CHAPTER 5

TOPOS STRUCTURE: FIRST STEPS

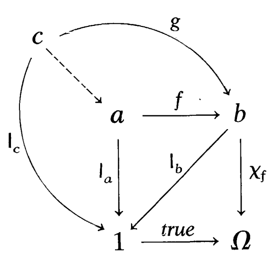
> *"The development of elementary topoi by Lawvere and Tierney strikes this writer as the most important event in the history of categorical algebra since its creation... It is not just that they proved these things, its that they dared to believe them provable."*
>
> Peter Freyd

5.1. Monics equalise

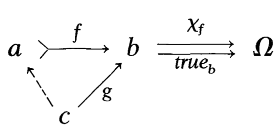
In §3.10 it was stated that an injective function $f: A \rightarrowtail B$ is an equaliser for a pair of functions g and h. We now see that g is $\chi_{\mathrm{Im} f}: B \to 2$ and h is the composite of $!: B \to \{0\}$ and $true: \{0\} \to \{0, 1\}$. This situation generalises directly:—

THEOREM 1: *If $f: a \rightarrowtail b$ is a monic $\mathscr{E}$-arrow ($\mathscr{E}$ any topos) then f is an equaliser of χ_f and $true_b = true \circ !_b$.*

PROOF: Since the pullback square of



commutes, and $!_a = !_b \circ f$, we have $\chi_f \circ f = true_b \circ f$. But if $\chi_f \circ g = true_b \circ g$



then the perimeter of the first diagram must commute, since $1_b \circ g = 1_c$. So, by the universal property of pullbacks, g factors uniquely through f as required. □

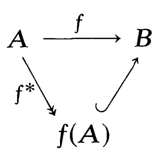
COROLLARY: *In any topos, an arrow is iso iff it is both epic and monic.*

PROOF. In any category, an iso is monic and epic (§3.3). On the other hand, in a topos an epic monic is, by the Theorem, an epic equaliser. Such a thing is always iso (§3.10). □

EXERCISE. $true: 1 \to \Omega$ equalises $\mathbf{1}_\Omega : \Omega \to \Omega$ and $true_\Omega : \Omega \to \Omega$. □

5.2. Images of arrows

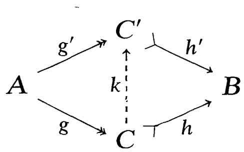
Any set function $f: A \to B$ can be factored into a surjection, followed by an injection. We have the commutative diagram



where $f(A) = \text{Im}\, f = \{f(x): x \in A\}$, and $f^*(x) = f(x)$, all $x \in A$.

This "epi-monic" factorisation of f is unique up to a *unique commuting* iso*morphism* as shown in the

EXERCISE 1. If $h \circ g: A \twoheadrightarrow C \rightarrowtail B$ and $h' \circ g': A \twoheadrightarrow C' \rightarrowtail B$ are any two epi-monic factorisations of f (i.e. $f = h \circ g = h' \circ g'$) then there is exactly one $k: C \to C'$ such that



commutes, and furthermore k is iso in **Set** (a bijection). □

The reader may care to develop a set-theoretic proof of this exercise and contrast it with the "arrows-only" approach to follow.

In all topoi, each arrow has an epi-monic factorisation. To see how this works, we turn first to a different description of factorisation in **Set**, one that has a categorial formulation. Given $f: A \to B$ we define, as in §3.13, the Kernel equivalence relation $R_f \subseteq A \times A$ by

$$xR_f y \quad \text{iff} \quad f(x) = f(y).$$

Now a map $h: A/R_f \to B$ is well-defined by $h([x]) = f(x)$. Moreover h is injective and

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle f_R}\searrow \quad \nearrow{\scriptstyle h} & \\ & A/R_f & \end{array}$$

commutes, where f_R is the surjective natural map $f_R(x) = [x]$.

