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INTEGRAL SUBMANIFOLDS IN THREE-SASAKIAN MANIFOLDS WHOSE MEAN CURVATURE VECTOR FIELDS ARE EIGENVECTORS OF THE LAPLACE OPERATOR

DOREL FETCU

Department of Mathematics, "Gh. Asachi" Technical University of Iaşi
11 Carol I Blvd., 700506 Iaşi, Romania

Abstract. We find the Legendre curves and a class of integral surfaces in a 7-dimensional three-Sasakian manifold whose mean curvature vectors are eigenvectors of the Laplacian or the normal Laplacian and we give the explicit expression for such surfaces in the sphere $\mathbb{S}^7$.

1. Introduction

The class of submanifolds of a (pseudo-) Riemannian manifold, satisfying the condition

$$\Delta H = \lambda H \tag{1}$$

where λ is a constant, H is the mean curvature vector field and Δ denotes the Laplace operator, has been studied by many authors. The study of Euclidean submanifolds with this property was initiated by Chen in [4]. In the same paper biharmonic submanifolds of the Euclidean space are defined as those with harmonic mean curvature vector field.

Results concerning integral submanifolds of a Sasakian manifold of dimension three or five satisfying (1) were obtained in [6, 8, 13, 14]. In some of these papers (see [8, 13, 14]) are studied also the integral curves and surfaces with

$$\Delta^{\perp} H = \lambda H \tag{2}$$

where $\Delta^{\perp}$ is the normal Laplacian.

On the other hand, curves in the 7-sphere endowed with its canonical three-Sasakian structure, which are Legendre curves for all three Sasakian structures, with constant first curvature κ_1 and unit second curvature κ_2, were classified in [1]. It is easy to see that for such curves (1) is verified.

It seems to be interesting to view if these curves are all curves satisfying (1) and, moreover, to study the general case of Legendre curves in an arbitrary 7-dimensional three-Sasakian manifold.

The goals of this paper are to find the Legendre curves (with respect to all three Sasakian structures on the manifold) and a class of integral surfaces (with respect to two of the Sasakian structures) in a 7-dimensional three-Sasakian manifold, satisfying (1) or (2).

2. Preliminaries

2.1. Sasakian Manifolds

Concerning the Sasakian manifolds let us recall some notions and results as they are presented in [3].

Let M be an odd dimensional differentiable manifold and let (φ, ξ, η) be a tensor field of type $(1,1)$ on M, a vector field and an one-form, respectively. If $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, then (φ, ξ, η) is called an **almost contact structure** on M. On such a manifold, one obtains, by some algebraic computations, $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $\varphi^3 + \varphi = 0$. If the tensor field S, of type $(1,2)$, defined by $S = N_\varphi + 2\mathrm{d}\eta \otimes \xi$, where $N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y]$, is the Nijenhuis tensor field of φ, vanishes, then the almost contact structure is said to be normal (for more details see [3]). Let g be a (semi-)Riemannian metric on M. Then g is called an **associated metric** to the almost contact structure if $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M. Let Ω be the fundamental two-form of the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, defined by $\Omega(X,Y) = g(X, \varphi Y)$. If $\Omega = \mathrm{d}\eta$ then M is called a **contact metric manifold**. A normal contact metric manifold is called a **Sasaki manifold**. In [3] it is proved that an almost contact metric structure (φ, ξ, η, g) is Sasakian if and only if $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$, where ∇ is the Levi-Civita connection of g.

If the manifold M admits three almost contact structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha = 1,2,3$, satisfying

$$\varphi_\gamma = \varphi_\alpha\varphi_\beta - \eta_\beta \otimes \xi_\alpha = -\varphi_\beta\varphi_\alpha + \eta_\alpha \otimes \xi_\beta$$
$$\xi_\gamma = \varphi_\alpha\xi_\beta = -\varphi_\beta\xi_\alpha, \qquad \eta_\gamma = \eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha$$

for an even permutation (α, β, γ) of $(1,2,3)$, then the manifold is said to have an *almost contact three-structure*. The dimension of such a manifold is of the form $4n+3$. It is proved that there exists an associated metric to each of this three structures. If all structures are Sasakian then we call the manifold M a **three-Sasakian manifold**. Note that every three-contact structure is three-Sasakian (see [3]).

Remark 1. Some authors use a different sign convention in definition of three-almost contact structures (see for example [1]). However, our results also remains valid in this case.

