

## ON MULTICOMPONENT MKDV EQUATIONS ON SYMMETRIC SPACES OF DIII-TYPE AND THEIR REDUCTIONS

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**Abstract.** New reductions for the multicomponent modified Korteweg de Vries (MMKdV) equations on the symmetric spaces of **DIII**-type are derived using the approach based on the reduction group introduced by A. Mikhailov. The relevant inverse scattering problem is studied and reduced to a Riemann-Hilbert problem. The minimal sets of scattering data  $\mathcal{T}_i$ ,  $i = 1, 2$  which allow one to reconstruct uniquely both the scattering matrix and the potential of the Lax operator are defined. The effect of the new reductions on the hierarchy of Hamiltonian structures of MMKdV and on  $\mathcal{T}_i$  are studied. We illustrate our results by the MMKdV equations related to the algebra  $\mathfrak{g} \simeq \mathfrak{so}(8)$  and derive several new MMKdV-type equations using group of reductions isomorphic to  $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ .

### 1. Introduction

The **modified Korteweg de-Vries equation** (MKdV) [27]

$$q_t + q_{xxx} + 6\epsilon q_x q^2(x, t) = 0, \quad \epsilon = \pm 1 \quad (1)$$

has natural multicomponent generalizations (MMKdV) related to the symmetric spaces [3]. They can be integrated by the ISM using the fact that they allow the following Lax representation

$$L\psi \equiv \left( i \frac{d}{dx} + Q(x, t) - \lambda J \right) \psi(x, t, \lambda) = 0 \quad (2)$$

$$Q(x, t) = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3)$$

$$\begin{aligned} M\psi &\equiv \left( i \frac{d}{dt} + V_0(x, t) + \lambda V_1(x, t) + \lambda^2 V_2(x, t) - 4\lambda^3 J \right) \psi(x, t, \lambda) \\ &= \psi(x, t, \lambda) C(\lambda) \end{aligned} \quad (4)$$

$$V_2(x, t) = 4Q(x, t), \quad V_1(x, t) = 2iJQ_x + 2JQ^2 \quad (5)$$

$$V_0(x, t) = -Q_{xx} - 2Q^3. \quad (6)$$

The corresponding MMKdV equations take the form

$$\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3(Q_x Q^2 + Q^2 Q_x) = 0. \quad (7)$$

The analysis in [2, 3, 11] reveals a number of important results. These include the corresponding multicomponent generalizations of KdV equations and the generalized Miura transformations interrelating them with the generalized MMKdV equations, two of their most important reductions as well as their Hamiltonians.

Our aim in this paper is to explore new types of reductions of the MMKdV equations. To this end we make use of the reduction group introduced by Mikhailov [22, 24] which allows one to impose algebraic reductions on the coefficients of  $Q(x, t)$  which will be automatically compatible with the evolution of MMKdV. Similar problems have been analyzed for the  $N$ -wave type equations related to the simple Lie algebras of rank 2 and 3 [16, 17] and the multicomponent NLS equations [18, 19]. Here we illustrate our analysis by the MMKdV equations related to the algebras  $\mathfrak{g} \simeq \mathfrak{so}(2r)$  which are linked to the **DIII**-type symmetric spaces series. Due to the fact that the dispersion law for MNLS is proportional to  $\lambda^2$  while for MMKdV it is proportional to  $\lambda^3$  the sets of admissible reductions for these two NLEE equations differ substantially.

In the next Section 2 we give some preliminaries on the scattering theory for  $L$ , the reduction group and graded Lie algebras. In Section 3 we construct the fundamental analytic solutions of  $L$ , formulate the corresponding Riemann-Hilbert problem and introduce the minimal sets of scattering data  $\mathcal{T}_i$ ,  $i = 1, 2$  which define uniquely both the scattering matrix and the solution of the MMKdV  $Q(x, t)$ . Some of these facts have been discussed in more details in [18], others had to be modified and extended so that they adequately take into account the peculiarities of the **DIII**-type symmetric spaces. In particular we modified the definition of the fundamental analytic solution which lead to changes in the formulation of the Riemann-Hilbert problem. In Section 4 we first briefly outline the hierarchy of Hamiltonian structures for the generic MMKdV equations. Next we list nontrivial examples of two classes of reductions of the MMKdV equations related to the algebra  $\mathfrak{so}(8)$ . The first class is performed with automorphisms of  $\mathfrak{so}(8)$  that preserve  $J$  and the second class uses automorphisms that map  $J$  into  $-J$ . While the reductions of first type can be applied both to MNLS and MMKdV equations, the reductions of second type can be applied only to MMKdV equations. Under them “half” of the members of the Hamiltonian hierarchy become degenerated [3, 9]. For both classes of reductions we find examples with groups of reductions isomorphic to  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$  and  $\mathbb{Z}_4$ . We also provide the corresponding reduced Hamiltonians and symplectic

forms and Poisson brackets. At the end of Section 4 we derive the effects of these reductions on the scattering matrix and on the minimal sets of scattering data. The last section contains some conclusions.

## 2. Preliminaries

In this section we outline some of the well known facts about the spectral theory of the Lax operators of the type (2).

### 2.1. The Scattering Problem for $L$

Here we briefly outline the basic facts about the direct and the inverse scattering problems [4,5,7,8,10,15,25,26,28,29] for the system (2) for the class of potentials  $Q(x, t)$  that are smooth enough and fall off to zero fast enough for  $x \rightarrow \pm\infty$  for all  $t$ . In what follows we treat **DIII**-type symmetric spaces which means that  $Q(x, t)$  is an element of the algebra  $\mathfrak{so}(2r)$ . In the examples below we take  $r = 4$  and  $\mathfrak{g} \simeq \mathfrak{so}(8)$ . For convenience we choose the following definition for the orthogonal algebras and groups

$$X \in \mathfrak{so}(2r) \longrightarrow X + S_0 X^T \hat{S}_0 = 0, \quad T \in \text{SO}(2r) \longrightarrow S_0 T^T \hat{S}_0 = \hat{T} \quad (8)$$

where the “hat” denotes the inverse matrix  $\hat{T} \equiv T^{-1}$  and

$$S_0 \equiv \sum_{k=1}^r (-1)^{k+1} (E_{k,\bar{k}} + E_{\bar{k},k}) = \begin{pmatrix} 0 & s_0 \\ \hat{s}_0 & 0 \end{pmatrix}, \quad \bar{k} = 2r + 1 - k. \quad (9)$$

Here and below by  $E_{jk}$  we denote a  $2r \times 2r$  matrix with just one non-vanishing and equal to 1 matrix element at  $j, k$ -th position:  $(E_{jk})_{mn} = \delta_{jm}\delta_{kn}$ . Obviously  $S_0^2 = \mathbb{1}$ .

The main tool for solving the direct and inverse scattering problems are the Jost solutions which are fundamental solutions defined by their asymptotics at  $x \rightarrow \pm\infty$

$$\lim_{x \rightarrow \infty} \psi(x, \lambda) e^{i\lambda Jx} = \mathbb{1}, \quad \lim_{x \rightarrow -\infty} \phi(x, \lambda) e^{i\lambda Jx} = \mathbb{1}. \quad (10)$$

Along with the Jost solutions we introduce

$$\xi(x, \lambda) = \psi(x, \lambda) e^{i\lambda Jx}, \quad \varphi(x, \lambda) = \phi(x, \lambda) e^{i\lambda Jx} \quad (11)$$

which satisfy the following linear integral equations

$$\xi(x, \lambda) = \mathbb{1} + i \int_{\infty}^x dy e^{-i\lambda J(x-y)} Q(y) \xi(y, \lambda) e^{i\lambda J(x-y)} \quad (12)$$

$$\varphi(x, \lambda) = \mathbb{1} + i \int_{-\infty}^x dy e^{-i\lambda J(x-y)} Q(y) \varphi(y, \lambda) e^{i\lambda J(x-y)}. \quad (13)$$

These are Volterra type equations which, have solutions providing one can ensure the convergence of the integrals in the right hand side. For  $\lambda$  real the exponential

factors in (12) and (13) are just oscillating and the convergence is ensured by the fact that  $Q(x, t)$  is rapidly vanishing for  $x \rightarrow \infty$ .

**Remark 1.** It is an well known fact that if the potential  $Q(x, t) \in \mathfrak{so}(2r)$  then the corresponding Jost solutions of equation (2) take values in the corresponding group, i.e.,  $\psi(x, \lambda), \phi(x, \lambda) \in SO(2r)$ .

The Jost solutions as whole can not be extended for  $\text{im } \lambda \neq 0$ . However, some of their columns can be extended for  $\lambda \in \mathbb{C}_+$ , others – for  $\lambda \in \mathbb{C}_-$ . More precisely we can write down the Jost solutions  $\psi(x, \lambda)$  and  $\phi(x, \lambda)$  in the following block-matrix form

$$\begin{aligned} \psi(x, \lambda) &= (|\psi^-(x, \lambda)\rangle, |\psi^+(x, \lambda)\rangle), & \phi(x, \lambda) &= (|\phi^+(x, \lambda)\rangle, |\phi^-(x, \lambda)\rangle) \\ |\psi^\pm(x, \lambda)\rangle &= \begin{pmatrix} \psi_1^\pm(x, \lambda) \\ \psi_2^\pm(x, \lambda) \end{pmatrix}, & |\phi^\pm(x, \lambda)\rangle &= \begin{pmatrix} \phi_1^\pm(x, \lambda) \\ \phi_2^\pm(x, \lambda) \end{pmatrix} \end{aligned} \quad (14)$$

where the superscript + and (respectively –) shows that the corresponding  $r \times r$  block-matrices allow analytic extension for  $\lambda \in \mathbb{C}_+$  (respectively  $\lambda \in \mathbb{C}_-$ ).

Solving the direct scattering problem means given the potential  $Q(x)$  to find the scattering matrix  $T(\lambda)$ . By definition  $T(\lambda)$  relates the two Jost solutions

$$\phi(x, \lambda) = \psi(x, \lambda)T(\lambda), \quad T(\lambda) = \begin{pmatrix} \mathbf{a}^+(\lambda) & -\mathbf{b}^-(\lambda) \\ \mathbf{b}^+(\lambda) & \mathbf{a}^-(\lambda) \end{pmatrix} \quad (15)$$

and has compatible block-matrix structure. In what follows we will need also the inverse of the scattering matrix

$$\psi(x, \lambda) = \phi(x, \lambda)\hat{T}(\lambda), \quad \hat{T}(\lambda) \equiv \begin{pmatrix} \mathbf{c}^-(\lambda) & \mathbf{d}^-(\lambda) \\ -\mathbf{d}^+(\lambda) & \mathbf{c}^+(\lambda) \end{pmatrix} \quad (16)$$

where

$$\mathbf{c}^-(\lambda) = \hat{\mathbf{a}}^+(\lambda)(\mathbf{1} + \rho^-\rho^+)^{-1} = (\mathbf{1} + \tau^+\tau^-)^{-1}\hat{\mathbf{a}}^+(\lambda) \quad (17a)$$

$$\mathbf{d}^-(\lambda) = \hat{\mathbf{a}}^+(\lambda)\rho^-(\lambda)(\mathbf{1} + \rho^+\rho^-)^{-1} = (\mathbf{1} + \tau^+\tau^-)^{-1}\tau^+(\lambda)\hat{\mathbf{a}}^-(\lambda) \quad (17b)$$

$$\mathbf{c}^+(\lambda) = \hat{\mathbf{a}}^-(\lambda)(\mathbf{1} + \rho^+\rho^-)^{-1} = (\mathbf{1} + \tau^-\tau^+)^{-1}\hat{\mathbf{a}}^-(\lambda) \quad (17c)$$

$$\mathbf{d}^+(\lambda) = \hat{\mathbf{a}}^-(\lambda)\rho^+(\lambda)(\mathbf{1} + \rho^-\rho^+)^{-1} = (\mathbf{1} + \tau^-\tau^+)^{-1}\tau^-(\lambda)\hat{\mathbf{a}}^+(\lambda). \quad (17d)$$

The diagonal blocks of  $T(\lambda)$  and  $\hat{T}(\lambda)$  allow analytic continuation off the real axis, namely  $\mathbf{a}^+(\lambda), \mathbf{c}^+(\lambda)$  are analytic functions of  $\lambda$  for  $\lambda \in \mathbb{C}_+$ , while  $\mathbf{a}^-(\lambda), \mathbf{c}^-(\lambda)$  are analytic functions of  $\lambda$  for  $\lambda \in \mathbb{C}_-$ . We introduced also  $\rho^\pm(\lambda)$  and  $\tau^\pm(\lambda)$  the multicomponent generalizations of the reflection coefficients (for the scalar case, see [1, 6, 21])

$$\rho^\pm(\lambda) = \mathbf{b}^\pm\hat{\mathbf{a}}^\pm(\lambda) = \hat{\mathbf{c}}^\pm\mathbf{d}^\pm(\lambda), \quad \tau^\pm(\lambda) = \hat{\mathbf{a}}^\pm\mathbf{b}^\mp(\lambda) = \mathbf{d}^\mp\hat{\mathbf{c}}^\pm(\lambda). \quad (18)$$

The reflection coefficients do not have analyticity properties and are defined only for  $\lambda \in \mathbb{R}$ .