Now as observed in §3.13, R_f as a set of ordered pairs yields a pullback

$$\begin{array}{ccc} R_f & \xrightarrow{q} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

where p and q, the projections, are the *kernel pair* of f. The considerations of §3.12 then show that f_R co-equalises the kernel pair (p, q) and that h is the unique arrow making

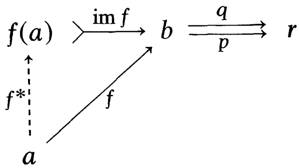
$$\begin{array}{ccccc} R_f & \overset{p}{\underset{q}{\rightrightarrows}} & A & \xrightarrow{f_R} & A/R_f \\ & & & {\scriptstyle f}\searrow & \vdots\,{\scriptstyle h} \\ & & & & B \end{array}$$

commute. This suggests that in a more general category we attempt to factor an arrow by co-equalising its pullback along itself. However, for technical reasons (the availability of the results of the last section) it is simpler now to dualise the construction, i.e. to equalise the pushout of the arrow with itself.

So, let $\mathscr{E}$ be any topos, and $f: a \to b$ any $\mathscr{E}$-arrow. We form the pushout

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ b & \xrightarrow{p} & r \end{array}$$

of f with f, and let $\operatorname{im} f : f(a) \rightarrowtail b$ be the equaliser of p and q ($\operatorname{im} f$ is monic by Theorem 3.10.1). Since $q \circ f = p \circ f$, there is a unique arrow $f^* : a \to f(a)$ making

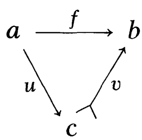


commute.

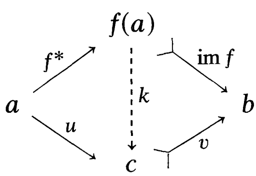
EXERCISE 2. Analyse this construction in concrete terms in **Set**.

EXERCISE 3. If $p = q$, then f is epic. □

THEOREM 1. $\operatorname{im} f$ *is the smallest subobject of* b *through which* f *factors.* That is, *if*

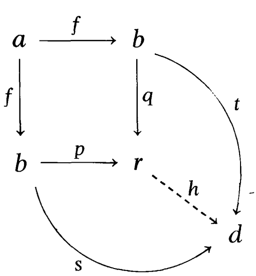


commutes, for any u *and monic* v *as shown, then there is a (unique)* $k : f(a) \to c$ *making*



commute, and hence $\operatorname{im} f \subseteq v$.

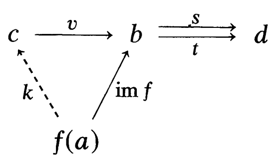
PROOF. Being monic, v equalises a pair $s, t : b \rightrightarrows d$ of $\mathscr{E}$-arrows (§5.1). Thus $s \circ f = s \circ v \circ u = t \circ v \circ u = t \circ f$, so



there is a unique $h: r \to d$ such that $h \circ p = s$ and $h \circ q = t$. But then

$$\begin{aligned} s \circ \operatorname{im} f &= h \circ p \circ \operatorname{im} f \\ &= h \circ q \circ \operatorname{im} f \\ &= t \circ \operatorname{im} f, \end{aligned}$$

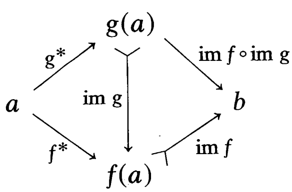
so, as v equalises s and t



we get a unique arrow k that has $v \circ k = \operatorname{im} f$. This k is the unique arrow making the right-hand triangle in the diagram in the statement of the theorem commute. But then $v \circ k \circ f^* = \operatorname{im} f \circ f^* = f = v \circ u$, and v is monic (left-cancellable), so $k \circ f^* = u$. Thus k makes the left-hand triangle commute as well. □

COROLLARY. *$f^*: a \to f(a)$ is epic.*

PROOF. Apply the image construction to f^* itself, giving the commuting diagram



where $g = f^*$.

But $\operatorname{im} f \circ \operatorname{im} g$ is monic, being a product of monics, and so, as $\operatorname{im} f$ is left cancellable, we must have $\operatorname{im} g$ as the unique arrow making $\operatorname{im} f \circ \operatorname{im} g \subseteq \operatorname{im} f$. But also, applying the Theorem to $\operatorname{im} f$ we must have $\operatorname{im} f \subseteq \operatorname{im} f \circ \operatorname{im} g$, and so $\operatorname{im} f \simeq \operatorname{im} f \circ \operatorname{im} g$ in $\mathrm{Sub}(b)$, hence $g(a) \cong f(a)$. Thus the unique arrow $\operatorname{im} g$ must be iso.