If $\mathcal{D}$ is the contact distribution in a contact manifold (M, φ, ξ, η), defined by the subspaces $\mathcal{D}_m = \{X \in T_m M\,;\, \eta(X) = 0\}$, then an one-dimensional integral submanifold of $\mathcal{D}$ will be called a **Legendre curve**. A curve $\gamma : I \to M$, parametrized by its arc length is a Legendre curve if and only if $\eta(\gamma') = 0$. In case of a $(4n+3)$-dimensional three-Sasakian manifold the maximum dimension of an integral submanifold with respect to all three structures is n. Thus, in dimension 7 these would be Legendre curves.

Let us consider a Sasakian manifold $(M, \varphi, \xi, \eta, g)$. The sectional curvature of a two-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called φ-sectional curvature determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a **Sasakian space form** and it is denoted by $M(c)$. Note that if one of the three sectional curvatures, for example the third, of a three-Sasakian manifold is constant then it is equal to one since ξ_1 and $\xi_2 = \varphi_3 \xi_1$ are orthogonal to ξ_3 and any two-plane containing one of the characteristic vector fields has sectional curvature one.

The curvature tensor field of a Sasakian space form $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z = \tfrac{c+3}{4}\{g(Z,Y)X - g(Z,X)Y\} + \tfrac{c-1}{4}\{&\eta(Z)\eta(X)Y \\ - \eta(Z)\eta(Y)X + g(Z,X)\eta(Y)\xi - g(Z,Y)\eta(X)\xi& \\ + g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}&. \end{aligned}$$

A contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called **regular** if for any point $p \in M$ there exists a cubic neighborhood of p such that any integral curve of ξ passes through the neighborhood at most once, and **strictly regular** if all integral curves are homeomorphic to each other.

Let $(M, \varphi, \xi, \eta, g)$ be a regular contact metric manifold. Then the orbit space $\bar{M}$ has a natural manifold structure and, moreover, if M is compact then M is a principal circle bundle over $\bar{M}$ (the Boothby-Wang Theorem). In this case the fibration $\pi : M \to \bar{M}$ is called the **Boothby-Wang fibration**. A very known example of a Boothby-Wang fibration is the Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$.

We end this section by recalling the following result obtained by Ogiue.

Theorem 1 ([11]). *Let $(M, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold. Then on $\bar{M}$ can be given the structure of a Kähler manifold. Moreover, if $(M, \varphi, \xi, \eta, g)$ is a Sasakian space form $M(c)$, then $\bar{M}$ has constant sectional holomorphic curvature $c + 3$.*

2.2. Biharmonic Maps

A **biharmonic map** $\phi : (N, h) \to (M, g)$ between Riemannian manifolds is a critical point of the **bienergy functional** $E_2(\phi) = \frac{1}{2}\int_N \|\tau(\phi)\|^2 \nu_h$. This notion is a generalization of that of *harmonic maps*, which are critical points of the **energy functional** $E(\phi) = \frac{1}{2}\int_N \|\mathrm{d}\phi\|^2 \nu_h$, and it was suggested by Eells and Sampson in [5]. Chen defined the **biharmonic submanifolds** in an Euclidean space as the submanifolds with harmonic mean curvature. If we apply the characterization formula of biharmonic maps to Riemannian immersions into Euclidean spaces, we recover Chen's notion of biharmonic submanifold.

The Euler-Lagrange equation for the energy functional is $\tau(\phi) = 0$, where $\tau(\phi) = \operatorname{trace} \nabla \mathrm{d}\phi$ is the tension field, and the **Euler-Lagrange equation** for the bienergy functional was derived by Jiang in [9]

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace} R^M(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi = 0.$$

Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps, which are called **proper-biharmonic**.

3. Legendre Curves in Three-Sasakian Manifolds

Definition 1. *Let (M^m, g) be a Riemannian manifold and $\gamma : I \to M$ a curve parametrized by arc length, that is $\|\gamma'\| = 1$. Then γ is called a Frenet curve of osculating order r, $1 \leq r \leq m$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along γ such that*

$$\nabla_T E_1 = \kappa_1 E_2, \quad \nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \quad \ldots, \quad \nabla_T E_r = -\kappa_{r-1} E_{r-1}$$

where $E_1 = \gamma' = T$ and $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on I.

Remark 2. A geodesic is a Frenet curve of osculating order one. A *circle* is a Frenet curve of osculating order two with $\kappa_1 = \text{constant}$, a *helix of order* r, $r \geq 3$, is a Frenet curve of osculating order r with $\kappa_1, \ldots, \kappa_{r-1}$ constants and a helix of order three is called simply *helix*.