From Remark 1 one concludes that  $T(\lambda) \in \text{SO}(2r)$ , therefore it must satisfy the second of the equations in (8). As a result we get the following relations between  $\mathbf{c}^\pm$ ,  $\mathbf{d}^\pm$  and  $\mathbf{a}^\pm$ ,  $\mathbf{b}^\pm$

$$\begin{aligned} \mathbf{c}^+(\lambda) &= \hat{\mathbf{s}}_0 \mathbf{a}^{+,T}(\lambda) \mathbf{s}_0, & \mathbf{c}^-(\lambda) &= \mathbf{s}_0 \mathbf{a}^{-,T}(\lambda) \hat{\mathbf{s}}_0 \\ \mathbf{d}^+(\lambda) &= -\hat{\mathbf{s}}_0 \mathbf{b}^{+,T}(\lambda) \mathbf{s}_0, & \mathbf{d}^-(\lambda) &= -\mathbf{s}_0 \mathbf{b}^{-,T}(\lambda) \hat{\mathbf{s}}_0 \end{aligned} \quad (19)$$

and in addition we have

$$\begin{aligned} \rho^+(\lambda) &= -\hat{\mathbf{s}}_0 \rho^{+,T}(\lambda) \mathbf{s}_0, & \rho^-(\lambda) &= -\mathbf{s}_0 \rho^{-,T}(\lambda) \hat{\mathbf{s}}_0 \\ \tau^+(\lambda) &= -\mathbf{s}_0 \tau^{+,T}(\lambda) \hat{\mathbf{s}}_0, & \tau^-(\lambda) &= -\hat{\mathbf{s}}_0 \tau^{-,T}(\lambda) \mathbf{s}_0. \end{aligned} \quad (20)$$

Next we need also the asymptotics of the Jost solutions and the scattering matrix for  $\lambda \rightarrow \infty$

$$\begin{aligned} \lim_{\lambda \rightarrow -\infty} \phi(x, \lambda) e^{i\lambda Jx} &= \lim_{\lambda \rightarrow \infty} \psi(x, \lambda) e^{i\lambda Jx} = \mathbf{1}, & \lim_{\lambda \rightarrow \infty} T(\lambda) &= \mathbf{1} \\ \lim_{\lambda \rightarrow \infty} \mathbf{a}^+(\lambda) &= \lim_{\lambda \rightarrow \infty} \mathbf{c}^-(\lambda) = \mathbf{1}, & \lim_{\lambda \rightarrow \infty} \mathbf{a}^-(\lambda) &= \lim_{\lambda \rightarrow \infty} \mathbf{c}^+(\lambda) = \mathbf{1}. \end{aligned} \quad (21)$$

The inverse to the Jost solutions  $\hat{\psi}(x, \lambda)$  and  $\hat{\phi}(x, \lambda)$  are solutions to

$$i \frac{d\hat{\psi}}{dx} - \hat{\psi}(x, \lambda)(Q(x) - \lambda J) = 0 \quad (22)$$

satisfying the conditions

$$\lim_{x \rightarrow \infty} e^{-i\lambda Jx} \hat{\psi}(x, \lambda) = \mathbf{1}, \quad \lim_{x \rightarrow -\infty} e^{-i\lambda Jx} \hat{\phi}(x, \lambda) = \mathbf{1}. \quad (23)$$

Now it is the collections of rows of  $\hat{\psi}(x, \lambda)$  and  $\hat{\phi}(x, \lambda)$  that possess analytic properties in  $\lambda$

$$\hat{\psi}(x, \lambda) = \begin{pmatrix} \langle \hat{\psi}^+(x, \lambda) | \\ \langle \hat{\psi}^-(x, \lambda) | \end{pmatrix}, \quad \hat{\phi}(x, \lambda) = \begin{pmatrix} \langle \hat{\phi}^-(x, \lambda) | \\ \langle \hat{\phi}^+(x, \lambda) | \end{pmatrix} \quad (24)$$

$$\langle \hat{\psi}^\pm(x, \lambda) | = (\mathbf{s}_0^{\pm 1} \boldsymbol{\psi}_2^\pm, \mathbf{s}_0^{\pm 1} \boldsymbol{\psi}_1^\pm)(x, \lambda), \quad \langle \hat{\phi}^\pm(x, \lambda) | = (\mathbf{s}_0^{\mp 1} \boldsymbol{\psi}_2^\pm, \mathbf{s}_0^{\mp 1} \boldsymbol{\psi}_1^\pm)(x, \lambda).$$

Just like the Jost solutions, their inverse (24) are solutions to linear equations (22) with regular boundary conditions (23) and therefore they have no singularities on the real axis  $\lambda \in \mathbb{R}$ . The same holds true also for the scattering matrix  $T(\lambda) = \hat{\psi}(x, \lambda) \phi(x, \lambda)$  and its inverse  $\hat{T}(\lambda) = \hat{\phi}(x, \lambda) \psi(x, \lambda)$ , i.e.,

$$\mathbf{a}^+(\lambda) = \langle \hat{\psi}^+(x, \lambda) | \phi^+(x, \lambda) \rangle, \quad \mathbf{a}^-(\lambda) = \langle \hat{\psi}^-(x, \lambda) | \phi^-(x, \lambda) \rangle \quad (25)$$

as well as

$$\mathbf{c}^+(\lambda) = \langle \hat{\phi}^+(x, \lambda) | \psi^+(x, \lambda) \rangle, \quad \mathbf{c}^-(\lambda) = \langle \hat{\phi}^-(x, \lambda) | \psi^-(x, \lambda) \rangle \quad (26)$$

are analytic for  $\lambda \in \mathbb{C}_\pm$  and have no singularities for  $\lambda \in \mathbb{R}$ . However they may become degenerate (i.e., their determinants may vanish) for some values  $\lambda_j^\pm \in \mathbb{C}_\pm$  of  $\lambda$ . Below we briefly analyze the structure of these degeneracies and show that they are related to discrete spectrum of  $L$ .

## 2.2. The Reduction Group of Mikhailov

The reduction group  $G_R$  is a finite group which preserves the Lax representation (2), i.e., it ensures that the reduction constraints are automatically compatible with the evolution.  $G_R$  must have two realizations: i)  $G_R \subset \text{Aut } \mathfrak{g}$  and ii)  $G_R \subset \text{Conf } \mathbb{C}$ , i.e., as conformal mappings of the complex  $\lambda$ -plane. To each  $g_k \in G_R$  we relate a reduction condition for the Lax pair as follows [24]

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda) \quad (27)$$

where  $C_k \in \text{Aut } \mathfrak{g}$  and  $\Gamma_k(\lambda) \in \text{Conf } \mathbb{C}$  are the images of  $g_k$  and  $\eta_k = 1$  or  $-1$  depending on the choice of  $C_k$ . Since  $G_R$  is a finite group then for each  $g_k$  there exist an integer  $N_k$  such that  $g_k^{N_k} = \mathbb{1}$ .

More specifically the automorphisms  $C_k$ ,  $k = 1, \dots, 4$  listed above lead to the following reductions for the potentials  $U(x, t, \lambda)$  and  $V(x, t, \lambda)$  of the Lax pair

$$U(x, t, \lambda) = Q(x, t) - \lambda J, \quad V(x, t, \lambda) = \sum_{k=0}^2 \lambda^k V_k(x, t) - 4\lambda^3 J \quad (28)$$

of the Lax representation

$$1) \ C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), \quad C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda) \quad (29)$$

$$2) \ C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), \quad C_2(V^T(\kappa_2(\lambda))) = -V(\lambda) \quad (30)$$

$$3) \ C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), \quad C_3(V^*(\kappa_1(\lambda))) = -V(\lambda) \quad (31)$$

$$4) \ C_4(U(\kappa_2(\lambda))) = U(\lambda), \quad C_4(V(\kappa_2(\lambda))) = V(\lambda). \quad (32)$$

The condition (27) is obviously compatible with the group action.

## 2.3. Cartan-Weyl Basis and Weyl Group for $\mathfrak{so}(2r)$

Here we fix the notations and the normalization conditions for the Cartan-Weyl generators of  $\mathfrak{g} \simeq \mathfrak{so}(2r)$ , see e.g. [20]. The root system  $\Delta$  of this series of simple Lie algebras consists of the roots  $\Delta \equiv \{\pm(e_i - e_j), \pm(e_i + e_j)\}$  where  $1 \leq i < j \leq r$ . We introduce an ordering in  $\Delta$  by specifying the set of positive roots  $\Delta^+ \equiv \{e_i - e_j, e_i + e_j\}$  for  $1 \leq i < j \leq r$ . Obviously all roots have the same length equal to 2.

We introduce the basis in the Cartan subalgebra by  $h_k \in \mathfrak{h}$ ,  $k = 1, \dots, r$  where  $\{h_k\}$  are the Cartan elements dual to the orthonormal basis  $\{e_k\}$  in the root space  $\mathbb{E}^r$ . Along with  $h_k$  we introduce also

$$H_\alpha = \sum_{k=1}^r (\alpha, e_k) h_k, \quad \alpha \in \Delta \quad (33)$$

where  $(\alpha, e_k)$  is the scalar product in the root space  $\mathbb{E}^r$  between the root  $\alpha$  and  $e_k$ . The basis in  $\mathfrak{so}(2r)$  is completed by adding the Weyl generators  $E_\alpha$ ,  $\alpha \in \Delta$ .

The commutation relations for the elements of the Cartan-Weyl basis are given in [20]

$$\begin{aligned} [h_k, E_\alpha] &= (\alpha, e_k) E_\alpha, & [E_\alpha, E_{-\alpha}] &= H_\alpha \\ [E_\alpha, E_\beta] &= \begin{cases} N_{\alpha, \beta} E_{\alpha+\beta} & \text{for } \alpha + \beta \in \Delta \\ 0 & \text{for } \alpha + \beta \notin \Delta \cup \{0\}. \end{cases} \end{aligned} \quad (34)$$

We will need also the typical  $2r$ -dimensional representation of  $\mathfrak{so}(2r)$ . In order to have the Cartan generators represented by diagonal matrices we modified the definition of orthogonal matrix, see (8). Using the matrices  $E_{jk}$  defined after equation (9) we get

$$\begin{aligned} h_k &= E_{kk} - E_{\bar{k}\bar{k}}, & E_{e_i - e_j} &= E_{ij} - (-1)^{i+j} E_{\bar{j}\bar{i}} \\ E_{e_i + e_j} &= E_{i\bar{j}} - (-1)^{i+j} E_{\bar{j}\bar{i}}, & E_{-\alpha} &= E_\alpha^T \end{aligned} \quad (35)$$

where  $\bar{k} = 2r + 1 - k$ .

We will denote by  $\vec{a} = \sum_{k=1}^r e_k$  the  $r$ -dimensional vector dual to  $J \in \mathfrak{h}$  where  $J = \sum_{k=1}^r h_k$ . If the root  $\alpha \in \Delta_+$  is positive (negative) then  $(\alpha, \vec{a}) \geq 0$  ( $(\alpha, \vec{a}) < 0$  respectively). The normalization of the basis is determined by

$$E_{-\alpha} = E_\alpha^T, \quad \langle E_{-\alpha}, E_\alpha \rangle = 2, \quad N_{-\alpha, -\beta} = -N_{\alpha, \beta}. \quad (36)$$

The root system  $\Delta$  of  $\mathfrak{g}$  is invariant with respect to the Weyl reflections  $S_\alpha$  which act on the vectors  $\vec{y} \in \mathbb{E}^r$  specified by the formula

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)} \alpha, \quad \alpha \in \Delta. \quad (37)$$

All Weyl reflections  $S_\alpha$  form a finite group  $W_{\mathfrak{g}}$  known as the **Weyl group**. On the root space this group is isomorphic to  $\mathcal{S}_r \otimes (\mathbb{Z}_2)^{r-1}$  where  $\mathcal{S}_r$  is the group of permutations of the basic vectors  $e_j \in \mathbb{E}^r$ . Each of the  $\mathbb{Z}_2$  groups acts on  $\mathbb{E}^r$  by changing simultaneously the signs of two of the basic vectors  $e_j$ .

One may introduce also an action of the Weyl group on the Cartan-Weyl basis, namely [20]

$$\begin{aligned} S_\alpha(H_\beta) &\equiv A_\alpha H_\beta A_\alpha^{-1} = H_{S_\alpha \beta} \\ S_\alpha(E_\beta) &\equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha, \beta} E_{S_\alpha \beta}, \quad n_{\alpha, \beta} = \pm 1. \end{aligned} \quad (38)$$

The matrices  $A_\alpha$  are given (up to a factor from the Cartan subgroup) by

$$A_\alpha = e^{E_\alpha} e^{-E_{-\alpha}} e^{E_\alpha} H_A \quad (39)$$

where  $H_A$  is a conveniently chosen element from the Cartan subgroup such that  $H_A^2 = \mathbb{1}$ . The formula (39) and the explicit form of the Cartan-Weyl basis in the typical representation will be used in calculating the reduction condition following from (27).

