f : a \to a$, or $a \circlearrowleft^{f}$. The category $\mathscr{C}\circlearrowleft$ has as objects the $\mathscr{C}$-endo's, with an arrow from $a \circlearrowleft^{f}$ to $b \circlearrowleft^{g}$ being a $\mathscr{C}$-arrow $h : a \to b$ such that

$$\begin{tikzcd}
a \arrow[r, "h"] \arrow[d, "f"'] & b \arrow[d, "g"] \\
a \arrow[r, "h"] & b
\end{tikzcd}$$

i.e.

$$a \circlearrowleft^{f} \xrightarrow{h} b \circlearrowleft^{g}$$

commutes. Let $G : \mathscr{C}\circlearrowleft \to \mathscr{C}$ be the forgetful functor taking $f : a \to a$ to its domain a.

Suppose G has a left adjoint

$$\frac{a \to G(b)}{F(a) \to b},$$

and let the endo $F(1)$ be denoted $N \circlearrowleft^{\sigma}$ and the unit $\eta_1 : 1 \to G(F(1))$ denoted $O : 1 \to N$. The notation is of course intentional:

the freeness of $(F(1), \eta_1)$ over 1

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr] & G(F(1)) \arrow[d, dashed] & F(1) \arrow[d, dashed] \\
 & G(A) & A
\end{tikzcd}$$

means that for any endo $A : a \xrightarrow{f} a$ and any $\mathscr{C}$-arrow $x : 1 \to a = G(A)$ there is a unique arrow $h : F(1) \to A$, i.e.

$$N \circlearrowleft^{\sigma} \xrightarrow{h} a \circlearrowleft^{f},$$

such that

$$\begin{tikzcd} 1 \arrow[r, "O"] \arrow[dr, "x"'] & N \arrow[d, "h"] \\ & a \end{tikzcd}$$

and hence

$$\begin{tikzcd} & N \arrow[loop right, "s"] \arrow[dd, "h"] \\ 1 \arrow[ur, "O"] \arrow[dr, "x"'] & \\ & a \arrow[loop right, "f"] \end{tikzcd}$$

commutes. Thus $(F(1), \eta_1)$ is a natural numbers object.

Conversely, if $\mathscr{C} \models \text{NNO}$, define $F:\mathscr{C} \to \mathscr{C}^{\circlearrowleft}$ to take a to the endo

$$a \times N \xrightarrow{\mathbf{1}_a \times s} a \times N$$

and $f: a \to b$ to $f \times \mathbf{1}_N$.

Then by the theorem 13.2.1 of Freyd, if $\mathscr{C}$ has exponentials, then for any endo $f: b \to b$ and any arrow $h_0: a \to b$ there is a unique h for which

$$\begin{tikzcd} & a \times N \arrow[loop right, "\mathbf{1}_a \times s"] \arrow[dd, "h"] \\ a \arrow[ur, "\langle \mathbf{1}_a, O_a \rangle"] \arrow[dr, "h_0"'] & \\ & b \arrow[loop right, "f"] \end{tikzcd}$$

commutes. We have the situation

$$\frac{a \xrightarrow{h_0} G(b \circlearrowleft^{f})}{F(a) \xrightarrow{h} b \circlearrowleft^{f}}$$

indicating that $F \dashv G$. The unit η_1 now becomes $\langle \mathbf{1}_1, O \rangle: 1 \to 1 \times N$ from which we recover $O: 1 \to N$ under the natural isomorphism $1 \times N \cong N$.

Altogether then, *a cartesian closed category $\mathscr{C}$ has a natural numbers object iff the forgetful functor from $\mathscr{C}^{\circlearrowleft}$ to $\mathscr{C}$ has a left adjoint.*

We also obtain the characterisation of a natural numbers object as a universal arrow from the terminal object to this functor.