But $\operatorname{im} g$ is, by definition, the equaliser

$$g(a) \xrightarrow{\operatorname{im} g} f(a) \underset{q}{\overset{p}{\rightrightarrows}} r,$$

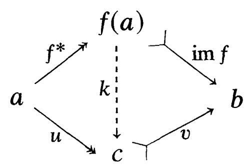
where p and q, the *cokernel pair* of $g = f^*$, form a pushout thus:

$$\begin{array}{ccc} a & \xrightarrow{f^*} & f(a) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle q} \\ f(a) & \xrightarrow{p} & r \end{array}$$

Since $p \circ \mathrm{im}\, g = q \circ \mathrm{im}\, g$, and im g is iso, hence epic, we cancel to get $p = q$. The co-universal property of pushouts then yields f^* as epic (as in Exercise 3, above). □

Bringing the work of this section together we have

THEOREM 2. $\mathrm{im}\, f \circ f^* : a \twoheadrightarrow f(a) \rightarrowtail b$ *is an epi-monic factorisation of* f *that is unique up to a unique commuting isomorphism. That is, if* $v \circ u : a \twoheadrightarrow c \rightarrowtail b$ *has* $v \circ u = f$, *then there is exactly one arrow* $k : f(a) \to c$ *such that*



commutes, and k *is iso.*

PROOF. The unique k exists by Theorem 1. But then $v \circ k = \mathrm{im}\, f$ is monic, so k is monic by Exercise 2, §3.1. Also $k \circ f^* = u$ is epic, so dually k is epic. Hence k, being epic and monic, is iso. (§5.1). □

EXERCISE 4. $f : a \to b$ is epic iff there exists $g : f(a) \cong b$ such that $g \circ f^* = f$. □

5.3. Fundamental facts

If $\mathscr{E}$ is a topos then the comma category $\mathscr{E} \downarrow a$ of objects over a is also a topos. As mentioned in Chapter 4, this is (part of) a result known as the Fundamental Theorem of Topoi. The proof of this theorem involves a construction too advanced for our present stage of development, but yielding some important information that we shall need now. We therefore record these consequences of the Fundamental Theorem without proof:

Fact 1. *Pullbacks preserve epics. If*

$$\begin{array}{ccc} a & \longrightarrow & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ c & \longrightarrow & d \end{array}$$

is a pullback square in a topos, and f is epic, then g, the pullback of f, is also epic.

Fact 2. *Coproducts preserve pullbacks. If*

$$\begin{array}{ccc} a & \xrightarrow{f} & d \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ b & \xrightarrow{h} & e \end{array} \quad \textit{and} \quad \begin{array}{ccc} a' & \xrightarrow{f'} & d \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle k} \\ b' & \xrightarrow{h'} & e \end{array}$$

are pullbacks in a topos, then so is

$$\begin{array}{ccc} a+a' & \xrightarrow{[f,f']} & d \\ {\scriptstyle g+g'}\downarrow & & \downarrow{\scriptstyle k} \\ b+b' & \xrightarrow{[h,h']} & e \end{array}$$

Proofs of these results may be found in Kock and Wraith [71], Freyd [72], and Brook [74].

5.4. Extensionality and bivalence

Since a general topos $\mathscr{E}$ is supposed to be "**Set**-like", its initial object 0 ought to behave like the null set $\emptyset$, and have no elements. This in fact obtains, except in one case. If there is an arrow $x:1 \to 0$, then by the work in §3.16 on Cartesian closed categories, $\mathscr{E}$ is degenerate, i.e. all $\mathscr{E}$-objects are isomorphic. This happens for example in the category **1** with one object and one arrow – **1** is a degenerate topos. So in a non-degenerate topos, 0 has no elements.