## 2.4. Graded Lie Algebras

One of the important notions in constructing integrable equations and their reductions is the one of graded Lie algebra and Kac-Moody algebras [20]. The standard construction is based on a finite order automorphism  $C \in \text{Aut } \mathfrak{g}$ ,  $C^N = \mathbb{1}$ . The eigenvalues of  $C$  are  $\omega^k$ ,  $k = 0, 1, \dots, N-1$ , where  $\omega = \exp(2\pi i/N)$ . To each eigenvalue there corresponds a linear subspace  $\mathfrak{g}^{(k)} \subset \mathfrak{g}$  determined by

$$\mathfrak{g}^{(k)} \equiv \{X; X \in \mathfrak{g}, C(X) = \omega^k X\}. \quad (40)$$

Then  $\mathfrak{g} = \bigoplus_{k=0}^{N-1} \mathfrak{g}^{(k)}$  and the grading condition holds

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(n)}] \subset \mathfrak{g}^{(k+n)} \quad (41)$$

where  $k+n$  is taken modulo  $N$ . Thus to each pair  $\{\mathfrak{g}, C\}$  one can relate an infinite-dimensional algebra of Kac-Moody type  $\widehat{\mathfrak{g}}_C$  whose elements are

$$X(\lambda) = \sum_k X_k \lambda^k, \quad X_k \in \mathfrak{g}^{(k)}. \quad (42)$$

The series in (42) must contain only finite number of negative (positive) powers of  $\lambda$  and  $\mathfrak{g}^{(k+N)} \equiv \mathfrak{g}^{(k)}$ . This construction is a most natural one for Lax pairs and we will see that due to the grading condition (41) we can always impose a reduction on  $L(\lambda)$  and  $M(\lambda)$  such that both  $U(x, t, \lambda)$  and  $V(x, t, \lambda) \in \widehat{\mathfrak{g}}_C$ . In the case of symmetric spaces  $N = 2$  and  $C$  is the Cartan involution. Then one can choose the Lax operator  $L$  in such a way that

$$Q \in \mathfrak{g}^{(1)}, \quad J \in \mathfrak{g}^{(0)} \quad (43)$$

as it is the case in (2). Here the subalgebra  $\mathfrak{g}^{(0)}$  consists of all elements of  $\mathfrak{g}$  commuting with  $J$ . The special choice of  $J = \sum_{k=1}^r h_k$  taken above allows us to split the set of all positive roots  $\Delta^+$  into two subsets

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+, \quad \Delta_0^+ = \{e_i - e_j\}_{i < j}, \quad \Delta_1^+ = \{e_i + e_j\}_{i < j}. \quad (44)$$

Obviously the elements  $\alpha \in \Delta_1^+$  have the property  $\alpha(J) = (\alpha, \vec{a}) = 2$ , while the elements  $\beta \in \Delta_0^+$  have the property  $\beta(J) = (\beta, \vec{a}) = 0$ .



### 3. The Fundamental Analytic Solutions and the Riemann-Hilbert Problem

#### 3.1. The Fundamental Analytic Solutions

The next step is to construct the **fundamental analytic solutions** (FAS)  $\chi^\pm(x, \lambda)$  of (2). Here we slightly modify the definition in [18] to ensure that  $\chi^\pm(x, \lambda) \in \text{SO}(2r)$ . Thus we define

$$\chi^+(x, \lambda) \equiv (|\phi^+, |\psi^+ \hat{c}^+)) (x, \lambda) = \phi(x, \lambda) \mathbf{S}^+(\lambda) = \psi(x, \lambda) \mathbf{T}^-(\lambda) D^+(\lambda) \quad (45)$$

$$\chi^-(x, \lambda) \equiv (|\psi^- \hat{c}^-, |\phi^-)) (x, \lambda) = \phi(x, \lambda) \mathbf{S}^-(\lambda) = \psi(x, \lambda) \mathbf{T}^+(\lambda) D^-(\lambda)$$

where the block-triangular functions  $\mathbf{S}^\pm(\lambda)$  and  $\mathbf{T}^\pm(\lambda)$  are given by

$$\begin{aligned} \mathbf{S}^+(\lambda) &= \begin{pmatrix} \mathbb{1} & \mathbf{d}^- \hat{c}^+(\lambda) \\ 0 & \mathbb{1} \end{pmatrix}, & \mathbf{T}^-(\lambda) &= \begin{pmatrix} \mathbb{1} & 0 \\ \mathbf{b}^+ \hat{\mathbf{a}}^+(\lambda) & \mathbb{1} \end{pmatrix} \\ \mathbf{S}^-(\lambda) &= \begin{pmatrix} \mathbb{1} & 0 \\ -\mathbf{d}^+ \hat{c}^-(\lambda) & \mathbb{1} \end{pmatrix}, & \mathbf{T}^+(\lambda) &= \begin{pmatrix} \mathbb{1} & -\mathbf{b}^- \hat{\mathbf{a}}^-(\lambda) \\ 0 & \mathbb{1} \end{pmatrix}. \end{aligned} \quad (46)$$

The matrices  $D^\pm(\lambda)$  are block-diagonal and equal

$$D^+(\lambda) = \begin{pmatrix} \mathbf{a}^+(\lambda) & 0 \\ 0 & \hat{c}^+(\lambda) \end{pmatrix}, \quad D^-(\lambda) = \begin{pmatrix} \hat{c}^-(\lambda) & 0 \\ 0 & \mathbf{a}^-(\lambda) \end{pmatrix}. \quad (47)$$

The upper scripts  $\pm$  here refer to their analyticity properties for  $\lambda \in \mathbb{C}_\pm$ .

In view of the relations (19) it is easy to check that all factors  $\mathbf{S}^\pm$ ,  $\mathbf{T}^\pm$  and  $D^\pm$  take values in the group  $\text{SO}(2r)$ . Besides, since

$$\begin{aligned} T(\lambda) &= \mathbf{T}^-(\lambda) D^+(\lambda) \hat{\mathbf{S}}^+(\lambda) = \mathbf{T}^+(\lambda) D^-(\lambda) \hat{\mathbf{S}}^-(\lambda) \\ \hat{T}(\lambda) &= \mathbf{S}^+(\lambda) \hat{D}^+(\lambda) \hat{\mathbf{T}}^-(\lambda) = \mathbf{S}^-(\lambda) \hat{D}^-(\lambda) \hat{\mathbf{T}}^+(\lambda) \end{aligned} \quad (48)$$

we can view the factors  $\mathbf{S}^\pm$ ,  $\mathbf{T}^\pm$  and  $D^\pm$  as generalized Gauss decompositions (see [20]) of  $T(\lambda)$  and its inverse.

The relations between  $\mathbf{c}^\pm(\lambda)$ ,  $\mathbf{d}^\pm(\lambda)$  and  $\mathbf{a}^\pm(\lambda)$ ,  $\mathbf{b}^\pm(\lambda)$  in equation (17) ensure that equations (48) become identities. From equations (45), (46) we derive

$$\chi^+(x, \lambda) = \chi^-(x, \lambda) G_0(\lambda), \quad \chi^-(x, \lambda) = \chi^+(x, \lambda) \hat{G}_0(\lambda) \quad (49)$$

$$G_0(\lambda) = \begin{pmatrix} \mathbb{1} & \tau^+ \\ \tau^- & \mathbb{1} + \tau^- \tau^+ \end{pmatrix}, \quad \hat{G}_0(\lambda) = \begin{pmatrix} \mathbb{1} + \tau^+ \tau^- & -\tau^+ \\ -\tau^- & \mathbb{1} \end{pmatrix} \quad (50)$$

valid for  $\lambda \in \mathbb{R}$ . Below we introduce

$$X^\pm(x, \lambda) = \chi^\pm(x, \lambda) e^{i\lambda Jx}. \quad (51)$$

Strictly speaking it is  $X^\pm(x, \lambda)$  that allow analytic extension for  $\lambda \in \mathbb{C}_\pm$ . They have also another nice property, namely their asymptotic behavior for  $\lambda \rightarrow \pm\infty$  is given by

$$\lim_{\lambda \rightarrow \infty} X^\pm(x, \lambda) = \mathbf{1}. \quad (52)$$

Along with  $X^\pm(x, \lambda)$  we can use another set of FAS  $\tilde{X}^\pm(x, \lambda) = X^\pm(x, \lambda)\hat{D}^\pm$ , which also satisfy equation (52) due to the fact that

$$\lim_{\lambda \rightarrow \infty} D^\pm(\lambda) = \mathbf{1}. \quad (53)$$

The analyticity properties of  $X^\pm(x, \lambda)$  and  $\tilde{X}^\pm(x, \lambda)$  for  $\lambda \in \mathbb{C}_\pm$  along with equation (52) are crucial for our considerations.

### 3.2. The Riemann-Hilbert Problem

The equations (49) and (50) can be written down as

$$X^+(x, \lambda) = X^-(x, \lambda)G(x, \lambda), \quad \lambda \in \mathbb{R} \quad (54)$$

where

$$G(x, \lambda) = e^{-i\lambda Jx} G_0(\lambda) e^{i\lambda Jx}. \quad (55)$$

Likewise the second pair of FAS satisfy

$$\tilde{X}^+(x, \lambda) = \tilde{X}^-(x, \lambda)\tilde{G}(x, \lambda), \quad \lambda \in \mathbb{R} \quad (56)$$

with

$$\tilde{G}(x, \lambda) = e^{-i\lambda Jx} \tilde{G}_0(\lambda) e^{i\lambda Jx}, \quad \tilde{G}_0(\lambda) = \begin{pmatrix} \mathbf{1} + \rho^- \rho^+ & \rho^- \\ \rho^+ & \mathbf{1} \end{pmatrix}. \quad (57)$$

Equation (54) (respectively equation (56)) combined with (52) is known in the literature [12] as a Riemann-Hilbert problem (RHP) with canonical normalization. It is well known that RHP with canonical normalization has unique regular solution while the matrix-valued solutions  $X_0^+(x, \lambda)$  and  $X_0^-(x, \lambda)$  in (54), obeying (52) are called regular if  $\det X_0^\pm(x, \lambda)$  does not vanish for any  $\lambda \in \mathbb{C}_\pm$ .

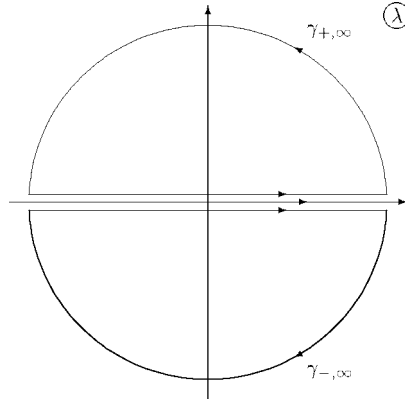
Let us now apply the contour-integration method to derive the integral decompositions of  $X^\pm(x, \lambda)$ . To this end we consider the contour integrals

$$\mathcal{J}_1(\lambda) = \frac{1}{2\pi i} \oint_{\gamma_+} \frac{d\mu}{\mu - \lambda} X^+(x, \mu) - \frac{1}{2\pi i} \oint_{\gamma_-} \frac{d\mu}{\mu - \lambda} X^-(x, \mu) \quad (58)$$

and

$$\mathcal{J}_2(\lambda) = \frac{1}{2\pi i} \oint_{\gamma_+} \frac{d\mu}{\mu - \lambda} \tilde{X}^+(x, \mu) - \frac{1}{2\pi i} \oint_{\gamma_-} \frac{d\mu}{\mu - \lambda} \tilde{X}^-(x, \mu) \quad (59)$$

where  $\lambda \in \mathbb{C}_+$  and the contours  $\gamma_\pm$  are shown in Fig. 1.



**Figure 1.** The contours  $\gamma_{\pm} = \mathbb{R} \cup \gamma_{\pm\infty}$ .

Each of these integrals can be evaluated by Cauchy residue theorem. The result for  $\lambda \in \mathbb{C}_+$  are

$$\mathcal{J}_1(\lambda) = X^+(x, \lambda) + \sum_{j=1}^N \operatorname{Res}_{\mu=\lambda_j^+} \frac{X^+(x, \mu)}{\mu - \lambda} + \sum_{j=1}^N \operatorname{Res}_{\mu=\lambda_j^-} \frac{X^-(x, \mu)}{\mu - \lambda} \quad (60)$$

$$\mathcal{J}_2(\lambda) = \tilde{X}^+(x, \lambda) + \sum_{j=1}^N \operatorname{Res}_{\mu=\lambda_j^+} \frac{\tilde{X}^+(x, \mu)}{\mu - \lambda} + \sum_{j=1}^N \operatorname{Res}_{\mu=\lambda_j^-} \frac{\tilde{X}^-(x, \mu)}{\mu - \lambda}. \quad (61)$$

The discrete sums in the right hand sides of equations (60) and (61) naturally provide the contribution from the discrete spectrum of  $L$ . For the sake of simplicity we assume that  $L$  has a finite number of simple eigenvalues  $\lambda_j^{\pm} \in \mathbb{C}_{\pm}$  and for additional details see [18]. Let us clarify the above statement. For the  $2 \times 2$  Zakharov-Shabat problem it is well known that the discrete eigenvalues of  $L$  are provided by the zeroes of the transmission coefficients  $a^{\pm}(\lambda)$ , which in that case are scalar functions. For the more general  $2r \times 2r$  Zakharov-Shabat system (2) the situation becomes more complex because now  $a^{\pm}(\lambda)$  are  $r \times r$  matrices. The discrete eigenvalues  $\lambda_j^{\pm}$  now are the points at which  $a^{\pm}(\lambda)$  become degenerate and their inverse develop pole singularities. More precisely, we assume that in the vicinities of  $\lambda_j^{\pm}$   $a^{\pm}(\lambda)$ ,  $c^{\pm}(\lambda)$  and their inverse  $\hat{a}^{\pm}(\lambda)$ ,  $\hat{c}^{\pm}(\lambda)$  have the following decompositions in Taylor series

$$\mathbf{a}^{\pm}(\lambda) = \mathbf{a}_j^{\pm} + (\lambda - \lambda_j^{\pm})\hat{\mathbf{a}}_j^{\pm} + \dots, \quad \mathbf{c}^{\pm}(\lambda) = \mathbf{c}_j^{\pm} + (\lambda - \lambda_j^{\pm})\hat{\mathbf{c}}_j^{\pm} + \dots \quad (62)$$

$$\hat{\mathbf{a}}^{\pm}(\lambda) = \frac{\hat{\mathbf{a}}_j^{\pm}}{\lambda - \lambda_j^{\pm}} + \hat{\mathbf{a}}_j^{\pm} + \dots, \quad \hat{\mathbf{c}}^{\pm}(\lambda) = \frac{\hat{\mathbf{c}}_j^{\pm}}{\lambda - \lambda_j^{\pm}} + \hat{\mathbf{a}}_j^{\pm} + \dots \quad (63)$$

where all the leading coefficients  $\mathbf{a}_j^\pm$ ,  $\hat{\mathbf{a}}_j^\pm$ ,  $\mathbf{c}_j^\pm$ ,  $\hat{\mathbf{c}}_j^\pm$  are degenerate matrices such that

$$\hat{\mathbf{a}}_j^\pm \mathbf{a}_j^\pm = \mathbf{a}_j^\pm \hat{\mathbf{a}}_j^\pm = 0, \quad \hat{\mathbf{c}}_j^\pm \mathbf{c}_j^\pm = \mathbf{c}_j^\pm \hat{\mathbf{c}}_j^\pm = 0. \quad (64)$$

In addition we have more relations such as

$$\hat{\mathbf{a}}_j^\pm \hat{\mathbf{a}}_j^\pm + \hat{\mathbf{a}}_j^\pm \mathbf{a}_j^\pm = \mathbf{1}, \quad \hat{\mathbf{c}}_j^\pm \mathbf{c}_j^\pm + \hat{\mathbf{c}}_j^\pm \hat{\mathbf{c}}_j^\pm = \mathbf{1} \quad (65)$$

that are needed to ensure the identities  $\hat{\mathbf{a}}^\pm(\lambda)\mathbf{a}^\pm(\lambda) = \mathbf{1}$ ,  $\hat{\mathbf{c}}^\pm(\lambda)\mathbf{c}^\pm(\lambda) = \mathbf{1}$ , etc for all values of  $\lambda$ .