Adjoints in posets

Let $(P, \sqsubseteq)$ and $(Q, \sqsubseteq)$ be posets. A functor from P to Q is a function $f: P \to Q$ that is monotonic, i.e. has

$$p \sqsubseteq q \quad \text{only if} \quad f(p) \sqsubseteq f(q).$$

Then $g:Q \to P$ will be right adjoint to f,

$$\frac{p \to g(r)}{f(p) \to r}$$

iff for all $p \in P$ and $r \in Q$,

$$p \sqsubseteq g(r) \quad \text{iff} \quad f(p) \sqsubseteq r.$$

On the other hand g will be left adjoint to f,

$$\frac{r \to f(p)}{g(r) \to p},$$

when

$$g(r) \sqsubseteq p \quad \text{iff} \quad r \sqsubseteq f(p).$$

For example, given a function $f:A \to B$, and subsets $X \subseteq A$, $Y \subseteq B$, we have

$$X \subseteq f^{-1}(Y) \quad \textit{iff} \quad f(X) \subseteq Y$$

and so the functor $f^{-1}:\mathcal{P}(B) \to \mathcal{P}(A)$ taking $Y \subseteq B$ to $f^{-1}(Y)$ is right adjoint to the functor $\mathcal{P}(f):\mathcal{P}(A) \to \mathcal{P}(B)$ of §9.1, that takes $X \subseteq A$ to $f(X) \subseteq B$.

As well as having a left adjoint, $\mathcal{P}(f) \dashv f^{-1}$, f^{-1} has a right adjoint

$$f^{+}:\mathcal{P}(A) \to \mathcal{P}(B)$$

given by $f^{+}(X) = \{y \in B : f^{-1}\{y\} \subseteq X\}$ where $f^{-1}\{y\} = \{x : f(x) = y\}$ is the inverse image of $\{y\}$. That $f^{-1} \dashv f^{+}$ follows from the fact that

$$f^{-1}(Y) \subseteq X \quad \text{iff} \quad Y \subseteq f^{+}(X).$$

Subobject classifier

The display (Lawvere [72])

$$\frac{d \to \Omega}{? \rightarrowtail d}$$

where $? \rightarrowtail d$ denotes an arbitrary subobject of d, indicates that the Ω-axiom expresses a property related to adjointness.

The functor $\mathrm{Sub}:\mathscr{C} \to \mathbf{Set}$ described in §9.1, Example 11, assigns to each object d the collection of subobjects of d, and to each arrow $f:c \to d$ the function $\mathrm{Sub}(f):\mathrm{Sub}(d) \to \mathrm{Sub}(c)$ that takes each subobject of d to its pullback along f. As it stands, Sub is contravariant. However, by switching to the opposite category of $\mathscr{C}$ we can regard Sub as a covariant functor

$$\text{Sub}: \mathscr{C}^{\text{op}} \to \mathbf{Set}.$$

Now in the case $\mathscr{C} = \mathscr{E}$ (a topos) the arrow $true: 1 \to \Omega$ is a subobject of Ω and so corresponds to a function $\eta: 1 = \{0\} \to \text{Sub}(\Omega)$.

Now consider the diagram

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "g"'] & \text{Sub}(\Omega) \arrow[d, dashed, "\text{Sub}(f)"] & \Omega \arrow[d, dashed, "f"] \\ & \text{Sub}(d) & d \end{tikzcd}$$

A function g as shown picks out a subobject $g_0: a \rightarrowtail d$ of d, for which we have a character χ_{g_0}, and pullback

$$\begin{tikzcd} a \arrow[r, tail, "g_0"] \arrow[d] & d \arrow[d, "\chi_{g_0}"] \\ 1 \arrow[r, "true"] & \Omega \end{tikzcd}$$

in $\mathscr{E}$. Thus $f = (\chi_{g_0})^{\text{op}}$ is an $\mathscr{E}^{\text{op}}$ arrow from Ω to d. Then Sub(f)($=$Sub(χ_{g_0}) originally) takes *true* to its pullback along χ_{g_0}, i.e. to the subobject g_0, and so the above triangle commutes. But by the uniqueness of the character of g_0, the only arrow along which *true* pulls back to give g_0 is χ_{g_0} and so the only $\mathscr{E}^{\text{op}}$ arrow for which the triangle commutes is $f = (\chi_{g_0})^{\text{op}}$.

Thus the pair $\langle \Omega, \eta \rangle$, i.e. $\langle \Omega, true: 1 \to \Omega \rangle$ is free over 1 with respect to Sub.

Conversely the freeness of $\langle \Omega, \eta \rangle$ implies that $\eta(0)$ classifies subobjects and so we can say that *any category $\mathscr{C}$ with pullbacks has a subobject classifier iff there exists a universal arrow from* 1 *to* Sub$: \mathscr{C}^{\text{op}} \to \mathbf{Set}$. (cf. Herrlich and Strecker [73], Theorem 30.14).