Now if we call an object a *non-zero* if it is not isomorphic to 0, $a \not\cong 0$, and *non-empty* if there is at least one $\mathscr{E}$-arrow $1 \to a$, then when $\mathscr{E} = \mathbf{Set}$, "non-zero" and "non-empty" are co-extensive. But when $\mathscr{E} = \mathbf{Set}^2$, the topos of pairs of sets, the situation is different. The object $\langle \emptyset, \{0\} \rangle$ is not isomorphic to the initial object $\langle \emptyset, \emptyset \rangle$, hence is *non-zero*. But an element $\langle f, g \rangle : \langle \{0\}, \{0\} \rangle \to \langle \emptyset, \{0\} \rangle$ of $\langle \emptyset, \{0\} \rangle$ would require f to be a set function $\{0\} \to \emptyset$, of which there is no such thing. Thus $\langle \emptyset, \{0\} \rangle$ is non-zero but empty.

EXERCISE 1. Are there any other non-zero empty objects in **Set**^{2}? What about non-empty zero objects?

EXERCISE 2. Are there non-zero empty objects in **Set**$^{\to}$? In **Bn**(I)? □

The question of the existence of elements of objects relates to the notion of *extensionality*, the principle that sets with the same elements are identical. For functions, this principle takes the following form (which we have used repeatedly): two parallel functions $f, g : A \rightrightarrows B$ are equal if they give the same output for the same input, i.e. if for each $x \in A$, $f(x) = g(x)$. Categorially this takes the form of the:

EXTENSIONALITY PRINCIPLE FOR ARROWS. *If $f, g : a \rightrightarrows b$ are a pair of* distinct *parallel arrows, then there is an element $x : 1 \to a$ of a such that $f \circ x \neq g \circ x$.*

(Category-theorist will recognise this as the statement "1 is a generator".) This principle holds in **Set**, but not in **Set**^{2}. It is easy to see that in the latter there are two distinct arrows from $\langle \emptyset, \{0\} \rangle$ to $\langle \emptyset, 2 \rangle$. But $\langle \emptyset, \{0\} \rangle$ has no elements at all to distinguish them.

A non-degenerate topos that satisfies the extensionality principle for arrows is called *well-pointed*. The purpose of this section is to examine the properties of such categories.

THEOREM 1. *If $\mathscr{E}$ is well-pointed, then every non-zero $\mathscr{E}$-object is non-empty.*

PROOF. If a is non-zero then $0_a : 0 \rightarrowtail a$ and $1_a : a \rightarrowtail a$ have different domains, and so are distinct. Hence $\chi_{0_a} : a \to \Omega$ and $\chi_{1_a} : a \to \Omega$ are distinct (otherwise $0_a \simeq 1_a$, hence $0 \cong a$). By extensionality it follows that there is some $x : 1 \to a$ such that $\chi_{0_a} \circ x \neq \chi_{1_a} \circ x$. In particular a has an element, so is non-empty. □

False

In **Set** there are exactly two arrows from $1 = \{0\}$ to $\Omega = \{0, 1\}$. One of course is the map *true*, with $true(0) = 1$. The other we call *false*, and is defined by $false(0) = 0$. This map, having codomain Ω is the characteristic function of

$$\{x : false(x) = 1\} = \emptyset, \quad \text{the null set,}$$

so in **Set** we have a pullback

$$\begin{array}{ccc} \emptyset & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle false} \\ 1 & \xrightarrow{\;true\;} & \Omega \end{array}$$

Abstracting this, we define in any topos $\mathscr{E}$, $false : 1 \to \Omega$ to be the unique $\mathscr{E}$-arrow such that

$$\begin{array}{ccc} 0 & \xrightarrow{\;0_1\;} & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle false} \\ 1 & \xrightarrow{\;\top\;} & \Omega \end{array}$$

is a pullback in $\mathscr{E}$. Thus $false = \chi_{0_1}$. We will also use the symbol "$\perp$" for this arrow.

Example 1. In **Set**^{2}, $\perp : 1 \to \Omega$ is $\langle false, false \rangle : \langle \{0\}, \{0\} \rangle \to \langle 2, 2 \rangle$.

Example 2. In **Bn**(I), $\perp : 1 \to \Omega$ *is* $\perp : I \to 2 \times I$ where $\perp(\mathrm{i}) = \langle 0, i \rangle$, all $i \in I$.

Example 3. In **Top**(I), $\perp : I \to \hat{I}$ has $\perp(i) = \langle i, [\emptyset]_i \rangle$, the germ of $\emptyset$ at i.

