PROSPECTUS

> *"... all sciences including the most evolved are characterised by a state of perpetual becoming."*
>
> Jean Piaget

The purpose of this book is to introduce the reader to the notion of a *topos*, and to explain what its implications are for logic and the foundations of mathematics.

The study of topoi arises within *category theory*, itself a relatively new branch of mathematical enquiry. One of the primary perspectives offered by category theory is that the concept of *arrow*, abstracted from that of *function* or *mapping*, may be used instead of the set membership relation as the basic building block for developing mathematical constructions, and expressing properties of mathematical entities. Instead of defining properties of a collection by reference to its members, i.e. *internal* structure, one can proceed by reference to its *external* relationships with other collections. The links between collections are provided by functions, and the axioms for a category derive from the properties of functions under composition.

A category may be thought of in the first instance as a universe for a particular kind of mathematical discourse. Such a universe is determined by specifying a certain kind of "object", and a certain kind of "arrow" that links different objects. Thus the study of topology takes place in a universe of discourse (category) with topological spaces as the objects and continuous functions as the arrows. Linear algebra is set in the category whose arrows are linear transformations between vector spaces (the objects); group theory in the category whose arrows are group homomorphisms; differential topology where the arrows are smooth maps of manifolds, and so on.

We may thus regard the broad mathematical spectrum as being blocked out into a number of 'subject matters' or categories (a useful way of lending coherence and unity to an ever proliferating and diversifying discipline). Category theory provides the language for dealing with these domains and for developing methods of passing from one to the other. The subject was initiated in the early 1940's by Samuel Eilenberg and Saunders Maclane. Its origins lie in algebraic topology, where constructions are developed that connect the domain of topology with that of algebra, specifically group theory. The study of categories has rapidly become however an abstract discipline in its own right and now constitutes a substantial branch of pure mathematics. But further than this it has had a considerable impact on the conceptual basis of mathematics and the language of mathematical practice. It provides an elegant and powerful means of expressing relationships across wide areas of mathematics, and a range of tools that seem to be becoming more and more a part of the mathematician's stock in trade. New light is shed on existing theories by recasting them in arrow-theoretic terms (witness the recent unification of computation and control theories described in Manes [75]). Moreover category theory has succeeded in identifying and explicating a number of extremely fundamental and powerful mathematical ideas (universal property, adjointness). And now after a mere thirty years it offers a new theoretical framework for mathematics itself!

The most general universe of current mathematical discourse is the category known at **Set**, whose objects are the sets and whose arrows are the set functions. Here the fundamental mathematical concepts (number, function, relation) are given formal descriptions, and the specification of axioms legislating about the properties of sets leads to a so called *foundation* of mathematics. The basic set-theoretic operations and attributes (empty set, intersection, product set, surjective function e.g.) can be described by reference to the arrows in **Set**, and these descriptions interpreted in any category. However the category axioms are "weak", in the sense that they hold in contexts that differ wildly from the initial examples cited above. In such contexts the interpretations of set-theoretic notions can behave quite differently to their counterparts in **Set**. So the question arises as to when this situation is avoided, i.e. when does a category look and behave like **Set**? A vague answer is – when it is (at least) a topos. This then gives our first indication of what a topos is. It is a category whose structure is sufficiently like **Set** that in it the interpretations of basic set-theoretical constructions behave much as they do in **Set** itself.

The word *topos* ("place", or "site" in Greek) was originally used by Alexander Grothendieck in the context of algebraic geometry. Here there is a notion called a "sheaf" over a topological space. The collection of sheaves over a topological space form a category. Grothendieck and his colleagues extended this construction by replacing the topological space by a more general categorial structure. The resulting generalised notion of category of sheaves was given the name "topos" (cf. Artin et al. [SGA4]).

Independently of this, F. William Lawvere tackled the question as to what conditions a category must satisfy in order for it to be "essentially the same" as **Set**. His first answer was published in 1964. A shortcoming of this work was that one of the conditions was set-theoretic in nature. Since the aim was to categorially axiomatise set theory, i.e. to produce set-theory out of category theory, the result was not satisfactory, in that it made use of set-theory from the outset.