EXERCISE 1. Let $Rel(\text{-}, a): \mathscr{C} \to \mathbf{Set}$ take each $\mathscr{C}$-object b to the collection of all $\mathscr{C}$-arrows of the form $R \rightarrowtail b \times a$ ("relations" from b to a). For any $f: c \to b$, $Rel(f, a)$ maps $R \rightarrowtail b \times a$ to its pullback along $f \times 1_a$, so that $Rel(\text{-}, a)$ as defined is contravariant. Show that $\mathscr{C}$ (finitely complete) has power objects iff for each $\mathscr{C}$-object a, there is a universal arrow from 1 to

$$Rel(\text{-}, a): \mathscr{C}^{\text{op}} \to \mathbf{Set}.$$

EXERCISE 2. Can you characterise the partial arrow classifier $\eta_a: a \rightarrowtail \tilde{a}$ in terms of universal arrows? □

Notice that the Ω-axiom states that

$$\mathrm{Sub}(d) \cong \mathscr{E}(d, \Omega) \cong \mathscr{E}^{\mathrm{op}}(\Omega, d)$$

and similarly we have

$$Rel(b, a) \cong \mathscr{E}(b, \Omega^a) \cong \mathscr{E}^{\mathrm{op}}(\Omega^a, b),$$

and so the covariant $\mathscr{E}^{\mathrm{op}} \to \mathbf{Set}$ versions of Sub and $Rel(\text{-}, a)$ are naturally isomorphic to "hom-functors" of the form $\mathscr{C}(d, \text{-})$ (§9.1, Example (7)). In general a **Set**-valued functor isomorphic to a hom-functor is called *representable*. Representable functors are always characterised by their possession of objects free over 1 in **Set**.

15.3. The fundamental theorem

Let $\mathscr{C}$ be a category with pullbacks, and $f: a \to b$ a $\mathscr{C}$-arrow. Then f induces a "pulling-back" functor $f^*: \mathscr{C} \downarrow b \to \mathscr{C} \downarrow a$ which generalises the $f^{-1}: \mathscr{P}(B) \to \mathscr{P}(A)$ example of the last section. f^* acts as in the diagram

$$\begin{tikzcd}
c \arrow[rr] \arrow[dr, dashed] \arrow[dd, "f^*(g)"'] & & d \arrow[dr, "k"] \arrow[dd, "g"] & \\
& m \arrow[rr] \arrow[dl, "f^*(h)"] & & n \arrow[dl, "h"] \\
a \arrow[rr, "f"] & & b &
\end{tikzcd}$$

k is a $\mathscr{C} \downarrow b$ arrow from g to h, $f^*(g)$ and $f^*(h)$ are the pullbacks of g and h along f, yielding a unique arrow $c \to m$ as shown which we take as $f^*(k): f^*(g) \to f^*(h)$.

The "composing with f" functor

$$\Sigma_f: \mathscr{C} \downarrow a \to \mathscr{C} \downarrow b$$

takes object $g: c \to a$ to $f \circ g: c \to b$, and arrow

$$\begin{tikzcd}
c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\
& a &
\end{tikzcd}$$

to

$$\begin{tikzcd}
c \arrow[rr, "k"] \arrow[dr, "f \circ g"'] & & d \arrow[dl, "f \circ h"] \\
& b &
\end{tikzcd}$$

Now an arrow k

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[dr, "f \circ g"'] & & d \arrow[dl, "t"] \\ & b & \end{tikzcd}$$

from $\Sigma_f(g)$ to $t: b \to d$ in $\mathscr{C} \downarrow b$ corresponds to a unique $\mathscr{C} \downarrow a$ arrow k'

$$\begin{tikzcd} c \arrow[drr, bend left, "k"] \arrow[dr, dashed, "k'"'] \arrow[ddr, bend right, "g"'] & & \\ & p \arrow[r] \arrow[d, "f^*(t)"] & d \arrow[d, "t"] \\ & a \arrow[r, "f"] & b \end{tikzcd}$$

from g to $f^*(t)$, by the universal property of the pullback $f^*(t)$, and so we have the adjunction

$$\frac{g \to f^*(t)}{\Sigma_f(g) \to t}$$

showing $\Sigma_f \dashv f^*$.