The assumption that the eigenvalues are simple here means that we have considered only first order pole singularities of  $\hat{\mathbf{a}}_j^\pm(\lambda)$  and  $\hat{\mathbf{c}}_j^\pm(\lambda)$ . After some additional considerations we find that the ‘‘halves’’ of the Jost solutions  $|\psi^\pm(x, \lambda)\rangle$  and  $|\phi^\pm(x, \lambda)\rangle$  satisfy the following relationships for  $\lambda = \lambda_j^\pm$

$$|\psi_j^\pm(x)\hat{\mathbf{c}}_j^\pm\rangle = \pm|\phi_j^\pm(x)\tau_j^\pm\rangle, \quad |\phi_j^\pm(x)\hat{\mathbf{a}}_j^\pm\rangle = \pm|\psi_j^\pm(x)\rho_j^\pm\rangle \quad (66)$$

where  $|\psi_j^\pm(x)\rangle = |\psi^\pm(x, \lambda_j^\pm)\rangle$ ,  $|\phi_j^\pm(x)\rangle = |\phi^\pm(x, \lambda_j^\pm)\rangle$

$$\rho_j^\pm = \hat{\mathbf{c}}_j^\pm \mathbf{d}_j^\pm = \mathbf{b}_j^\pm \hat{\mathbf{a}}_j^\pm, \quad \tau_j^\pm = \hat{\mathbf{a}}_j^\pm \mathbf{b}_j^\pm = \mathbf{d}_j^\pm \hat{\mathbf{c}}_j^\pm \quad (67)$$

and the additional coefficients  $\mathbf{b}_j^\pm$  and  $\mathbf{d}_j^\pm$  are constant  $r \times r$  nondegenerate matrices which, as we shall see below, are also part of the minimal sets of scattering data needed to determine the potential  $Q(x, t)$ .

These considerations allow us to calculate explicitly the residues in equations (60), (61) with the result

$$\operatorname{Res}_{\mu=\lambda_j^+} \frac{X^+(x, \mu)}{\mu - \lambda} = \frac{(|0\rangle, |\phi_j^+(x)\tau_j^+\rangle)}{\lambda_j^+ - \lambda}, \quad \operatorname{Res}_{\mu=\lambda_j^+} \frac{\tilde{X}^+(x, \mu)}{\mu - \lambda} = \frac{(|\psi_j^+(x)\rho_j^+\rangle, |0\rangle)}{\lambda_j^+ - \lambda} \quad (68)$$

$$\operatorname{Res}_{\mu=\lambda_j^+} \frac{X^-(x, \mu)}{\mu - \lambda} = -\frac{(|\phi_j^-(x)\tau_j^-\rangle, |0\rangle)}{\lambda_j^+ - \lambda}, \quad \operatorname{Res}_{\mu=\lambda_j^+} \frac{\tilde{X}^-(x, \mu)}{\mu - \lambda} = -\frac{(|0\rangle, |\psi_j^-(x)\tau_j^-\rangle)}{\lambda_j^+ - \lambda}$$

where  $|0\rangle$  stands for a collection of  $r$  columns whose components are all equal to zero.

We can also evaluate  $\mathcal{J}_1(\lambda)$  and  $\mathcal{J}_2(\lambda)$  by integrating along the contours. In integrating along the infinite semi-circles of  $\gamma_{\pm, \infty}$  we use the asymptotic behavior of  $X^\pm(x, \lambda)$  and  $\tilde{X}^\pm(x, \lambda)$  for  $\lambda \rightarrow \infty$ . The results are

$$\mathcal{J}_1(\lambda) = \mathbf{1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \phi(x, \mu) e^{i\mu Jx} K(x, \mu) \quad (69)$$

$$\mathcal{J}_2(\lambda) = \mathbf{1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \psi(x, \mu) e^{i\mu Jx} \tilde{K}(x, \mu) \quad (70)$$

$$K(x, \mu) = e^{-i\mu Jx} K_0(\mu) e^{i\mu Jx}, \quad \tilde{K}(x, \mu) = e^{-i\mu Jx} \tilde{K}_0(\mu) e^{i\mu Jx} \quad (71)$$

$$K_0(\mu) = \begin{pmatrix} 0 & \tau^+(\mu) \\ \tau^-(\mu) & 0 \end{pmatrix}, \quad \tilde{K}_0(\mu) = \begin{pmatrix} 0 & \rho^+(\mu) \\ \rho^-(\mu) & 0 \end{pmatrix} \quad (72)$$

where in evaluating the integrands we made use of equations (15), (17), (54) and (56).

Equating the right hand sides of (60) and (69), and (61) and (70) we get the following integral decomposition for  $X^\pm(x, \lambda)$

$$X^+(x, \lambda) = \mathbf{1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} X^-(x, \mu) K_1(x, \mu) + \sum_{j=1}^N \frac{X_j^-(x) K_{1,j}(x)}{\lambda_j^- - \lambda} \quad (73)$$

$$X^-(x, \lambda) = \mathbf{1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} X^-(x, \mu) K_2(x, \mu) - \sum_{j=1}^N \frac{X_j^+(x) K_{2,j}(x)}{\lambda_j^+ - \lambda} \quad (74)$$

where  $X_j^\pm(x) = X^\pm(x, \lambda_j^\pm)$  and

$$K_{1,j}(x) = e^{-i\lambda_j^- Jx} \begin{pmatrix} 0 & \rho_j^+ \\ \tau_j^- & 0 \end{pmatrix} e^{i\lambda_j^- Jx}, \quad K_{2,j}(x) = e^{-i\lambda_j^+ Jx} \begin{pmatrix} 0 & \tau_j^+ \\ \rho_j^- & 0 \end{pmatrix} e^{i\lambda_j^+ Jx}. \quad (75)$$

Equations (73), (74) can be viewed as a set of singular integral equations which are equivalent to the RHP. For the MNLS these were first derived in [23].

We end this section by a brief explanation of how the potential  $Q(x, t)$  can be recovered provided we have solved the RHP and know the solutions  $X^\pm(x, \lambda)$ . First we take into account that  $X^\pm(x, \lambda)$  satisfy the differential equation

$$i \frac{dX^\pm}{dx} + Q(x, t) X^\pm(x, \lambda) - \lambda [J, X^\pm(x, \lambda)] = 0 \quad (76)$$

which must hold true for all  $\lambda$ . From equation (52) and also from the integral equations (73), (74) one concludes that  $X^\pm(x, \lambda)$  and their inverse  $\hat{X}^\pm(x, \lambda)$  are regular for  $\lambda \rightarrow \infty$  and allow asymptotic expansions of the form

$$X^\pm(x, \lambda) = \mathbf{1} + \sum_{s=1}^{\infty} \lambda^{-s} X_s^\pm(x), \quad \hat{X}^\pm(x, \lambda) = \mathbf{1} + \sum_{s=1}^{\infty} \lambda^{-s} \hat{X}_s^\pm(x). \quad (77)$$

Inserting these into equation (76) and taking the limit  $\lambda \rightarrow \infty$  we get

$$Q(x, t) = \lim_{\lambda \rightarrow \infty} \lambda (J - X^\pm(x, \lambda) J \hat{X}^\pm(x, \lambda)) = [J, X_1(x)]. \quad (78)$$

### 3.3. The Minimal Set of Scattering Data

Obviously, given the potential  $Q(x)$  one can solve the integral equations for the Jost solutions which determine them uniquely. The Jost solutions in turn determine uniquely the scattering matrix  $T(\lambda)$  and its inverse  $\hat{T}(\lambda)$ . But the potential  $Q(x)$  contains  $r(r-1)$  independent complex-valued functions of  $x$ . Thus it is natural

to expect that at most  $r(r-1)$  of the coefficients in  $T(\lambda)$  for  $\lambda \in \mathbb{R}$  will be independent and the rest must be functions of those.

The set of independent coefficients of  $T(\lambda)$  are known as the minimal set of scattering data. As such we may use any of the following two sets  $\mathcal{T}_i \equiv \mathcal{T}_{i,c} \cup \mathcal{T}_{i,d}$

$$\begin{aligned} \mathcal{T}_{1,c} &\equiv \{\rho^+(\lambda), \rho^-(\lambda), \quad \lambda \in \mathbb{R}\}, & \mathcal{T}_{1,d} &\equiv \left\{ \rho_j^\pm, \lambda_j^\pm \right\}_{j=1}^N \\ \mathcal{T}_{2,c} &\equiv \{\tau^+(\lambda), \tau^-(\lambda), \quad \lambda \in \mathbb{R}\}, & \mathcal{T}_{1,d} &\equiv \left\{ \tau_j^\pm, \lambda_j^\pm \right\}_{j=1}^N \end{aligned} \quad (79)$$

where the reflection coefficients  $\rho^\pm(\lambda)$  and  $\tau^\pm(\lambda)$  were introduced in equation (17),  $\lambda_j^\pm$  are (simple) discrete eigenvalues of  $L$  and  $\rho_j^\pm$  and  $\tau_j^\pm$  characterize the norming constants of the corresponding Jost solutions.

**Remark 2.** A consequence of equation (20) is the fact that  $\mathbf{S}^\pm(\lambda), \mathbf{S}^\pm(\lambda) \in \text{SO}(2r)$ . These factors can be written also in the form

$$\mathbf{S}^\pm(\lambda) = \exp \left( \sum_{\alpha \in \Delta_1^+} \tau_\alpha^\pm(\lambda) E_{\pm\alpha} \right), \quad \mathbf{T}^\pm(\lambda) = \exp \left( \sum_{\alpha \in \Delta_1^+} \rho_\alpha^\pm(\lambda) E_{\pm\alpha} \right). \quad (80)$$

Taking into account that in the typical representation we have  $E_{\pm\alpha} E_{\pm\beta} = 0$  for all roots  $\alpha, \beta \in \Delta_1^+$  we find that

$$\begin{aligned} \sum_{\alpha \in \Delta_1^+} \tau_\alpha^+(\lambda) E_{\pm\alpha} &= \begin{pmatrix} 0 & \tau^+(\lambda) \\ 0 & 0 \end{pmatrix}, & \sum_{\alpha \in \Delta_1^+} \tau_\alpha^-(\lambda) E_{-\alpha} &= \begin{pmatrix} 0 & 0 \\ \tau^-(\lambda) & 0 \end{pmatrix} \\ \sum_{\alpha \in \Delta_1^+} \rho_\alpha^+(\lambda) E_{\pm\alpha} &= \begin{pmatrix} 0 & \rho^+(\lambda) \\ 0 & 0 \end{pmatrix}, & \sum_{\alpha \in \Delta_1^+} \rho_\alpha^-(\lambda) E_{-\alpha} &= \begin{pmatrix} 0 & 0 \\ \rho^-(\lambda) & 0 \end{pmatrix} \end{aligned} \quad (81)$$

where  $\Delta_1^+$  is a subset of the positive roots of  $\mathfrak{so}(2r)$  defined below in Subsection 3.3. The formulae (81) ensure that the number of independent matrix elements of  $\tau^+(\lambda)$  and  $\tau^-(\lambda)$  (respectively,  $\rho^+(\lambda)$  and  $\rho^-(\lambda)$ ) equals  $2|\Delta_1^+| = r(r-1)$  which coincides with the number of independent functions of  $Q(x)$ .