Example 4. In **M-Set**, $0 = (\emptyset, \emptyset)$, with $\emptyset : \mathrm{M} \times \emptyset \to \emptyset$, *the "empty action"*. $\perp : \{0\} \to L_{\mathrm{M}}$ has $\perp(0) = \{m : \lambda_0(m, 0) \in \emptyset\} = \emptyset$. □

Exercise 3. For any $\mathscr{E}$-object a,

$$\begin{array}{ccc} 0 & \xrightarrow{\;0_a\;} & a \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \perp \circ !_a} \\ 1 & \xrightarrow{\;\top\;} & \Omega \end{array}$$

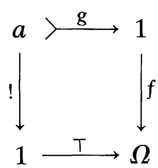
is a pullback, i.e. $\chi_{0_a} = \perp \circ !_a (= \perp_a = false_a)$.
(Hint: you may need the PBL)

EXERCISE 4. In a non-degenerate topos, *true* $\neq$ *false*. □

A non-degenerate topos $\mathscr{E}$ is called *bivalent* (two-valued) if *true* and *false* are its only truth-values (elements of Ω).

THEOREM 2. *If $\mathscr{E}$ is well-pointed, then $\mathscr{E}$ is bivalent.*

PROOF. Let $f: 1 \to \Omega$ be any element of Ω and form the pullback



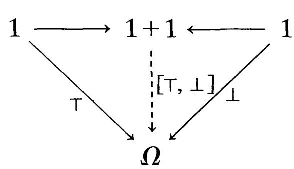
of f and $\top$.

Case 1: If $a \cong 0$, then a is an initial object, with $g \simeq 0_1$. Then $f = \chi_g = \chi_{0_1} = \mathit{false}$.

Case 2: If not $a \cong 0$, then as $\mathscr{E}$ is well-pointed, a has an element $x: 1 \to a$ (Theorem 1). We use this to show that g is epic. For, if $h, k: 1 \rightrightarrows b$ have $h \circ g = k \circ g$, then $h \circ g \circ x = k \circ g \circ x$. But $g \circ x: 1 \to 1$ can only be $\mathbf{1}_1$ (1 is terminal) so $h = k$. Thus g is right cancellable. Hence g is both epic and monic (being the pullback of a monic), giving $g: a \cong 1$. So a is terminal, yielding $g \simeq \mathbf{1}_1$, hence $f = \chi_g = \chi_{1_1} = \mathit{true}$.

Altogether then we have shown that an element of Ω must be either *true* or *false*. □

Now in **Set**, the co-product $1+1$ is a two-element set and hence isomorphic to $\Omega = 2$ (this was observed in §3.9). In fact the isomorphism is given by the co-product arrow $[\top, \bot]: 1+1 \to \Omega$



But any topos $\mathscr{E}$ has co-products, and so the arrow $[\top, \bot]$ is certainly *defined*. If $[\top, \bot]$ is an *iso* $\mathscr{E}$-arrow we will say that $\mathscr{E}$ is a *classical* topos. Shortly we shall see that there are non-classical topoi. However we do have

THEOREM 3. *In any topos,* $[\top, \bot]$ *is monic.*

To prove this we need to do some preliminary work with co-product arrows. If $f: a \to b$ and $g: c \to b$ are $\mathscr{E}$-arrows, we say that f and g are *disjoint* if their pullback is 0, i.e. if

$$\begin{array}{ccc} 0 & \xrightarrow{\;!\;} & c \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle g} \\ a & \xrightarrow{\;f\;} & b \end{array}$$

is a pullback square in $\mathscr{E}$. (In **Set** this means precisely that $\operatorname{Im} f \cap \operatorname{Im} g = \emptyset$.)