For set functions, f^* also has a right adjoint

$$\Pi_f : \mathbf{Set} \downarrow A \to \mathbf{Set} \downarrow B.$$

Given $g: X \to A$, then $\Pi_f(g)$ has the form $k: Z \to B$, which we regard as a bundle over B. Thinking likewise of g, the stalk in Z over $b \in B$, i.e. $k^{-1}\{b\}$, is the set of all local sections of g defined on $f^{-1}\{b\} \subseteq A$.

Formally Z is the set of all pairs (b, h) such that h is a function with domain $f^{-1}\{b\}$, such that

$$\begin{tikzcd} f^{-1}\{b\} \arrow[rr, "h"] \arrow[dr, hook] & & X \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

commutes. k is the projection to B.

Notice that if g is an inclusion $g: X \hookrightarrow A$ then the only possible section h as above is the inclusion $f^{-1}\{b\} \hookrightarrow X$, provided that $f^{-1}\{b\} \subseteq X$. Thus the stalk over b in Z is empty if not $f^{-1}\{b\} \subseteq X$, and has one element otherwise. Thus k can be identified with the inclusion of the set

$$\{b: f^{-1}\{b\} \subseteq X\} = f^+(X)$$

into B, and so the functor f^+ is a special case of Π_f.

Now given arrows $g:X \to A$ and $h:Y \to B$, consider

$$\begin{tikzcd}
P \arrow[rr] \arrow[dd, "f^*(h)"'] \arrow[dr, dashed, "t'"] & & Y \arrow[dd, "h"'] \arrow[dr, "t"] & \\
& X \arrow[dl, "g"] & & Z \arrow[dl, "\pi_f(g)"] \\
A \arrow[rr, "f"] & & B &
\end{tikzcd}$$

t is an arrow from h to $\Pi_f(g)$ in $\mathbf{Set}\downarrow B$. $f^*(h)$, the pullback of h along f, is the projection to A of the set

$$P = \{\langle a, y\rangle : f(a) = h(y)\}.$$

Thus if $\langle a, y\rangle \in P$, y lies in the stalk over $f(a)$ in B, and so $t(y)$ is in the stalk over $f(a)$ of $\Pi_f(g)$. Thus $t(y)$ is a section s of g over $f^{-1}\{f(a)\}$, which includes a. Put $t'(\langle a, y\rangle) = s(a)$. Then t' is an arrow from $f^*(h)$ to g in $\mathbf{Set}\downarrow A$.

In this way we establish a correspondence

$$\frac{h \xrightarrow{t} \Pi_f(g)}{f^*(h) \xrightarrow{t'} g}$$

which gives $f^* \dashv \Pi_f$.

Exercise. How do you go from $t':f^*(h) \to g$ to $t:h \to \Pi_f(g)$? □

The full statement of the *Fundamental Theorem of Topoi* (Freyd [72], Theorem 2.31) is this:

For any topos $\mathscr{E}$, and $\mathscr{E}$-object b, the comma category $\mathscr{E}\downarrow b$ is a topos, and for any arrow $f: a \to b$ the pulling-back functor $f^*: \mathscr{E}\downarrow b \to \mathscr{E}\downarrow a$ has both a left adjoint Σ_f and a right adjoint Π_f.

The existence of Σ_f requires only pullbacks. The construction of Π_f is special to topoi, in that it uses partial arrow classifiers (N.B. local sections are partial functions).

Given $f: a \to b$, let k be the unique arrow for which

$$\begin{tikzcd}
a \arrow[r, tail, "\langle f, 1_a\rangle"] \arrow[d, "1_a"'] & b \times a \arrow[d, "k"] \\
a \arrow[r, tail, "\eta_a"] & \tilde{a}
\end{tikzcd}$$

is a pullback, where now η_a denotes the partial arrow classifier of §11.8 (why is $\langle f, 1_a\rangle$ monic?). Let $h: b \to \tilde{a}^a$ be the exponential adjoint to k. (In **Set** h takes $b \in B$ to the arrow corresponding to the partial function $f^{-1}\{b\} \hookrightarrow A$ from A to A).