The reflection coefficients  $\rho^\pm(\lambda)$  and  $\tau^\pm(\lambda)$  are defined only on the real  $\lambda$ -axis, while the diagonal blocks  $\mathbf{a}^\pm(\lambda)$  and  $\mathbf{c}^\pm(\lambda)$  (or, equivalently,  $D^\pm(\lambda)$ ) allow analytic extensions for  $\lambda \in \mathbb{C}_\pm$ . From the equations (17) there follows that

$$\mathbf{a}^+(\lambda) \mathbf{c}^-(\lambda) = (\mathbf{1} + \rho^- \rho^+(\lambda))^{-1}, \quad \mathbf{a}^-(\lambda) \mathbf{c}^+(\lambda) = (\mathbf{1} + \rho^+ \rho^-(\lambda))^{-1} \quad (82)$$

$$\mathbf{c}^-(\lambda) \mathbf{a}^+(\lambda) = (\mathbf{1} + \tau^+ \tau^-(\lambda))^{-1}, \quad \mathbf{c}^+(\lambda) \mathbf{a}^-(\lambda) = (\mathbf{1} + \tau^- \tau^+(\lambda))^{-1}. \quad (83)$$

Given  $\mathcal{T}_1$  (respectively,  $\mathcal{T}_2$ ) we determine the right hand sides of (82) (respectively (83)) for  $\lambda \in \mathbb{R}$ . Combined with the facts about the limits

$$\lim_{\lambda \rightarrow \infty} \mathbf{a}^+(\lambda) = \lim_{\lambda \rightarrow \infty} \mathbf{c}^-(\lambda) = \lim_{\lambda \rightarrow \infty} \mathbf{a}^-(\lambda) = \lim_{\lambda \rightarrow \infty} \mathbf{c}^+(\lambda) = \mathbf{1} \quad (84)$$

each of the relations (82), (83) can be viewed as a RHP with canonical normalization. Such RHP can be solved explicitly in the one-component case (provided we know the locations of their zeroes) by using the Plemelj-Sokhotsky formulae [12]. These zeroes are in fact the discrete eigenvalues of  $L$ . One possibility to make use of these facts is to take log of the determinants of both sides of (82) which leads to

$$A^+(\lambda) + C^-(\lambda) = -\ln \det(\mathbf{1} + \rho^- \rho^+(\lambda)), \quad \lambda \in \mathbb{R} \quad (85)$$

where

$$A^\pm(\lambda) = \ln \det \mathbf{a}^\pm(\lambda), \quad C^\pm(\lambda) = \ln \det \mathbf{c}^\pm(\lambda). \quad (86)$$

Then Plemelj-Sokhotsky formulae allows us to recover  $A^\pm(\lambda)$  and  $C^\pm(\lambda)$

$$\mathcal{A}(\lambda) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln \det(\mathbf{1} + \rho^- \rho^+(\mu)) + \sum_{j=1}^N \ln \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-} \quad (87)$$

where  $\mathcal{A}(\lambda) = A^+(\lambda)$  for  $\lambda \in \mathbb{C}_+$  and  $\mathcal{A}(\lambda) = -C^-(\lambda)$  for  $\lambda \in \mathbb{C}_-$ . In deriving (87) we have also assumed that  $\lambda_j^\pm$  are simple zeroes of  $A^\pm(\lambda)$  and  $C^\pm(\lambda)$ .

Let us consider the reduction condition (29) with  $C_1$  from the Cartan subgroup  $C_1 = \text{diag}(B_+, B_-)$  where the diagonal matrices  $B_\pm$  are such that  $B_\pm^2 = \mathbf{1}$ . Then we get the following constraints on the sets  $\mathcal{T}_{1,2}$

$$\rho^-(\lambda) = (B_- \rho^+(\lambda) B_+)^{\dagger}, \quad \rho_j^- = (B_- \rho_j^+ B_+)^{\dagger}, \quad \lambda_j^- = (\lambda_j^+)^* \quad (88)$$

$$\tau^-(\lambda) = (B_+ \tau^+(\lambda) B_-)^{\dagger}, \quad \tau_j^- = (B_+ \tau_j^+ B_-)^{\dagger}, \quad \lambda_j^- = (\lambda_j^+)^* \quad (89)$$

where  $j = 1, \dots, N$ . For more details see Subsection 4.4 and Subsection 4.5.

**Remark 3.** For certain reductions such as, e.g.  $Q = -Q^\dagger$  the generalized Zakharov-Shabat system  $L(\lambda)\psi = 0$  can be written down as an eigenvalue problem  $\mathcal{L}\psi = \lambda\psi(x, \lambda)$  where  $\mathcal{L}$  is a self-adjoint operator. The continuous spectrum of  $\mathcal{L}$  fills up the whole real  $\lambda$ -axis thus “leaving no space” for discrete eigenvalues. Such Lax operators have no discrete spectrum and the corresponding MNLS or MMKdV equations do not have soliton solutions.

From the general theory of RHP [12] one may conclude that (82), (83) allow unique solutions provided the number and types of the zeroes  $\lambda_j^\pm$  are properly chosen. Thus we can outline a procedure which allows one to reconstruct not only  $T(\lambda)$  and  $\hat{T}(\lambda)$  and the corresponding potential  $Q(x)$  from each of the sets  $\mathcal{T}_i$ ,  $i = 1, 2$ :

- i) Given  $\mathcal{T}_2$  (respectively  $\mathcal{T}_1$ ) solve the RHP (82) (respectively (83)) and construct  $\mathbf{a}^\pm(\lambda)$  and  $\mathbf{c}^\pm(\lambda)$  for  $\lambda \in \mathbb{C}_\pm$ .

ii) Given  $\mathcal{T}_1$  we determine  $\mathbf{b}^\pm(\lambda)$  and  $\mathbf{d}^\pm(\lambda)$  as

$$\mathbf{b}^\pm(\lambda) = \rho^\pm(\lambda)\mathbf{a}^\pm(\lambda), \quad \mathbf{d}^\pm(\lambda) = \mathbf{c}^\pm(\lambda)\rho^\pm(\lambda) \quad (90)$$

or if  $\mathcal{T}_2$  is known then

$$\mathbf{b}^\pm(\lambda) = \mathbf{a}^\pm(\lambda)\tau^\pm(\lambda), \quad \mathbf{d}^\pm(\lambda) = \tau^\pm(\lambda)\mathbf{c}^\pm(\lambda). \quad (91)$$

iii) The potential  $Q(x)$  can be recovered from  $\mathcal{T}_1$  by solving the RHP (54) and using equation (78).

Another method for reconstructing  $Q(x)$  from  $\mathcal{T}_j$  uses the interpretation of the ISM as generalized Fourier transform, see [1, 13, 21].

## 4. Finite Order Reductions of MMKdV Equations

In order that the potential  $Q(x, t)$  be relevant for a **DIII**-type symmetric space it must be of the form

$$Q(x, t) = \sum_{\alpha \in \Delta_1^+} (q_\alpha(x, t)E_\alpha + p_\alpha(x, t)E_{-\alpha}) \quad (92)$$

or, equivalently

$$Q(x, t) = \sum_{1 \leq i < j \leq r} \left( q_{ij}(x, t)E_{e_i+e_j} + p_{ij}(x, t)E_{-e_i-e_j} \right). \quad (93)$$

### 4.1. Hamiltonian Formulations for the Generic MMKdV Type Equations

Let us, before going into the non-trivial reductions, briefly discuss the Hamiltonian formulations for the generic (i.e., non-reduced) MMKdV type equations. It is well known (see [18] and the numerous references therein) that the class of these equations is generated by the so-called recursion operator  $\Lambda = 1/2(\Lambda_+ + \Lambda_-)$  which act on generic block-off-diagonal matrix valued function  $Z(x)$  by

$$\Lambda_\pm Z = i \operatorname{ad}_J^{-1} \left\{ \frac{dZ}{dx} + \left[ Q(x), \int_{\pm\infty}^x dy [Q(y), Z(y)] \right] \right\}. \quad (94)$$

Any nonlinear evolution equation (NLEE) integrable via the inverse scattering method applied to the Lax operator  $L$  (2) can be written in the form

$$i \operatorname{ad}_J^{-1} \frac{\partial Q}{\partial t} + 2f(\Lambda)Q(x, t) = 0 \quad (95)$$

where the function  $f(\lambda)$  is known as the dispersion law of this NLEE. The generic MMKdV equation is a member of this class and is obtained by choosing  $f(\lambda) = -4\lambda^3$ . If  $Q(x, t)$  is a solution to (95) then the corresponding scattering matrix satisfy the linear evolution equation

$$i \frac{dT}{dt} + f(\lambda)[J, T(\lambda, t)] = 0 \quad (96)$$



and vice versa. In particular from (96) there follows that  $\alpha^\pm(\lambda)$  and  $c^\pm(\lambda)$  are time-independent and therefore can be considered as generating functionals of integrals of motion for the NLEE.

If no additional reduction is imposed one can write each of the equations in (95) in Hamiltonian form. The corresponding Hamiltonian and symplectic form for the MMKV equation are given by

$$H_{\text{MMKdV}}^{(0)} = \frac{1}{4} \int_{-\infty}^{\infty} dx \left( \text{tr}(JQ_x Q_{xx}) - 3 \text{tr}(JQ^3 Q_x) \right) \quad (97)$$

$$\begin{aligned} \Omega^{(0)} &= \frac{1}{i} \int_{-\infty}^{\infty} dx \text{tr} \left( \text{ad}_J^{-1} \delta Q(x) \wedge \left[ J, \text{ad}_J^{-1} \delta Q(x) \right] \right) \\ &= \frac{1}{2i} \int_{-\infty}^{\infty} dx \text{tr} (J \delta Q(x) \wedge \delta Q(x)). \end{aligned} \quad (98)$$

The Hamiltonian can be identified as proportional to the fourth coefficient  $I_4$  in the asymptotic expansion of  $A^+(\lambda)$  (84) over the negative powers of  $\lambda$

$$A^+(\lambda) = \sum_{k=1}^{\infty} i I_k \lambda^{-k}. \quad (99)$$

This series of integrals of motion is known as the principal one. The first three of these integrals take the form

$$\begin{aligned} I_1 &= \frac{1}{4} \int_{-\infty}^{\infty} dx \text{tr}(Q^2(x, t)), & I_2 &= -\frac{i}{4} \int_{-\infty}^{\infty} dx \text{tr}(Q \text{ad}_J^{-1} Q_x) \\ I_3 &= -\frac{1}{8} \int_{-\infty}^{\infty} dx \text{tr}(QQ_{xx} + 2Q^4) \\ I_4 &= \frac{1}{32} \int_{-\infty}^{\infty} dx \left( \text{tr}(JQ_x Q_{xx}) - 3 \text{tr}(JQ^3 Q_x) \right). \end{aligned} \quad (100)$$

We will remind also another important result, namely that the gradient of  $I_k$  is expressed through  $\Lambda$  as

$$\nabla_{Q^T(x)} I_k = -\frac{1}{2} \Lambda^{k-1} Q(x, t). \quad (101)$$

Then the Hamiltonian equations written through  $\Omega^{(0)}$  and the Hamiltonian vector field  $X_{H^{(0)}}$  in the form

$$\Omega^{(0)}(\cdot, X_{H^{(0)}}) + \delta H^{(0)} = 0 \quad (102)$$

for  $H^{(0)}$  given by (97) coincides with the MMKdV equation.

An alternative way to formulate Hamiltonian equations of motion is to introduce along with the Hamiltonian the Poisson brackets on the phase space  $\mathcal{M}$  which is

the space of smooth functions taking values in  $\mathfrak{g}^{(0)}$  and vanishing fast enough for  $x \rightarrow \pm\infty$ , see (93). These brackets can be introduced by

$$\{F, G\}_{(0)} = i \int_{-\infty}^{\infty} dx \operatorname{tr} \left( \nabla_{Q^T(x)} F, \left[ J, \nabla_{Q^T(x)} G \right] \right). \quad (103)$$

Then the Hamiltonian equations of motions

$$\frac{dq_{ij}}{dt} = \{q_{ij}, H^{(0)}\}_{(0)}, \quad \frac{dp_{ij}}{dt} = \{p_{ij}, H^{(0)}\}_{(0)} \quad (104)$$

with the above choice for  $H^{(0)}$  again give the MMKdV equation.

Along with this standard Hamiltonian formulation there exist a whole hierarchy of them. This is a special property of the integrable NLEE. The hierarchy is generated again by the recursion operator and has the form

$$H_{\text{MMKdV}}^{(m)} = -8I_{4+m} \quad (105)$$

$$\Omega^{(m)} = \frac{1}{i} \int_{-\infty}^{\infty} dx \operatorname{tr} \left( \operatorname{ad}_J^{-1} \delta Q(x) \wedge \left[ J, \Lambda^m \operatorname{ad}_J^{-1} \delta Q(x) \right] \right). \quad (106)$$

Of course there is also a hierarchy of Poisson brackets

$$\{F, G\}_{(m)} = i \int_{-\infty}^{\infty} dx \operatorname{tr} \left( \nabla_{Q^T(x)} F, \left[ J, \Lambda^{-m} \nabla_{Q^T(x)} G \right] \right). \quad (107)$$

For a fixed value of  $m$  the Poisson bracket  $\{\cdot, \cdot\}_{(m)}$  is dual to the symplectic form  $\Omega^{(m)}$  in the sense that combined with a given Hamiltonian they produce the same equations of motion. Note that since  $\Lambda$  is an integro-differential operator in general it is not easy to evaluate explicitly its negative powers. Using this duality one can avoid the necessity to evaluate negative powers of  $\Lambda$ .