Since, by definition the mean curvature of γ is $H = \nabla_T T$, from the previous Frenet equations one obtains

$$\begin{aligned} \Delta H &= -\nabla_T^2 H = -\nabla_T(\nabla_T H) = -\nabla_T[\nabla_T(\kappa_1 E_2)] \\ &= -\nabla_T(\kappa_1' E_2 - \kappa_1^2 T + \kappa_1\kappa_2 E_3) \\ &= 3\kappa_1\kappa_1' T + (-\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2)E_2 - (\kappa_1\kappa_2' + 2\kappa_1'\kappa_2)E_3 - \kappa_1\kappa_2\kappa_3 E_4. \end{aligned} \tag{3}$$

If we claim that $\Delta H = \lambda H$, for λ being a constant, then, from (3), it follows

$$\kappa_1 \in \mathbb{R}, \qquad \kappa_1\kappa_2' = 0, \qquad \kappa_1\kappa_2\kappa_3 = 0.$$

We can state

Proposition 1. *The equation $\Delta H = \lambda H$ holds for a curve $\gamma : I \to M$, where dimension of M is greater than three, if and only if either γ is a geodesic or*

$$\kappa_1 \in \mathbb{R} \setminus \{0\}, \qquad \kappa_2 \in \mathbb{R}, \qquad \kappa_2\kappa_3 = 0. \tag{4}$$

Moreover, if γ is not a geodesic then $\lambda = \kappa_1^2 + \kappa_2^2$.

Assume that M^7 is a 7-dimensional three-Sasakian manifold with structure tensors $(\varphi_a, \xi_a, \eta_a)$, $a = 1, 2, 3$, and let $\gamma : I \to M^7$ be a Legendre Frenet curve of osculating order r in M^7, with respect to all three Sasakian structures on M^7, which is not a geodesic. Then, using the same notations as above, we have $g(T, \xi_a) = \eta_a(T) = 0$, for $a = 1, 2, 3$, where g is an associated Riemannian metric to the three-Sasakian structure. Differentiating this equation along γ we obtain, from $\nabla_X \xi_a = -\varphi_a X$, that $\nabla_T T = E_2$ is orthogonal to ξ_a for any $a = 1, 2, 3$. Now, it is easy to see that $\{T, \varphi_1 T, \varphi_2 T, \varphi_3 T, \xi_1, \xi_2, \xi_3\}$ is an orthogonal basis. It follows that, in this basis, the expression of $\nabla_T T$ is

$$\nabla_T T = \kappa_1 E_2 = \sum_{a=1}^{3} \lambda_a \varphi_a T$$

where $\lambda_a = \lambda_a(s)$, $a = 1, 2, 3$, are real valued functions along γ called the relative curvatures of γ. Hence, $\kappa_1 = \|\nabla_T T\| = \sqrt{\sum_{a=1}^{3} \lambda_a^2}$ and $E_2 = \sum_{a=1}^{3} \frac{\lambda_a}{\kappa_1} \varphi_a T$. Since M^7 is a three-Sasakian manifold, we easily get

$$\begin{aligned}
\nabla_T \varphi_1 T &= \xi_1 - \lambda_1 T + \lambda_2 \varphi_3 T - \lambda_3 \varphi_2 T \\
\nabla_T \varphi_2 T &= \xi_2 - \lambda_1 \varphi_3 T - \lambda_2 T + \lambda_3 \varphi_1 T \\
\nabla_T \varphi_3 T &= \xi_3 + \lambda_1 \varphi_2 T - \lambda_2 \varphi_1 T - \lambda_3 T.
\end{aligned} \tag{5}$$

After a straightforward computation, using (5), one obtains

$$\nabla_T E_2 = -\kappa_1 T + \sum_{a=1}^{3} \left[\left(\frac{\lambda_a}{\kappa_1} \right)' \varphi_a T + \frac{\lambda_a}{\kappa_1} \xi_a \right].$$

If $\kappa_2 = 0$ it follows that $\lambda_a = 0$ for any $a = 1, 2, 3$, and $\kappa_1 = 0$. So, in this case, γ is a geodesic. Since γ is not a geodesic, from the second Frenet equation, we have

$$\kappa_2 E_3 = \sum_{a=1}^{3} \left[\left(\frac{\lambda_a}{\kappa_1} \right)' \varphi_a T + \frac{\lambda_a}{\kappa_1} \xi_a \right].$$