Then, for any $g: c \to a$, define $\Pi_f(g)$ to be the pullback

$$\begin{tikzcd} \pi_f(c) \arrow[r] \arrow[d, "\pi_f(g)"'] & \tilde{c}^a \arrow[d, "\tilde{g}^a"] \\ b \arrow[r, "h"] & \tilde{a}^a \end{tikzcd}$$

where $\tilde{g}$ is the unique arrow making the pullback

$$\begin{tikzcd} c \arrow[r, tail, "\eta_c"] \arrow[d, "g"'] & \tilde{c} \arrow[d, "\tilde{g}"] \\ a \arrow[r, tail, "\eta_a"] & \tilde{a} \end{tikzcd}$$

and $\tilde{g}^a$ is the image of $\tilde{g}$ under the functor $(\ )^a: \mathscr{E} \to \mathscr{E}$.

It is left to the reader to show how this reflects the definition of Π_f in **Set**.

The Π_f functor is also used to verify that $\mathscr{E} \downarrow b$ has exponentials. Illustrating with **Set** once more, given objects $f: A \to B$ and $h: Y \to B$ in **Set** $\downarrow B$, their exponential is of the form $h^f: E \to B$. According to the description in Chapter 4, the stalk in E over b consists of all pairs $\langle b, t\rangle$ where $t: f^{-1}\{b\} \to Y$ makes

$$\begin{tikzcd} f^{-1}\{b\} \arrow[rr, "t"] \arrow[dr, "f"'] & & Y \arrow[dl, "h"] \\ & B & \end{tikzcd}$$

commute. Now if we form the pullback $f^*(h)$

$$\begin{tikzcd} & P \arrow[r, "g"] \arrow[d, "f^*(h)"] & Y \arrow[d, "h"] \\ f^{-1}\{b\} \arrow[ur, "t'"] \arrow[r, hook] & A \arrow[r, "f"] & B \end{tikzcd}$$

and define t' as shown by $t'(a) = \langle a, t(a)\rangle$, then recalling the description of P given earlier, t' is seen to be a section of $f^*(h)$ over $f^{-1}\{b\}$, i.e. a germ at b of the bundle $\Pi_f(f^*(h))$. Moreover t is recoverable as $g \circ t'$, giving an exact correspondence, and an isomorphism, between h^f and $\Pi_f(f^*(h))$ in **Set**.

In $\mathscr{E}\downarrow b$ then, given $f:a\to b$ and $h:c\to b$ we find that $\Pi_f(f^*(h))$ serves as the exponential h^f. We can alternatively express this in the language of adjointness, since the product functor

$$-\times f:\mathscr{E}\downarrow b\to\mathscr{E}\downarrow b$$

is the composite functor of

$$\mathscr{E}\downarrow b\xrightarrow{f^*}\mathscr{E}\downarrow a\xrightarrow{\Sigma_f}\mathscr{E}\downarrow b.$$

This is because the product of h and f, $h\times f$, in $\mathscr{E}\downarrow b$ is their pullback

$$\begin{tikzcd} p \arrow[r, "f^*(h)"] \arrow[d] & a \arrow[d, "f"] \\ c \arrow[r, "h"] & b \end{tikzcd}$$

$f\circ f^*(h)=\Sigma_f(f^*(h))$ in $\mathscr{E}$.

But each of f^* and Σ_f has a right adjoint, Π_f and f^* respectively, and their composite $\Pi_f\circ f^*$ provides a right adjoint to $-\times f$.

The details of the Fundamental Theorem may be found in Freyd [72] or Kock and Wraith [71].

15.4. Quantifiers

If $\mathfrak{A}=\langle A,\ldots\rangle$ is a first-order model, then a formula $\varphi(v_1, v_2)$ of index 2 determines the subset

$$X=\{\langle x, y\rangle:\mathfrak{A}\models\varphi[x, y]\}$$

of A^2. The formulae $\exists v_2\varphi$ and $\forall v_2\varphi$, being of index 1, determine in a corresponding fashion subsets of A. These can be defined in terms of X as

$$\exists_p(X)=\{x:\text{for some } y, \langle x, y\rangle\in X\}$$

$$\forall_p(X)=\{x:\text{for all } y, \langle x, y\rangle\in X\}.$$

The "p" refers to the first projection from A^2 to A, having $p(\langle x, y\rangle)=x$. $\exists_p(X)$ is in fact precisely the image $p(X)$ of X under p, and so we know that for any $X\subseteq A^2$ and $Y\subseteq A$,

$$X\subseteq p^{-1}(Y)\quad\textit{iff}\quad\exists_p(X)\subseteq Y,$$

i.e. $\exists_p : \mathscr{P}(A^2) \to \mathscr{P}(A)$ is left adjoint to the functor $p^{-1} : \mathscr{P}(A) \to \mathscr{P}(A^2)$ analysed in §15.2.