Lemma. *If* $f: a \rightarrowtail b$ and $g: c \rightarrowtail b$ *are disjoint* monics *in* $\mathscr{E}$, *then* $[f, g]: a + c \to b$ *is monic.*

Proof. g being monic means

$$\begin{array}{ccc} c & \xrightarrow{\;1_c\;} & c \\ {\scriptstyle 1_c}\downarrow & & \downarrow{\scriptstyle g} \\ c & \xrightarrow[\;g\;]{} & b \end{array}$$

is a pullback. This, with the previous diagram, and Fact 2 of §5.3, gives the pullback

$$\begin{array}{ccc} 0+c & \xrightarrow{\;[0_c,\,1_c]\;} & c \\ {\scriptstyle 0_a+1_c}\downarrow & & \downarrow{\scriptstyle g} \\ a+c & \xrightarrow[\;[f,\,g]\;]{} & b \end{array}$$

Now $[0_c, 1_c]: 0 + c \cong c$ (dual of Exercise 3.8.4), from which it can be shown that

$$\begin{array}{ccc} c & \xrightarrow{\;1_c\;} & c \\ {\scriptstyle i_c}\downarrow & & \downarrow{\scriptstyle g} \\ a+c & \xrightarrow[\;[f,\,g]\;]{} & b \end{array}$$

is a pullback (i_c being the injection associated with $a+c$).

Analogously we get

$$\begin{array}{ccc} a & \xrightarrow{\;1_a\;} & a \\ {\scriptstyle i_a}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ a+c & \xrightarrow[{[f,\,g]}]{} & b \end{array}$$

as a pullback. These last two diagrams (suitably rotated and reflected), with Fact 2 again, give

$$\begin{array}{ccc} a+c & \xrightarrow{\;[i_a,\, i_c]\;} & a+c \\ {\scriptstyle 1_a+1_c}\big\downarrow & & \big\downarrow{\scriptstyle [f,\,g]} \\ a+c & \xrightarrow[{[f,\,g]}]{} & b \end{array}$$

as a pullback. But $[i_a, i_c] = 1_{a+c} = 1_a + 1_c$ (dual of Exercises 1, 4, §3.8), and from this it follows that $[f, g]$ is monic (cf. Example 9, §3.13). □

Now, for the proof of Theorem 3 we observed that

$$\begin{array}{ccc} 0 & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle \bot} \\ 1 & \xrightarrow{\;\top\;} & \Omega \end{array}$$

is a pullback, indeed this diagram gives the definition of $\bot$. Thus $\top$ and $\bot$ are *disjoint* monics, and so by the Lemma, $[\top, \bot]: 1+1 \to \Omega$ is monic.

□

THEOREM 4. *If $\mathscr{E}$ is well-pointed, then* $[\top, \bot]$: $1+1 \cong \Omega$, i.e. $\mathscr{E}$ *is classical.*

PROOF. In view of Theorem 3, we need only establish that $[\top, \bot]$ is epic, when $\mathscr{E}$ is well-pointed. So, suppose $f \circ [\top, \bot] = g \circ [\top, \bot]$.

$$\begin{array}{ccccc} 1 & \xrightarrow{\;i\;} & 1+1 & \xleftarrow{\;j\;} & 1 \\ & {\scriptstyle \top}\searrow & \big\downarrow{\scriptstyle [\top,\,\bot]} & \swarrow{\scriptstyle \bot} & \\ & & \Omega & & \\ & & {\scriptstyle f}\big\downarrow\big\downarrow{\scriptstyle g} & & \\ & & a & & \end{array}$$

Then

$$\begin{aligned} f\circ\top &= f\circ[\top,\bot]\circ i \\ &= g\circ[\top,\bot]\circ i \\ &= g\circ\top \end{aligned}$$

and similarly, (using j), $f\circ\bot = g\circ\bot$. Since $\top$ and $\bot$ are the only elements of Ω (Theorem 2), and neither of them distinguish f and g, the extensionality principle for arrows implies that $f=g$. Thus $[\top,\bot]$ *is right-cancellable.* □

The major link between the concepts of this section is:

THEOREM 5. *A topos $\mathscr{E}$ is well-pointed iff it is classical and every non-zero $\mathscr{E}$-object is non-empty in $\mathscr{E}$.*

The "only if" part of this theorem is given by Theorems 4 and 1. The proof of the "if" part requires some notions to be introduced in subsequent chapters, and will be held in abeyance until §7.6.