In 1969 Lawvere, in conjunction with Myles Tierney, began the study of categories having a special kind of arrow, called a "subobject classifier" (briefly, this is an embodiment of the correspondence between subsets and characteristic functions in **Set**). This notion proved to be, in Lawvere's words, the "principle struggle" – the key to the earlier problem. He discovered that the Grothendieck topoi all had subobject classifiers, and so took over the name. The outcome is the abstract axiomatic concept of an *elementary topos*, formulated entirely in the basic language of categories and independently of set theory. Subsequently William Mitchell [72] and Julian Cole [73] produced a full and detailed answer to the above question by identifying the elementary topoi that are equivalent to **Set**.

As mentioned earlier set theory provides a general conceptual framework for mathematics. Now, since category theory, through the notion of topos, has succeeded in axiomatising set-theory, the outcome is an entirely new *categorial foundation of mathematics*! The category-theorists attitude that "function" rather than "set membership" can be seen as the fundamental mathematical concept has been entirely vindicated. The pre-eminent role of set theory in contemporary mathematics is suddenly challenged. A revolution has occurred in the history of mathematical ideas (albeit a peaceful one) that will undoubtedly influence the direction of the path to the future.

The notion of topos has great unifying power. It encompasses **Set** as well as the Grothendieck categories of sheaves, and so brings together the domains of set theory and algebraic geometry. But it also has ramifications for another area of rational inquiry, namely *logic*, the study of the canons of deductive reasoning. The principles of classical logic are represented in **Set** by operations on a certain set – the two element Boolean algebra. Each topos has an analogue of this algebra and so one can say that each topos carries its own logical calculus. It turns out that this calculus may differ from classical logic, and in general the logical princples that hold in a topos are those of *intuitionistic* logic. Now Intuitionism is a constructivist philosophy about the nature of mathematical entities and the meaning and validity of mathematical statements. It has nothing to do, per se, with logic in a topos, since the latter arises from a reformulation in categorial language of the set-theoretical account of classical logic. And yet we have this remarkable discovery that the two enterprises lead to the same logical structure. An inkling of how this can be comes on reflection that there is a well-known link between intuitionistic logic and topology, and that sheaves are initially topological entities. Furthermore the set-theoretical modelling of intuitionistic logic due to Saul Kripke [65] can be used to construct topoi in which the logic, as generalised from **Set**, turns out to be a reformulation of Kripke's semantic theory. Moreover these topoi of Kripke models can be construed as categories of sheaves.

These developments have yielded significant insights and new perspectives concerning the nature of sets and the connection between intuitionistic and classical logic. For example, one property enjoyed by the arrows in **Set** is *extensionality*; a function is uniquely determined by the values it gives to its arguments. Now the individuals of a topos may be thought of as 'generalised' sets and functions that may well be non-extensional. Interestingly, the imposition of extensionality proves to be one way of ensuring that the topos logic is classical. Another way, equally revealing, is to invoke (in arrow language) the axiom of choice.

Our aim then is to present the details of the story just sketched. The currently available literature on topoi takes the form of graduate level lecture notes, research papers and theses, wherein the mathematical sophisticate will find his needs adequately served. The present work on the other hand is an attempt at a fully introductory exposition, aimed at a wide audience. The author shares the view that the emergence of topos theory is an event of supreme importance, that has major implications for the advancement of conceptual understanding as well as technical knowledge in mathematics. It should therefore be made available to the philosopher–logician as well as the mathematician. Hence there are very few prerequisites for this book. Everything – set theory, logic, and category theory – begins at square one. Although some material may be very familiar, it should be remembered that one of our main themes is the development of new perspectives for familiar concepts. Hence it would seem quite appropriate that these concepts be re-appraised and that explicit discussion be provided of things that to many will have become second nature.

There are a number of proofs of theorems whose length and detail may be discouraging. A similar comment applies to the *verification* of the structural properties of some of the more complex categories (sheaves, Kripke models). The reader is recommended to skip over all of this detail initially and concentrate on the flow of ideas. It can often happen that although the verifications are long and tedious, the facts and ideas are themselves clear and readily comprehensible. Hopefully by steering a judiciously chosen course through elementary expositions that will bore the cognoscente, abstruse constructions that will tax the novice, and detailed justifications that will exhaust anyone, the reader will emerge with some insight into the "what" and "why" of this fascinating new area of logical-mathematical–philosophical study.