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ONE REMARK ON VARIATIONAL PROPERTIES OF GEODESICS IN PSEUDORIEMANNIAN AND GENERALIZED FINSLER SPACES

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Abstract. A new variational property of geodesics in (pseudo-)Riemannian and Finsler spaces has been found.

1. Introduction

Let us assume an n-dimensional **Finsler space** F_n with local coordinates $x \equiv (x^1, \dots, x^n)$ on the underlying manifold M_n, and a (positive definite) metric form with local expression

$$\mathrm{d}s^2 = g_{ij}(x, \dot{x})\mathrm{d}x^i\mathrm{d}x^j\,. \tag{1}$$

Here $g_{ij}(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n)$ are components of the metric tensor, and $(x, \dot{x})$ denote adapted local coordinates on the tangent bundle TM, i.e., $(\dot{x}^1, \dots, \dot{x}^n)$ are coordinates of the "tangent vector" $\dot{x}$ at x. Metric depends on "positions" and "velocities" in general.

In the Finsler space F_n there exists a (fundamental) function $F(x, \dot{x})$ which is homogeneous of the second degree in $\dot{x}^i$ and satisfies

$$g_{ij}(x, \dot{x}) = \frac{\partial^2 F(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}\,.$$

Particularly, the equality

$$F(x, \dot{x}) = g_{ij}(x, \dot{x})\mathrm{d}x^i\mathrm{d}x^j$$

holds [3]. As it is well known, in the particular case when components of the metric tensor depend only on position coordinates (i.e., are independent of "velocity coordinates" $\dot{x}$) the Finsler space F_n turns out to be a **Riemannian space** V_n.

2. Pseudo-Riemannian and (Generalized) Finslerian Spaces

In what follows, the signature of the (non-degenerate) metric form is supposed to be arbitrary (we no more restrict ourselves onto positive definite metrics only) so that we can write

$$\mathrm{d}s^2 = e g_{ij}(x,\dot{x})\,\mathrm{d}x^i\mathrm{d}x^j, \qquad e=\pm 1 \tag{2}$$

and the sign is determined in such a way that $\mathrm{d}s^2 \geq 0$.

In short, we will call such metrics and spaces **Finslerian metrics** and **Finsler spaces** again, or *Riemannian*, respectively (more usually, they are called pseudo-Riemannian, or semi-Riemannian).

The arc length of a curve γ, given by parametrization $x^i = x^i(t)$, is given in a Finsler or Riemannian space (in our sense) by the integral

$$s = \int_{t_0}^{t_1} \sqrt{e g_{ij}(x(t),\dot{x}(t))\dot{x}^i(t)\dot{x}^j(t)}\,\mathrm{d}t, \qquad \dot{x}^i(t) = \frac{\mathrm{d}x^i(t)}{\mathrm{d}t}\cdot \tag{3}$$

It is well known [3], that this integral is stationary in a Finsler space if and only if its extremals are **geodesic curves** determined by the equations

$$\ddot{x}^h + 2G^h(x,\dot{x}) = \varrho(t)\dot{x}^h \tag{4}$$

where $\varrho(t)$ is a function, g^{ij} are components of the matrix inverse to (g_{ij}), and

$$G^h = \frac{1}{2}g^{ij}\left(\frac{\partial^2 F(x,\dot{x})}{\partial\dot{x}^j\partial x^k}\dot{x}^k - \frac{\partial F(x,\dot{x})}{\partial \dot{x}^j}\right)$$

are components of the Berwald connection. Let us emphasize that extremals of the integral of length are independent of reparametrization of geodesics. In Riemannian spaces, [2, 3], the components read

$$G^h = \frac{1}{2}\Gamma^h_{ij}(x)\,\dot{x}^i\dot{x}^j$$

where Γ^h_{ij} are the Christoffels of second type.

Many authors define a **geodesic** in V_n as an extremal curve of the integral

$$I = \int_{t_0}^{t_1} g_{ij}(x)\dot{x}^i\dot{x}^j\,\mathrm{d}t. \tag{5}$$

Extremals of this variational problem are those geodesics which satisfy the equations (4) with $\varrho(t) \equiv 0$.

Analogous situation is in Finsler spaces (in our generalized sense). Extremal curves of the integral (5) are determined together with their parameter, which is used to be called *canonical*. Note that particularly, arc length in V_n or F_n, respectively, is always canonical.

3. Generalized Variational Problem of Geodesics

In a Riemannian or in a Finsler space (in a more general sense explained above) consider the following more general **variational problem**

$$I = \int_{t_0}^{t_1} f(e\, g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j)\,\mathrm{d}\tau \tag{6}$$

where e takes the values ± 1, and $f(\tau)$ is a differentiable real-valued function (at least of class two) defined on some open domain $D \subset \mathbb{R}$ which is regular on D in the sense that $f'(\tau) \neq 0$ for all $\tau \in D$.

As an immediate consequence of the Euler-Lagrange equations for the Lagrange function $\mathcal{L} = f(e\, g_{ij}\dot{x}^i \dot{x}^j)$, it can be checked that the extremals satisfy the equations

$$\ddot{x}^h + 2G^h(x, \dot{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}(\ln |f'(e g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)|)\dot{x}^h. \tag{7}$$

We can prove the following theorem.

Theorem 1. *In* (*generalized*) *Finsler or Riemannian spaces, respectively, geodesic lines parameterized by a canonical parameter, which satisfy the condition*

$$e g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = k \in D$$

are extremals of the integral (6).

Theorem 2. *Consider* (*all*) *extremals of the integral* (6) *in a Finsler space* (*or in a Riemannian space, respectively*). *All curves arising under all possible regular reparameterizations of extremal curves belong to extremals, too, if and only if the function f takes the form $f(x) \equiv \alpha\sqrt{x}$ where α is some non-zero constant.*

Theorem 3. *All possible extremals of the integral* (6) *are just those geodesics which figure in Theorem 1 and Theorem 2. More precisely, in the particular case* $f(x) \equiv \alpha\sqrt{x}$, $0 \neq \alpha = \mathrm{const}$, *they are represented by all unparameterized geodesics (i.e., geodesics under all possible regular reparameterizations), while for all other functions f, satisfying the above assumptions of the problem* (6), *extremals are represented just by canonically parameterized geodesics only.*

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References

[1] Eisenhart L., *Riemannian Geometry*, Princeton Univ. Press, Princeton, 1926.

[2] Radulovich Zh., Mikesh J. and Gavril'chenko M., *Geodesic Mappings and Deformations of Riemannian Spaces*, Podgorica, Odessa, 1997.

[3] Rund H., *The Differential Geometry of Finsler Spaces*, Springer, Berlin, 1959.

[4] Sinyukov N., *Geodesic Mappings of Riemannian Spaces*, Moscow, Nauka, 1979.