Then the analogs of (102) and (104) take the form

$$\Omega^{(m)}(\cdot, X_{H^{(m)}}) + \delta H^{(m)} = 0 \quad (108)$$

$$\frac{dq_{ij}}{dt} = \{q_{ij}, H^{(-m)}\}_{(m)}, \quad \frac{dp_{ij}}{dt} = \{p_{ij}, H^{(-m)}\}_{(m)} \quad (109)$$

where the hierarchy of Hamiltonians is given by

$$H^{(m)} = -4 \sum_k f_k I_{k+1-m}. \quad (110)$$

The equations (108) and (109) with the Hamiltonian  $H^{(m)}$  given by (110) will produce the NLEE (95) with dispersion law  $f(\lambda) = \sum_k f_k \lambda^k$  for any value of  $m$ .

**Remark 4.** It is a separate issue to prove that the hierarchies of symplectic structures and Poisson brackets have all the necessary properties. This is done using the spectral decompositions of the recursion operators  $\Lambda_{\pm}$  which are known also as the expansions over the “squared solutions” of  $L$ . We refer the reader to the review papers [14, 18] where he/she can find the proof of the completeness relation for

the “squared solutions” along with the proof that any two of the symplectic forms introduced above are compatible.

In the next two subsections we display new reductions of the MMKdV equations.

## 4.2. Class A Reductions Preserving $J$

The class A reductions can be applied also to the NLS type equations. The corresponding automorphisms  $C$  preserve  $J$ , i.e.,  $C^{-1}JC = J$  and are of the form

$$C^{-1}U^\dagger(x, \lambda^*)C = U(x, \lambda), \quad U(x, \lambda) = Q(x, t) - \lambda J \quad (111)$$

where  $J$  is an element of the Cartan subalgebra dual to the vector  $e_1 + e_2 + e_3 + e_4$ . In the typical representation of  $\mathfrak{so}(8)$   $U(x, \lambda)$  takes the form

$$U(x, t, \lambda) = \begin{pmatrix} \lambda \mathbb{1} & q(x, t) \\ p(x, t) & -\lambda \mathbb{1} \end{pmatrix} \quad (112)$$

$$q(x, t) = \begin{pmatrix} q_{14} & q_{13} & q_{12} & 0 \\ q_{24} & q_{23} & 0 & q_{12} \\ q_{34} & 0 & q_{23} & -q_{13} \\ 0 & q_{34} & -q_{24} & q_{14} \end{pmatrix}, \quad p(x, t) = \begin{pmatrix} p_{14} & p_{24} & p_{34} & 0 \\ p_{13} & p_{23} & 0 & p_{34} \\ p_{12} & 0 & p_{23} & -p_{24} \\ 0 & p_{12} & -p_{13} & p_{14} \end{pmatrix}.$$

**Remark 5.** The automorphisms that satisfy  $C^{-1}JC = J$  naturally preserve the eigensubspaces of  $\text{ad}_J$ ; in other words their action on the root space maps the subsets of roots  $\Delta_1^\pm$  onto themselves:  $C\Delta_1^\pm = \Delta_1^\pm$ .

We list here several inequivalent reductions of the ZS system. In the first one we choose  $C = C_0$  to be an element of the Cartan subgroup

$$C_0 = \exp \left( \pi i \sum_{k=1}^4 s_k h_k \right) \quad (113)$$

where  $s_k$  take the values 0 and 1. This condition means that  $C_0^2 = \mathbb{1}$ , so this will be a  $\mathbb{Z}_2$ -reduction, or involution. Then the first example of  $\mathbb{Z}_2$ -reduction is

$$C_0^{-1}Q^\dagger(x, t)C_0 = Q(x, t) \quad (114)$$

or in components

$$p_{ij} = \epsilon_{ij} q_{ij}^*, \quad \epsilon_{ij} = \epsilon_i \epsilon_j, \quad \epsilon_j = e^{\pi i s_j} = \pm 1. \quad (115)$$

Obviously  $\epsilon_j$  takes values  $\pm 1$  depending on whether  $s_j$  equals 0 or 1.

The next examples of  $\mathbb{Z}_2$ -reduction correspond to several choices of  $C$  as elements of the Weyl group eventually combined with the Cartan subgroup element  $C_0$

$$C_1 = S_{e_1 - e_2} S_{e_3 - e_4} C_0 \quad (116)$$

where  $S_{e_i - e_j}$  is the Weyl reflection related to the root  $e_i - e_j$ . Again we have a  $\mathbb{Z}_2$ -reduction, or an involution

$$C_1^{-1}Q^\dagger(x, t)C_1 = Q(x, t). \quad (117)$$

Written in components it takes the form ( $\epsilon_{12} = \epsilon_{34} = 1$ )

$$\begin{aligned} p_{12} &= -q_{12}^*, & p_{24} &= -\epsilon_{23}q_{13}^*, & p_{23} &= -\epsilon_{13}q_{14}^* \\ p_{14} &= -\epsilon_{13}q_{23}^*, & p_{13} &= -\epsilon_{23}q_{24}^*, & p_{34} &= -q_{34}^*. \end{aligned} \quad (118)$$

The corresponding Hamiltonian and symplectic form take the form

$$\begin{aligned} 8I_3 &= - \int_{-\infty}^{\infty} dx (\partial_x q_{12}^* \partial_x q_{12} + \partial_x q_{34}^* \partial_x q_{34} + \epsilon_{13}(q_{23}^* \partial_x q_{14} + \partial_x q_{14}^* \partial_x q_{23}) \\ &+ \epsilon_{23}(\partial_x q_{24}^* \partial_x q_{13} + \partial_x q_{13}^* \partial_x q_{24})) + \int_{-\infty}^{\infty} dx (\epsilon_{12}q_{12}^* q_{12} + q_{34}^* q_{34} \\ &+ \epsilon_{13}(q_{23}^* q_{14} + q_{24}^* q_{13}) + \epsilon_{23}(q_{14}^* q_{23} + q_{13}^* q_{24}))^2 \\ &+ \int_{-\infty}^{\infty} dx |q_{13}q_{24} + q_{12}q_{34} - q_{14}q_{23}|^2 \end{aligned} \quad (119)$$

$$\begin{aligned} \Omega^{(0)} &= \frac{1}{i} \int_{-\infty}^{\infty} dx (\delta q_{12}^* \wedge \delta q_{12} + \epsilon_{13}(q_{23}^* \wedge \delta q_{14} + \delta q_{14}^* \wedge \delta q_{23}) \\ &+ \epsilon_{23}(\delta q_{24}^* \wedge \delta q_{13} + \delta q_{13}^* \wedge \delta q_{24}) + \delta q_{34}^* \wedge \delta q_{34}). \end{aligned} \quad (120)$$

Another inequivalent examples of  $\mathbb{Z}_2$ -reduction corresponds to

$$C_2 = S_{e_1 - e_2} C_0. \quad (121)$$

The involution is

$$C_2^{-1}Q^\dagger(x, t)C_2 = Q(x, t) \quad (122)$$

or in components it takes the form

$$\begin{aligned} p_{12} &= -\epsilon_{12}q_{12}^*, & p_{24} &= -\epsilon_{13}q_{14}^*, & p_{23} &= -\epsilon_{14}q_{13}^* \\ p_{14} &= -\epsilon_{23}q_{24}^*, & p_{13} &= -\epsilon_{24}q_{23}^*, & p_{34} &= -\epsilon_{34}q_{34}^*. \end{aligned} \quad (123)$$

As a consequence we get

$$\begin{aligned} 8I_3 &= - \int_{-\infty}^{\infty} dx (\epsilon_{12}\partial_x q_{12}^* \partial_x q_{12} + \epsilon_{34}\partial_x q_{34}^* \partial_x q_{34} + \epsilon_{14}\partial_x q_{13}^* \partial_x q_{23} \\ &+ \epsilon_{23}\partial_x q_{24}^* \partial_x q_{14} + \epsilon_{24}\partial_x q_{23}^* \partial_x q_{13} + \epsilon_{13}\partial_x q_{14}^* \partial_x q_{24}) \\ &+ \int_{-\infty}^{\infty} dx (\epsilon_{12}|q_{12}|^2 + \epsilon_{34}|q_{34}|^2 + \epsilon_{23}q_{24}^* q_{14} + \epsilon_{24}q_{23}^* q_{13} + \epsilon_{14}q_{13}^* q_{23} \\ &+ \epsilon_{13}q_{14}^* q_{24})^2 + \epsilon_{12}\epsilon_{34} \int_{-\infty}^{\infty} dx |q_{13}q_{24} + q_{12}q_{34} - q_{14}q_{23}|^2 \end{aligned} \quad (124)$$

$$\begin{aligned} \Omega^{(0)} &= \frac{1}{i} \int_{-\infty}^{\infty} dx (\epsilon_{12}\delta q_{12}^* \wedge \delta q_{12} + \epsilon_{34}q_{34}^* \wedge \delta q_{34} + \epsilon_{23}\delta q_{24}^* \wedge \delta q_{14} \\ &+ \epsilon_{14}\delta q_{13}^* \wedge \delta q_{23} + \epsilon_{24}\delta q_{23}^* \wedge \delta q_{13} + \epsilon_{13}\delta q_{14}^* \wedge \delta q_{24}). \end{aligned} \quad (125)$$

Next we consider a  $\mathbb{Z}_3$ -reduction generated by  $C_3 = S_{e_1-e_2}S_{e_2-e_3}$  which also maps  $J$  into  $J$ . It splits each of the sets  $\Delta_1^\pm$  into two orbits which are

$$\begin{aligned} (O)_1^\pm &= \{\pm(e_1 + e_2), \pm(e_2 + e_3), \pm(e_1 + e_3)\} \\ (O)_2^\pm &= \{\pm(e_1 + e_4), \pm(e_2 + e_4), \pm(e_3 + e_4)\}. \end{aligned} \quad (126)$$

In order to be more efficient we make use of the following basis in  $\mathfrak{g}^{(0)}$

$$\mathcal{E}_\alpha^{(k)} = \sum_{p=0}^2 \omega^{-kp} C_3^{-k} E_\alpha C_3^k, \quad \mathcal{F}_\alpha^{(k)} = \sum_{p=0}^2 \omega^{-kp} C_3^{-k} F_\alpha C_3^k \quad (127)$$

where  $\omega = \exp(2\pi i/3)$  and  $\alpha$  takes values  $e_1 + e_2$  and  $e_1 + e_4$ . Obviously

$$C_3^{-1} \mathcal{E}_\alpha^{(k)} C_3 = \omega^k \mathcal{E}_\alpha^{(k)}, \quad C_3^{-1} \mathcal{F}_\alpha^{(k)} C_3 = \omega^k \mathcal{F}_\alpha^{(k)}. \quad (128)$$

In addition, since  $\omega^* = \omega^{-1}$  we get  $(\mathcal{E}_\alpha^{(0)})^\dagger = \mathcal{F}_\alpha^{(0)}$  and  $(\mathcal{E}_\alpha^{(k)})^\dagger = \mathcal{F}_\alpha^{(3-k)}$  for  $k = 1, 2$ . Then we introduce the potential

$$Q(x, t) = \sum_{k=0}^3 \sum_{\alpha} \left( q_\alpha^{(k)}(x, t) \mathcal{E}_\alpha^{(k)} + p_\alpha^{(k)}(x, t) \mathcal{F}_\alpha^{(k)} \right). \quad (129)$$

In view of equation (128) the reduction condition (29) leads to the following relations between the coefficients

$$\begin{aligned} p_{12}^{(0)} &= (q_{12}^{(0)})^*, & p_{12}^{(k)} &= \omega^k (q_{12}^{(3-k)})^*, & q_{12}^{(k)} &= \omega^k (p_{12}^{(3-k)})^* \\ p_{14}^{(0)} &= (q_{14}^{(0)})^*, & p_{14}^{(k)} &= \omega^k (q_{14}^{(3-k)})^*, & q_{14}^{(k)} &= \omega^k (p_{14}^{(3-k)})^* \end{aligned} \quad (130)$$

where  $k = 1, 2$ . It is easy to check that from the conditions (130) there follows  $p_{12}^{(k)} = q_{12}^{(k)} = p_{14}^{(k)} = q_{14}^{(k)} = 0$ . So we are left with only one pair of independent functions  $q_{12}^{(0)}$  and  $q_{14}^{(0)}$  and their complex conjugate  $p_{14}^{(0)}$ ,  $q_{14}^{(0)}$ .

Similarly the reduction (30) leads to

$$\begin{aligned} q_{12}^{(0)} &= -(q_{12}^{(0)})^*, & q_{12}^{(k)} &= -\omega^{3-k} (q_{12}^{(3-k)})^*, & q_{14}^{(k)} &= -\omega^{3-k} (q_{14}^{(3-k)})^* \\ p_{12}^{(0)} &= -(p_{12}^{(0)})^*, & p_{12}^{(k)} &= -\omega^{3-k} (p_{12}^{(3-k)})^*, & p_{14}^{(k)} &= -\omega^{3-k} (p_{14}^{(3-k)})^* \end{aligned} \quad (131)$$

where  $k = 1, 2$ . Again from the conditions (131) it follows that  $p_{12}^{(k)} = q_{12}^{(k)} = p_{14}^{(k)} = q_{14}^{(k)} = 0$ . So we are left with two pairs of purely imaginary independent functions:  $q_{12}^{(0)}$ ,  $q_{14}^{(0)}$  and  $p_{12}^{(0)}$ ,  $q_{14}^{(0)}$ .