Since, for any $x \in A$, $p^{-1}\{x\} = \{\langle x, y\rangle : y \in A\}$ we see that

$$\forall_p(X) = \{x : p^{-1}\{x\} \subseteq X\} = p^+(X)$$

(cf. §15.2) and so we have

$$p^{-1}(Y) \subseteq X \quad \textit{iff} \quad Y \subseteq \forall_p(X)$$

and altogether $\exists_p \dashv p^{-1} \dashv \forall_p$.

In general then, for any $f : A \to B$, the left adjoint $\mathscr{P}(f)$ to $f^{-1} : \mathscr{P}(B) \to \mathscr{P}(A)$ will be renamed $\exists_f$, and the right adjoint f^+ will be denoted $\forall_f$. The link with the quantifiers is made explicit by the characterisations of $\exists_f(X) = f(X)$ and $\forall_f(X) = f^+(X)$ as

$$\exists_f(X) = \{y : \exists x (x \in X \text{ and } f(x) = y)\}$$

$$\forall_f(X) = \{y : \forall x (f(x) = y \text{ implies } x \in X)\}.$$

Moving now to a general topos $\mathscr{E}$, an arrow $f : a \to b$ induces a functor

$$f^{-1} : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$$

that takes a subobject of b to its pullback along f (pullbacks preserve monics).

A left adjoint $\exists_f : \mathrm{Sub}(a) \to \mathrm{Sub}(b)$ to f^{-1} is obtained by defining $\exists_f(g)$, for $g : c \rightarrowtail a$ to be the image arrow $\mathrm{im}(f \circ g)$ of $f \circ g$, so we have

$$\begin{tikzcd} c \arrow[rr, "f \circ g"] \arrow[dr] & & b \\ & f \circ g(c) \arrow[ur, tail, "\exists_f(g)"'] & \end{tikzcd}$$

Using the fact that the image of an arrow is the smallest subobject through which it factors (Theorem 5.2.1) the reader may attempt the

EXERCISE 1. Show that $g \subseteq h$ implies $\exists_f(g) \subseteq \exists_f(h)$, i.e. $\exists_f$ is a functor.

EXERCISE 2. Analyse the adjoint situation

$$\frac{g \to f^{-1}(h)}{\exists_f(g) \to h}$$

for $g : c \rightarrowtail a$ and $h : d \rightarrowtail b$, that gives $\exists_f \dashv f^{-1}$. □

The right adjoint $\forall_f : \mathrm{Sub}(a) \to \mathrm{Sub}(b)$ to f^{-1} is obtained from the functor $\Pi_f : \mathscr{E} \downarrow a \to \mathscr{E} \downarrow b$ (recall that in **Set**, f^+ is a special case of Π_f).

$\forall_f$ assigns to the subobject $g: c \rightarrowtail a$ the subobject $\Pi_f(g)$. Strictly speaking, g, as a *subobject*, is an equivalence class of arrows. Any ambiguity however is taken care of by

EXERCISE 3. If $g \subseteq h$ then $\forall_f(g) \subseteq \forall_f(h)$, and so

EXERCISE 4. If $g \simeq h$ then $\forall_f(g) \simeq \forall_f(h)$. □

The adjunction

$$\frac{h \to \forall_f(g)}{f^{-1}(h) \to g}$$

showing $f^{-1} \dashv \forall_f$, derives from the fact that $f^* \dashv \Pi_f$.