The category **Set**^{2} is classical, but not bivalent (it has four truth-values – what are they?) The category **Set**$^{\rightarrow}$ of functions on the other hand is neither bivalent (having three truth-values) nor classical (cf. Chapter 10). To construct an example of a non-classical but bivalent topos we use the following interesting fact:

THEOREM 6. *If* **M** *is a monoid, then the category* **M-Set** *is classical iff* **M** *is a group.*

PROOF. In **M-Set**, $1=(\{0\},\lambda_0)$ is the one-element **M**-set. $1+1$ can be described as the disjoint union of 1 with itself, i.e. two copies of 1 acting independently. To be specific we put $1+1=(\{0,1\},\gamma)$, where $\gamma(m,0)=0$ and $\gamma(m,1)=1$, all $m\in M$. We then have the co-product diagram

$$\begin{array}{ccccc} 1 & \xrightarrow{\;i\;} & 1+1 & \xleftarrow{\;j\;} & 1 \\ & {\scriptstyle\top}\searrow & \downarrow{\scriptstyle[\top,\bot]} & \swarrow{\scriptstyle\bot} & \\ & & \Omega & & \end{array}$$

where the injections are $i(0)=0$ and $j(0)=1$, with $[\top,\bot]$ mapping 0 to M and 1 to $\emptyset$ in $\Omega=(L_M,\omega)$. Now if $[\top,\bot]$ is iso, it is a bijection of sets, and so L_M has only two elements. Hence $L_M=\{M,\emptyset\}$. Conversely if $L_M=\{M,\emptyset\}$ then as $\omega(m,M)=M$ and $\omega(m,\emptyset)=\emptyset$, $[\top,\bot]$ is an equivariant bijection, i.e. an iso arrow in **M-Set**. Thus **M-Set** is classical iff $L_M = \{M, \emptyset\}$. But this last condition holds precisely when **M** is a group, (Exercise 4.6.3). □

So to construct a non-classical topos we need only select a monoid that is not a group. The natural thing to do is pick the smallest one. This is a two element algebra which can be described simply as consisting of the numbers 0 and 1 under multiplication. Formally it is the structure $\mathbf{M}_2 = (2, \cdot, 1)$ where $2 = \{0, 1\}$ and $\cdot$ is defined by

$$1 \cdot 1 = 1, \qquad 1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0,$$

or in a table

·	1	0
1	1	0
0	0	0

$\mathbf{M}_2$ is a monoid with identity 1, in which 0 has no inverse. The category of $\mathbf{M}_2$-sets is a kind of "universal counterexample" that will prove extremely useful for illustrative purposes. We will call it simply "the topos $\mathbf{M}_2$".

The set L_2 of left ideals of $\mathbf{M}_2$ has three elements, 2, $\emptyset$, and $\{0\}$ (why is $\{1\}$ not a left ideal?). Thus in $\mathbf{M}_2$, $\Omega = (L_2, \omega)$, where the action

$$\omega : 2 \times L_2 \to L_2,$$

defined by

$$\omega(m, B) = \{n : n \in 2 \text{ and } n \cdot m \in B\},$$

can be presented by the table

ω	2	$\{0\}$	$\emptyset$
1	2	$\{0\}$	$\emptyset$
0	2	2	$\emptyset$

Now the map $[\top, \bot]$ as considered in Theorem 6 is not iso. To show explicitly that it is not epic, consider $f_\Omega : L_2 \to L_2$ defined by

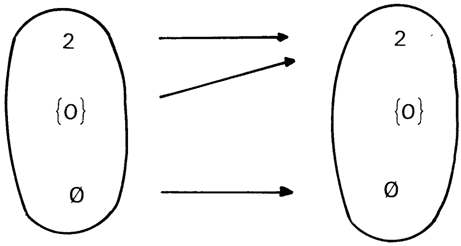


Fig. 5.1.