The corresponding Hamiltonian and symplectic forms are obtained from the slightly more general formulae below by imposing the constraints (130) and (131). Here for simplicity we skip the upper zeroes in  $q_{ij}$  and  $p_{ij}$

$$H_{\text{MMKdV}} = \frac{1}{6} \int_{-\infty}^{\infty} dx \left( \partial_x^2 q_{12} \partial_x p_{12} - \partial_x q_{12} \partial_x^2 p_{12} + \partial_x^2 q_{14} \partial_x p_{14} - \partial_x q_{14} \partial_x^2 p_{14} \right) \\ - \frac{1}{12} \int_{-\infty}^{\infty} dx \left( p_{12}^2 q_{12}^2 \partial_x - p_{12}^2 \partial_x q_{12}^2 + q_{14}^2 \partial_x p_{14}^2 - p_{12}^2 \partial_x q_{12}^2 \right) \quad (132)$$

$$8I_3 = \frac{4}{3} \int_{-\infty}^{\infty} dx \left( \partial_x q_{12} \partial_x p_{12} + \partial_x q_{14} \partial_x p_{14} \right) - \frac{8}{9} \int_{-\infty}^{\infty} dx \left( q_{14}^2 p_{14}^2 + q_{12}^2 p_{12}^2 \right) \quad (133)$$

$$\Omega^{(0)} = \frac{4}{3} \int_{-\infty}^{\infty} dx \left( \delta q_{14} \wedge \delta p_{14} + \delta q_{12} \wedge \delta p_{12} \right)$$

i.e., in this case we get two decoupled mKdV equations.

The  $\mathbb{Z}_4$ -reduction generated by  $C_4 = S_{e_1-e_2} S_{e_2-e_3} S_{e_3-e_4}$  also maps  $J$  into  $J$ . It splits each of the sets  $\Delta_1^\pm$  into two orbits which are

$$(O)_1^\pm = \{ \pm(e_1 + e_2), \pm(e_2 + e_3), \pm(e_3 + e_4), \pm(e_1 + e_4) \} \\ (O)_2^\pm = \{ \pm(e_1 + e_3), \pm(e_2 + e_4) \}. \quad (134)$$

Again we make use of a convenient basis in  $\mathfrak{g}^{(0)}$

$$\mathcal{E}_\alpha^{(k)} = \sum_{p=0}^3 i^{-kp} C_4^{-k} E_\alpha C_4^k, \quad \mathcal{F}_\alpha^{(k)} = \sum_{p=0}^3 i^{-kp} C_4^{-k} F_\alpha C_4^k \quad (135)$$

where  $\alpha$  takes values  $e_1 + e_2$  and  $e_1 + e_3$ . Obviously

$$C_4^{-1} \mathcal{E}_\alpha^{(k)} C_4 = i^k \mathcal{E}_\alpha^{(k)}, \quad C_4^{-1} \mathcal{F}_\alpha^{(k)} C_4 = i^k \mathcal{F}_\alpha^{(k)}. \quad (136)$$

and in addition,  $(\mathcal{E}_\alpha^{(0)})^\dagger = \mathcal{F}_\alpha^{(0)}$ ,  $(\mathcal{E}_\alpha^{(k)})^\dagger = \mathcal{F}_\alpha^{(4-k)}$  and  $(\mathcal{E}_\alpha^{(k)})^* = \mathcal{E}_\alpha^{(4-k)}$  for  $k = 1, 2, 3$ . Then we introduce the potential

$$Q(x, t) = \sum_{k=0}^3 \sum_{\alpha} \left( q_\alpha^{(k)}(x, t) \mathcal{E}_\alpha^{(k)} + p_\alpha^{(k)}(x, t) \mathcal{F}_\alpha^{(k)} \right). \quad (137)$$

In view of equation (136) the reduction condition (29) leads to the following relations between the coefficients

$$p_\alpha^{(0)} = (q_\alpha^{(0)})^*, \quad p_\alpha^{(k)} = i^k (q_\alpha^{(4-k)})^*, \quad q_\alpha^{(k)} = i^k (p_\alpha^{(4-k)})^* \quad (138)$$

for  $k = 1, 2, 3$ . Here  $p_\alpha, q_\alpha$  coincide with  $p_{12}, q_{12}$  (respectively  $p_{13}, q_{13}$ ) for  $\alpha = e_1 + e_2$  (respectively  $\alpha = e_1 + e_3$ ). Analogously the reduction (30) gives

$$q_\alpha^{(0)} = -(p_\alpha^{(0)})^*, \quad p_\alpha^{(0)} = -(q_\alpha^{(0)})^* \\ q_\alpha^{(k)} = -i^k (q_\alpha^{(4-k)})^*, \quad p_\alpha^{(k)} = -i^k (p_\alpha^{(4-k)})^* \quad (139)$$

for  $k = 1, 2, 3$ . Both conditions (138) and (139) lead to  $p_{12}^{(k)} = q_{12}^{(k)} = p_{14}^{(k)} = q_{14}^{(k)} = 0$  for  $k = 1, 3$ . In addition, it comes up that  $\mathcal{E}_{13}^{(0)} = \mathcal{E}_{13}^{(2)} = \mathcal{F}_{13}^{(0)} = \mathcal{F}_{13}^{(2)}$ . So

we are left with only two pairs of independent functions  $p_{12}^{(0)}, q_{12}^{(0)}$  and  $p_{12}^{(2)}, q_{12}^{(2)}$ . We provide below slightly more general formulae for the corresponding Hamiltonian and symplectic form which are obtained by imposing the constraints (138) or (139); again for simplicity of notations we skip the upper zeroes in  $q_{ij}^{(0)}$  and  $p_{ij}^{(0)}$  and replace  $q_{ij}^{(2)}$  and  $p_{ij}^{(2)}$  by  $\tilde{q}_{ij}$  and  $\tilde{p}_{ij}$

$$H_{\text{MMKdV}} = \frac{1}{4} \int_{-\infty}^{\infty} dx \left( \partial_x^2 q_{12} \partial_x p_{12} - \partial_x q_{12} \partial_x^2 p_{12} + \partial_x^2 \tilde{q}_{12} \partial_x \tilde{p}_{12} - \partial_x \tilde{q}_{12} \partial_x^2 \tilde{p}_{12} \right) - \frac{3}{32} \int_{-\infty}^{\infty} dx \left( \left( \partial_x (p_{12}^2) + \partial_x (\tilde{p}_{12}^2) \right) (q_{12}^2 + \tilde{q}_{12}^2) + p_{12} \tilde{p}_{12} \partial_x (q_{12} \tilde{q}_{12}) \right) \quad (140)$$

$$+ \left( \partial_x (q_{12}^2) + \partial_x (\tilde{q}_{12}^2) \right) (p_{12}^2 + \tilde{p}_{12}^2) + q_{12} \tilde{q}_{12} \partial_x (p_{12} \tilde{p}_{12}) \\ 8I_3 = 2 \int_{-\infty}^{\infty} dx \left( \partial_x q_{12} \partial_x p_{12} + \partial_x \tilde{q}_{12} \partial_x \tilde{p}_{12} \right) - \int_{-\infty}^{\infty} dx \left( (q_{12} p_{12} + \tilde{q}_{12} \tilde{p}_{12})^2 + (q_{12} \tilde{p}_{12} + \tilde{q}_{12} p_{12})^2 \right) \quad (141)$$

$$\Omega^{(0)} = 2 \int_{-\infty}^{\infty} dx \left( \delta q_{12} \wedge \delta p_{12} + \delta \tilde{q}_{12} \wedge \delta \tilde{p}_{12} \right). \quad (142)$$

Now we get two specially coupled mKdV-type equations.

### 4.3. Class B Reductions Mapping $J$ into $-J$

The class B reductions of the Zakharov-Shabat system change the sign of  $J$ , i.e.,  $C^{-1}JC = -J$ ; therefore we must have also  $\lambda \rightarrow -\lambda$ .

**Remark 6.** Note that  $\text{ad}_J$  has three eigensubspaces  $\mathcal{W}_a$ ,  $a = 0, \pm 1$  corresponding to the eigenvalues 0 and  $\pm 2$ . The automorphisms that satisfy  $C^{-1}JC = -J$  naturally preserve the eigensubspace  $\mathcal{W}_0$  but map  $\mathcal{W}_{-1}$  onto  $\mathcal{W}_1$  and vice versa. In other words their action on the root space maps the subset of roots  $\Delta_1^+$  onto  $\Delta_1^-$  and vice versa:  $C\Delta_1^\pm = \Delta_1^\mp$ .

Here we first consider

$$C_5 = S_{e_1 - e_2} S_{e_1 + e_2} S_{e_3 - e_4} S_{e_3 + e_4} C_0. \quad (143)$$

Obviously the product of the above four Weyl reflections will change the sign of  $J$ . Its effect on  $Q$  in components reads

$$q_{ij} = -\epsilon_{ij} q_{ij}^*, \quad p_{ij} = -\epsilon_{ij} p_{ij}^*, \quad \epsilon_{ij} = \epsilon_i \epsilon_j \quad (144)$$

i.e., some of the components of  $Q$  become purely imaginary, others may become real depending on the choice of the signs  $\epsilon_j$ .

There is no  $\mathbb{Z}_3$  reduction for the  $\mathfrak{so}(8)$  MMKdV that maps  $J$  into  $-J$ . So we go directly to the  $\mathbb{Z}_4$ -reduction generated by  $C_6 = S_{e_1+e_2}S_{e_2+e_3}S_{e_3+e_4}$  which maps  $J$  into  $-J$ . The orbits of  $C_3$  are

$$\begin{aligned} (O)_1^\pm &= \{\pm(e_1 + e_2), \mp(e_2 + e_3), \pm(e_3 + e_4), \mp(e_1 + e_4)\} \\ (O)_2^\pm &= \{\pm(e_1 + e_3), \mp(e_2 + e_4)\}. \end{aligned} \quad (145)$$

Again we make use of a convenient basis in  $\mathfrak{g}^{(0)}$

$$\mathcal{E}_\alpha^{(k)} = \sum_{p=0}^3 i^{-kp} C_6^{-k} E_\alpha C_6^k, \quad \mathcal{F}_\alpha^{(k)} = \sum_{p=0}^3 i^{-kp} C_6^{-k} F_\alpha C_6^k \quad (146)$$

where  $\alpha$  takes values  $e_1 + e_2$  and  $e_1 + e_3$ . Obviously

$$C_6^{-1} \mathcal{E}_\alpha^{(k)} C_6 = i^k \mathcal{E}_\alpha^{(k)}, \quad C_6^{-1} \mathcal{F}_\alpha^{(k)} C_6 = i^k \mathcal{F}_\alpha^{(k)} \quad (147)$$

where again  $(\mathcal{E}_\alpha^{(0)})^\dagger = \mathcal{F}_\alpha^{(0)}$ ,  $(\mathcal{E}_\alpha^{(k)})^\dagger = \mathcal{F}_\alpha^{(4-k)}$  and  $(\mathcal{E}_\alpha^{(k)})^* = \mathcal{E}_\alpha^{(4-k)}$  for  $k = 1, 2, 3$ . Then we introduce the potential

$$Q(x, t) = \sum_{k=0}^3 \sum_{\alpha} \left( q_\alpha^{(k)}(x, t) \mathcal{E}_\alpha^{(k)} + p_\alpha^{(k)}(x, t) \mathcal{F}_\alpha^{(k)} \right). \quad (148)$$

In view of equation (147) the reduction condition (29) leads to the following relations between the coefficients

$$p_\alpha^{(0)} = (q_\alpha^{(0)})^*, \quad p_\alpha^{(k)} = i^k (q_\alpha^{(4-k)})^*, \quad q_\alpha^{(k)} = i^k (p_\alpha^{(4-k)})^*, \quad k = 1, 2, 3 \quad (149)$$

while the reduction (30) gives

$$\begin{aligned} q_\alpha^{(0)} &= -(q_\alpha^{(0)})^*, & q_\alpha^{(k)} &= -i^k (q_\alpha^{(4-k)})^*, & k &= 1, 2, 3 \\ p_\alpha^{(0)} &= -(p_\alpha^{(0)})^*, & p_\alpha^{(k)} &= -i^k (p_\alpha^{(4-k)})^*, & k &= 1, 2, 3 \end{aligned} \quad (150)$$

where  $\alpha$  takes values  $e_1 + e_2$  and  $e_1 + e_3$ . From the conditions (149) it follows that  $p_{12}^{(k)} = q_{12}^{(k)} = p_{13}^{(k)} = q_{13}^{(k)} = 0$  for  $k = 1, 3$ . In addition, however, it comes up that  $\mathcal{E}_{13}^{(0)} = \mathcal{E}_{13}^{(2)} = \mathcal{F}_{13}^{(0)} = \mathcal{F}_{13}^{(2)}$ . So we are left with only two pairs of independent functions  $p_{12}^{(0)}, q_{12}^{(0)}$  and  $p_{12}^{(2)}, q_{12}^{(2)}$ . We provide below slightly more general formulae for the corresponding Hamiltonian and symplectic form which are obtained by imposing the constraints (149) or (150) and again for simplicity we skip the upper zeroes in  $q_{ij}$  and  $p_{ij}$  and replace  $q_{12}^{(2)}$  and  $p_{12}^{(2)}$  by  $\tilde{q}_{12}$  and  $\tilde{p}_{12}$



$$\begin{aligned}
H_{\text{MMKdV}} &= \frac{1}{4} \int_{-\infty}^{\infty} dx \left( \partial_x^2 q_{12} \partial_x p_{12} - \partial_x q_{12} \partial_x^2 p_{12} + \partial_x^2 \tilde{q}_{12} \partial_x \tilde{p}_{12} - \partial_x \tilde{q}_{12} \partial_x^2 \tilde{p}_{12} \right) \\
&\quad - \frac{3}{32} \int_{-\infty}^{\infty} dx \left( (\partial_x(p_{12}^2) + \partial_x(\tilde{q}_{12}^2)) (q_{12}^2 + \tilde{p}_{12}^2) + \partial_x(q_{12}\tilde{p}_{12}) p_{12}\tilde{q}_{12} \right) \\
&\quad - \left( \partial_x(q_{12}^2) + \partial_x(\tilde{p}_{12}^2) \right) (p_{12}^2 + \tilde{q}_{12}^2) - \partial_x(p_{12}\tilde{q}_{12}) q_{12}\tilde{p}_{12}
\end{aligned} \tag{151}$$

$$\begin{aligned}
8I_3 &= 2 \int_{-\infty}^{\infty} (\partial_x q_{12} \partial_x p_{12} + \partial_x \tilde{q}_{12} \partial_x \tilde{p}_{12}) \\
&\quad - \int_{-\infty}^{\infty} dx \left( (q_{12}p_{12} + \tilde{q}_{12}\tilde{p}_{12})^2 + (q_{12}\tilde{p}_{12} + \tilde{q}_{12}p_{12})^2 \right)
\end{aligned} \tag{152}$$

$$\Omega^{(0)} = 2 \int_{-\infty}^{\infty} dx (\delta q_{12} \wedge \delta p_{12} + \delta \tilde{q}_{12} \wedge \delta \tilde{p}_{12}). \tag{153}$$