By selecting a particular monic to represent each subobject, we obtain a functor $i_a : \mathrm{Sub}(a) \to \mathscr{E}\downarrow a$. In the opposite direction, $\sigma_a : \mathscr{E}\downarrow a \to \mathrm{Sub}(a)$ takes $g: c \to a$ to $\sigma_a(g) = \mathrm{im}\, g : g(c) \rightarrowtail a$, and an $\mathscr{E}\downarrow a$ arrow

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \\ & a & \end{tikzcd}$$

to the inclusion $\sigma_a(k)$,

$$\begin{tikzcd} c \arrow[rr, "k"] \arrow[d] & & d \arrow[d] \\ g(c) \arrow[rr, "\sigma_a(k)"] \arrow[dr, tail] & & h(d) \arrow[dl, tail] \\ & a & \end{tikzcd}$$

which exists because $\mathrm{im}\, g$ is the smallest subobject through which g factors. For the same reason, given $g: c \to a$ and $h: d \to a$ we have that

$$\begin{tikzcd} & g(c) \arrow[dr, tail, "\mathrm{im}\, g"] \arrow[dd, dashed] & \\ c \arrow[ur, two heads] \arrow[dr, dashed] & & a \\ & d \arrow[ur, tail, "h"'] & \end{tikzcd}$$

$\mathrm{im}\, g$ factors through h, i.e. $\sigma_a(g) \subseteq h$, precisely when g factors through h, i.e. precisely when there is an arrow

$$\begin{tikzcd} c \arrow[rr] \arrow[dr, "g"'] & & d' \arrow[dl, "i_a(h)"] \\ & a & \end{tikzcd}$$

in $\mathscr{E}\downarrow a$. So we have the situation

$$\frac{g \to i_a(h)}{\sigma_a(g) \to h}$$

making σ_a left adjoint to i_a.

Putting the work of these last two sections together we have the "doctrinal diagram" of Kock and Wraith [71] for the arrow $f: a \to b$

$$\begin{tikzcd}
\mathscr{E}\downarrow a \arrow[rr, bend left, "\Sigma_f"] \arrow[rr, bend right, "\Pi_f"'] \arrow[d, shift right, "\sigma_a"'] & & \mathscr{E}\downarrow b \arrow[ll, "f^*"'] \arrow[d, shift right, "\sigma_b"'] \\
\mathrm{Sub}(a) \arrow[rr, bend left, "\exists_f"] \arrow[rr, bend right, "\forall_f"'] \arrow[u, shift right, "i_a"'] & & \mathrm{Sub}(b) \arrow[ll, "f^{-1}"'] \arrow[u, shift right, "i_b"']
\end{tikzcd}$$

with

$$\exists_f \dashv f^{-1} \dashv \forall_f$$
$$\Sigma_f \dashv f^* \dashv \Pi_f$$
$$\sigma \dashv i$$

EXERCISE 5. Show that

$$\exists_f \circ \sigma_a = \sigma_b \circ \Sigma_f$$
$$i_b \circ \forall_f = \Pi_f \circ i_a$$
$$i_a \circ f^{-1} = f^* \circ i_b$$
$$f^{-1} \circ \sigma_b = \sigma_a \circ f^* \qquad \square$$

An even more general analysis of quantifiers than this is possible. Given a relation $R \subseteq A \times B$ in **Set** we define quantifiers

$$\exists_R : \mathscr{P}(A) \to \mathscr{P}(B)$$
$$\forall_R : \mathscr{P}(A) \to \mathscr{P}(B)$$

"along R" by

$$\exists_R(X) = \{y : \exists x (x \in X \text{ and } xRy)\}$$
$$\forall_R(X) = \{y : \forall x (xRy \text{ implies } x \in X)\}$$

Given an arrow $r: R \rightarrowtail a \times b$ in a topos there are actual arrows

$$\forall_r : \Omega^a \to \Omega^b$$

$$\exists_r : \Omega^a \to \Omega^b$$

which correspond *internally* to $\exists_R$ and $\forall_R$ in **Set**. Constructions for these are given by Street [74] and they are further analysed by Brockway [76]. In particular, for a given $f: a \to b$, applying these constructions to the relation

$$\langle 1_a, f\rangle : a \rightarrowtail a \times b$$

(the "graph" of f) yields arrows of the form $\Omega^a \to \Omega^b$ which are internal counterparts to the functors $\forall_f$ and $\exists_f$.

Specialising further by taking f to be the arrow $!: a \to 1$, we obtain arrows $\Omega^a \to \Omega^1$, which under the isomorphism $\Omega^1 \cong \Omega$ become the quantifier arrows

$$\forall_a : \Omega^a \to \Omega \qquad \exists_a : \Omega^a \to \Omega$$

used for the semantics in a topos of Chapter 11.

The functors $\forall_f$ and $\exists_f$, in the case that f is a projection, are used in the topos semantics developed by the Montréal school. More information about their basic properties is given by Reyes [74].