$$f_\Omega(2) = f_\Omega(\{0\}) = 2$$

$$f_\Omega(\emptyset) = \emptyset$$

By the table for ω, f_Ω is equivariant, so is an arrow $f_\Omega : \Omega \to \Omega$ in $\mathbf{M_2}$. But $f_\Omega \circ [\top, \bot] = 1_\Omega \circ [\top, \bot]$, while $f_\Omega \neq 1_\Omega$, hence $[\top, \bot]$ is not right-cancellable. Though $\mathbf{M_2}$ is non-classical, it is bivalent. For if $h : 1 \to \Omega$ is an $\mathbf{M_2}$-arrow, then $h : \{0\} \to L_2$ is an equivariant map, so $\omega(0, h(0)) = h(\lambda_0(0, 0)) = h(0)$. Since $\omega(0, \{0\}) = 2 \neq \{0\}$, we cannot have $h(0) = \{0\}$. Thus either $h(0) = 2$, whence $h = \top$, or $h(0) = \emptyset$, whence $h = \bot$. So $\mathbf{M_2}$ has only two truth-values.

By Theorem 4, $\mathbf{M_2}$ is not well-pointed. To see this explicitly, observe that $f_\Omega \neq 1_\Omega$ (f_Ω as above), but $f_\Omega \circ \top = 1_\Omega \circ \top$ (both output 2) while $f_\Omega \circ \bot = 1_\Omega \circ \bot$ (both output $\emptyset$). Thus no element of Ω distinguishes the distinct arrows $f_\Omega, 1_\Omega : \Omega \rightrightarrows \Omega$.

Exercise 5. Show that if $a = (X, \lambda)$ is an object in **M-Set** (**M** any monoid) then an element $x : 1 \to a$ of a in **M-Set** can be identified with a *fixed point* of a, i.e. an element $y \in X$ such that $\lambda(m, y) = y$, all $m \in M$. □

In the light of this exercise we can show that the converse of Theorem 1 above is false. If $a = (X, \lambda)$ is a non-zero object in $\mathbf{M_2}$, then $X \neq \emptyset$. Take some $x \in X$, and put $y = \lambda(0, x)$. Then y is a fixed point of a, since $\lambda(m, y) = \lambda(m \cdot 0, x) = \lambda(0, x) = y$. In this way we see that every non-zero object in $\mathbf{M_2}$ is non-empty, even though $\mathbf{M_2}$ is not well-pointed.

5.5. Monics and epics by elements

Using our notion of elements as arrows of the form $1 \to a$ we can give categorial definitions of "injective" and "surjective".

A $\mathscr{C}$-arrow $f : a \to b$, where $\mathscr{C}$ is a category with 1, is *surjective* if for each $y : 1 \to b$ there is some $x : 1 \to a$ with $f \circ x = y$. f is *injective* if whenever $x, y : 1 \rightrightarrows a$ have $f \circ x = f \circ y$, then $x = y$.

THEOREM 1. *If $\mathscr{E}$ is a well-pointed topos then an $\mathscr{E}$-arrow $f : a \to b$ is*
(i) *surjective iff epic*
(ii) *injective iff monic.*

PROOF. (i) Suppose f surjective. Let $g,h : b \rightrightarrows c$ be such that $g \circ f = h \circ f$. If $g \neq h$ then there is some $y : 1 \to b$ such that $g \circ y \neq h \circ y$. But as f is surjective, $y = f \circ x$ for some $x : 1 \to a$. Then $g \circ y = g \circ f \circ x = h \circ f \circ x = h \circ y$, a contradiction. So we must conclude that $g = h$, and that f cancels on the right.

Conversely assume f epic. Given $y : 1 \to b$, form the pullback

$$\begin{array}{ccc} c & \xrightarrow{p} & 1 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle y} \\ a & \xrightarrow{f} & b \end{array}$$

Now p is epic, by Fact 1 of §5.3, so if $c \cong 0$, then p would be monic (Theorem 3.16.1), hence iso, making $0 \cong 1$ and $\mathscr{E}$ degenerate. So c must be non-zero, ergo (Theorem 1) there exists $z : 1 \to c$. Then putting $x = q \circ z$ we get $x : 1 \to a$ and $f \circ x = y$ (details?).

EXERCISE 1. Prove Part (ii) of the Theorem.

EXERCISE 2. Show that in $\mathbf{M_2}$, f_Ω is surjective, although not epic, and similarly for $[\top, \bot]$.

EXERCISE 3. Show that f_Ω is not monic, but is injective. □

We will return to the subject of well-pointed topoi and extensionality in Chapters 7 and 12.