Now we get four specially coupled mKdV-type equations given by

$$\begin{aligned}
\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(q_0 q_2 + p_0 p_2) \partial_x p_2 + \frac{3}{2}(p_2 q_2 + q_0 p_0) \partial_x q_0 &= 0 \\
\partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_2 p_2 + q_0 p_0) \partial_x q_2 + \frac{3}{2}(q_0 q_2 + p_0 p_2) \partial_x p_0 &= 0 \\
\partial_t p_0 + \partial_x^3 p_0 + \frac{3}{2}(q_2 p_2 + q_0 p_0) \partial_x p_0 + \frac{3}{2}(q_0 q_2 + p_0 p_2) \partial_x q_2 &= 0 \\
\partial_t p_2 + \partial_x^3 p_2 + \frac{3}{2}(q_2 q_0 + p_0 p_2) \partial_x q_0 + \frac{3}{2}(q_0 p_0 + q_2 p_2) \partial_x p_2 &= 0
\end{aligned} \tag{154}$$

where we use for simplicity

$$q_{12} = q_0, \quad \tilde{q}_{12} = q_2, \quad p_{12} = p_0, \quad \tilde{p}_{12} = p_2 \tag{155}$$

and with second reduction (149)  $p_0 = q_0^*$  and  $p_2 = -q_2^*$  we have

$$\begin{aligned}
\partial_t q_0 + \partial_x^3 q_0 - \frac{3}{2}(q_0 q_2 - q_0^* q_2^*) \partial_x q_2^* - \frac{3}{2}(q_2^* q_2 - q_0 q_0^*) \partial_x q_0 &= 0 \\
\partial_t q_2 + \partial_x^3 q_2 - \frac{3}{2}(q_2 q_2^* - q_0 q_0^*) \partial_x q_2 + \frac{3}{2}(q_0 q_2 - q_0^* q_2^*) \partial_x q_0^* &= 0.
\end{aligned} \tag{156}$$

Obviously in the system (156) we can put both  $q_0, q_2$  real with the result

$$\begin{aligned}
\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(q_0^2 - q_2^2) \partial_x q_0 &= 0 \\
\partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_0^2 - q_2^2) \partial_x q_2 &= 0.
\end{aligned} \tag{157}$$

The reduction (150) means that  $q_\alpha^{(0)} = iq_0^\vee$ ,  $p_\alpha^{(0)} = ip_0^\vee$ ,  $q_\alpha^{(2)} = q_2^\vee$ ,  $p_\alpha^{(2)} = ip_2^\vee$  with real valued  $p_i^\vee$ ,  $q_i^\vee$ ,  $i = 0, 2$ . Thus we get

$$\begin{aligned}
 \partial_t q_0^\vee + \partial_x^3 q_0^\vee + \frac{3}{2}(q_0^\vee q_2^\vee + p_0^\vee p_2^\vee) \partial_x p_2^\vee + \frac{3}{2}(p_2^\vee q_2^\vee - q_0^\vee p_0^\vee) \partial_x q_0^\vee &= 0 \\
 \partial_t q_2^\vee + \partial_x^3 q_2^\vee + \frac{3}{2}(q_2^\vee p_2^\vee - q_0^\vee p_0^\vee) \partial_x q_2^\vee - \frac{3}{2}(q_0^\vee q_2^\vee + p_0^\vee p_2^\vee) \partial_x p_0^\vee &= 0 \\
 \partial_t p_0^\vee + \partial_x^3 p_0^\vee + \frac{3}{2}(q_2^\vee p_2^\vee - q_0^\vee p_0^\vee) \partial_x p_0^\vee + \frac{3}{2}(q_0^\vee q_2^\vee + p_0^\vee p_2^\vee) \partial_x q_2^\vee &= 0 \\
 \partial_t p_2^\vee + \partial_x^3 p_2^\vee - \frac{3}{2}(q_2^\vee q_0^\vee + p_0^\vee p_2^\vee) \partial_x q_0^\vee - \frac{3}{2}(q_0^\vee p_0^\vee - q_2^\vee p_2^\vee) \partial_x p_2^\vee &= 0.
 \end{aligned} \tag{158}$$

The class B reductions also render all the symplectic forms and Hamiltonians in the hierarchy real-valued. They allow to render the corresponding systems of MMKdV equations into ones involving only real-valued fields. ‘‘Half’’ of the Hamiltonian structures do not survive these reductions and become degenerate. This holds true for all symplectic forms  $\Omega^{(2m)}$  and integrals of motion  $I_{2m}$  with even indices. However the other ‘‘half’’ of the hierarchy with  $\Omega^{(2m+1)}$  and integrals of motion  $I_{2m+1}$  remains and provides Hamiltonian properties of the MMKdV.

#### 4.4. Effects of Class A Reductions on the Scattering Data

Let in this subsection all automorphisms  $C_i$  are of class A. Therefore acting on the root space they preserve the vector  $\sum_{k=1}^r e_k$  which is dual to  $J$ , and as a consequence, the corresponding Weyl group elements map the subset of roots  $\Delta_1^+$  onto itself.

**Remark 7.** An important consequence of this is that  $C_i$  will map block-upper-triangular (respectively block-lower-triangular) matrices like in equation (46) into matrices with the same block structure. The block-diagonal matrices will be mapped again into block-diagonal ones.

From the reduction conditions (29)–(32) one gets, in the limit  $x \rightarrow \infty$  that

$$\begin{aligned}
 \text{a) } C_1((\kappa_1(\lambda)J)^\dagger) &= \lambda J, & \text{b) } C_2((\kappa_2(\lambda)J)^T) &= -\lambda J \\
 \text{c) } C_3((\kappa_3(\lambda)J)^*) &= \lambda J, & \text{d) } C_4((\kappa_4(\lambda)J)) &= \lambda J.
 \end{aligned} \tag{159}$$

Using equation (159) and  $C_i(J) = J$  one finds that

$$\text{a) } \kappa_1(\lambda) = \lambda^*, \quad \text{b) } \kappa_2(\lambda) = -\lambda, \quad \text{c) } \kappa_3(\lambda) = -\lambda^*, \quad \text{d) } \kappa_4(\lambda) = \lambda. \tag{160}$$

It remains to take into account that the reductions (29)–(32) for the potentials of  $L$  lead to the following constraints on the scattering matrix  $T(\lambda)$

$$\begin{aligned} \text{a)} \quad C_1(T^\dagger(\lambda^*)) &= \hat{T}(\lambda), & \text{b)} \quad C_2(T^T(-\lambda)) &= \hat{T}(\lambda) \\ \text{c)} \quad C_3((T^*(-\lambda^*)) &= T(\lambda), & \text{d)} \quad C_4(T(\lambda)) &= T(\lambda). \end{aligned} \quad (161)$$

These results along with remark 7 lead to the following results for the generalized Gauss factors of  $T(\lambda)$

$$\begin{aligned} \text{a)} \quad C_1(\mathbf{S}^{+, \dagger}(\lambda^*)) &= \hat{\mathbf{S}}^-(\lambda), & C_1(\mathbf{T}^{-, \dagger}(\lambda^*)) &= \hat{\mathbf{T}}^+(\lambda) \\ \text{b)} \quad C_2(\mathbf{S}^{+, T}(-\lambda)) &= \hat{\mathbf{S}}^-(\lambda), & C_2(\mathbf{T}^{-, T}(-\lambda)) &= \hat{\mathbf{T}}^+(\lambda) \\ \text{c)} \quad C_3(\mathbf{S}^{\pm, *(-\lambda^*)}) &= \mathbf{S}^\pm(\lambda), & C_3(\mathbf{T}^{\pm, *(-\lambda^*)}) &= \mathbf{T}^\pm(\lambda) \\ \text{d)} \quad C_4(\mathbf{S}^\pm(\lambda)) &= \mathbf{S}^\pm(\lambda), & C_4(\mathbf{T}^\pm(\lambda)) &= \mathbf{T}^\pm(\lambda) \end{aligned} \quad (162)$$

and

$$\begin{aligned} \text{a)} \quad C_1(D^{+, \dagger}(\lambda^*)) &= \hat{D}^-(\lambda), & \text{b)} \quad C_2(D^{+, T}(-\lambda)) &= \hat{D}^-(\lambda) \\ \text{c)} \quad C_3(D^{\pm, *(-\lambda)}) &= D^\pm(\lambda), & \text{d)} \quad C_4(D^\pm(\lambda)) &= D^\pm(\lambda). \end{aligned} \quad (163)$$

#### 4.5. Effects of Class B Reductions on the Scattering Data

In this subsection all automorphisms  $C_i$  are of class B. Therefore acting on the root space they map the vector  $\sum_{k=1}^r e_k$  dual to  $J$  into  $-\sum_{k=1}^r e_k$ . As a consequence, the corresponding Weyl group elements map the subset of roots  $\Delta_1^+$  onto  $\Delta_1^- \equiv -\Delta_1^+$ .

**Remark 8.** An important consequence of this is that  $C_i$  will map block-upper-triangular into block-lower-triangular matrices like in equation (46) and vice versa. The block-diagonal matrices will be mapped again into block-diagonal ones.

Now equation (159) with  $C_i(J) = -J$  leads to

$$\begin{aligned} \text{a)} \quad \kappa_1(\lambda) &= -\lambda^*, & \text{b)} \quad \kappa_2(\lambda) &= \lambda \\ \text{c)} \quad \kappa_3(\lambda) &= \lambda^*, & \text{d)} \quad \kappa_4(\lambda) &= -\lambda. \end{aligned} \quad (164)$$

The reductions (29)–(32) for the potentials of  $L$  lead to the following constraints on the scattering matrix  $T(\lambda)$

$$\begin{aligned} \text{a)} \quad C_1(T^\dagger(-\lambda^*)) &= \hat{T}(\lambda), & \text{b)} \quad C_2(T^T(\lambda)) &= \hat{T}(\lambda) \\ \text{c)} \quad C_3((T^*(\lambda^*)) &= T(\lambda), & \text{d)} \quad C_4(T(-\lambda)) &= T(\lambda). \end{aligned} \quad (165)$$

Then along with Remark 8 we find the following results for the generalized Gauss factors of  $T(\lambda)$

$$\begin{aligned}
 \text{a)} \quad & C_1(\mathbf{S}^{\pm, \dagger}(-\lambda^*)) = \hat{\mathbf{S}}^{\pm}(\lambda), & C_1(\mathbf{T}^{\pm, \dagger}(-\lambda^*)) &= \hat{\mathbf{T}}^{\pm}(\lambda) \\
 \text{b)} \quad & C_2(\mathbf{S}^{\pm, T}(\lambda)) = \hat{\mathbf{S}}^{\pm}(\lambda), & C_2(\mathbf{T}^{\pm, T}(\lambda)) &= \hat{\mathbf{T}}^{\pm}(\lambda) \\
 \text{c)} \quad & C_3(\mathbf{S}^{+, *}(\lambda^*)) = \mathbf{S}^{-}(\lambda), & C_3(\mathbf{T}^{-, *}(\lambda^*)) &= \mathbf{T}^{+}(\lambda) \\
 \text{d)} \quad & C_4(\mathbf{S}^{+}(-\lambda)) = \mathbf{S}^{-}(\lambda), & C_4(\mathbf{T}^{-}(-\lambda)) &= \mathbf{T}^{+}(\lambda)
 \end{aligned} \tag{166}$$

and

$$\begin{aligned}
 \text{a)} \quad & C_1(D^{\pm, \dagger}(-\lambda^*)) = \hat{D}^{\pm}(\lambda), & \text{b)} \quad & C_2(D^{\pm, T}(\lambda)) = \hat{D}^{\pm}(\lambda) \\
 \text{c)} \quad & C_3(D^{+, *}(\lambda^*)) = D^{-}(\lambda), & \text{d)} \quad & C_4(D^{+}(-\lambda)) = D^{-}(\lambda).
 \end{aligned} \tag{167}$$

## 5. Conclusions

In conclusion, we have considered the reduced multicomponent MKdV equations associated with the **DIII**-type symmetric spaces using Mikhailov's reduction group. Several examples of such nontrivial reductions leading to new MMKdV systems related to the  $\mathfrak{so}(8)$  Lie algebra are given. In particular we provide examples with reduction groups isomorphic to  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$  and derive their effects on the scattering matrix, the minimal sets of scattering data and on the hierarchy of Hamiltonian structures. These results can be generalized also for other types of symmetric spaces.

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