Again using (5) and the third Frenet equation we have

$$\kappa_3 E_4 = \kappa_2 E_2 + \nabla_T E_3 = \sum_{a=1}^{3} \frac{2\lambda_a' \kappa_1 \kappa_2 - \lambda_a (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')}{\kappa_1^2 \kappa_2^2} \xi_a$$

$$+\left[\left(\frac{1}{\kappa_2}\left(\frac{\lambda_1}{\kappa_1}\right)'\right)' + \frac{\lambda_2'\lambda_3 - \lambda_3'\lambda_2 - \lambda_1(1-\kappa_2^2)}{\kappa_1\kappa_2}\right]\varphi_1 T$$
$$+\left[\left(\frac{1}{\kappa_2}\left(\frac{\lambda_2}{\kappa_1}\right)'\right)' + \frac{\lambda_3'\lambda_1 - \lambda_1'\lambda_3 - \lambda_2(1-\kappa_2^2)}{\kappa_1\kappa_2}\right]\varphi_2 T$$
$$+\left[\left(\frac{1}{\kappa_2}\left(\frac{\lambda_3}{\kappa_1}\right)'\right)' + \frac{\lambda_1'\lambda_2 - \lambda_2'\lambda_1 - \lambda_3(1-\kappa_2^2)}{\kappa_1\kappa_2}\right]\varphi_3 T.$$

Assume that (1) holds for γ. Since γ is not a geodesic then (4) holds too. Moreover, since $\kappa_2 \neq 0$, it follows that, in this case, (4) gives

$$\kappa_1, \kappa_2 \in \mathbb{R} \setminus \{0\}, \qquad \kappa_3 = 0.$$

But, if $\kappa_3 = 0$ it is easy to see that $\lambda_a' = 0$ for any $a = 1, 2, 3$. That means $\lambda_a \in \mathbb{R}$ are constants. From the expression of κ_2 one obtains $\kappa_2 = 1$.

Conversely, suppose that $\kappa_1 \in \mathbb{R} \setminus \{0\}$ and $\kappa_2 = 1$. Obviously, using again the expression of κ_2, it follows that $\lambda_a \in \mathbb{R}$ are constants. Taking account of that in the formula of $\kappa_3 E_4$, it can be easily verified that $\kappa_3 = 0$ and, then, due to Proposition 1, γ has the property (1).

Thus, we obtained

Theorem 2. *A Legendre Frenet curve of osculating order r, $\gamma : I \to M^7$, in a 7-dimensional three-Sasakian manifold with respect to all three Sasakian structures, verifies $\Delta H = \lambda H$ if and only if either*

i) *γ is a geodesic. In this case $\lambda = 0$*

or

ii) *$\kappa_1 \in \mathbb{R} \setminus \{0\}$ and $\kappa_2 = 1$. In this case $\lambda = 1 + \kappa_1^2$.*

Corollary 1. *The mean curvature of a curve γ as in the previous theorem is harmonic if and only if γ is a geodesic.*

The most canonical example of a three-Sasakian manifold is the 7-sphere $\mathbb{S}^7 = \{z \in \mathbb{C}^4\,;\ \|z\| = 1\}$ endowed with a three-Sasakian structure obtained as follows (see [1]).

Consider the Euclidean space E^8 with three complex structures,

$$\mathcal{I} = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{K} = -\mathcal{I}\mathcal{J}$$

where I_n denotes the $n \times n$ identity matrix. Let z denote the position vector of the unit sphere in E^8 and define three vector fields on $\mathbb{S}^7$ by

$$\xi_1 = -\mathcal{I}z, \qquad \xi_2 = -\mathcal{J}z, \qquad \xi_3 = -\mathcal{K}z.$$

The dual one-forms η_a are three independent contact structures on $\mathbb{S}^7$. We consider tensor fields φ_a, $a = 1, 2, 3$, of type $(1, 1)$ given by $\varphi_1 = s \circ \mathcal{I}$, $\varphi_2 = s \circ \mathcal{J}$, $\varphi_3 = s \circ \mathcal{K}$, where $s : T_zE^8 \to T_z\mathbb{S}^7$ denotes the orthogonal projection. Then $(\varphi_a, \xi_a, \eta_a)$ is a three-almost contact structure on $\mathbb{S}^7$. The standard metric on $\mathbb{S}^7$, g, of constant curvature one, is an associated metric for all three considered structures and $\mathbb{S}^7$ becomes a three-Sasakian manifold.

In [1] it is proved the following result

Theorem 3 ([1]). *Curves in $\mathbb{S}^7$, parametrized by arc length, which are Legendre curves for all three contact structures with $\kappa_1 = \text{constant}$ and $\kappa_2 = 1$ are either geodesics or lie in a totally geodesic 3-sphere and are given by*

$$\gamma(s) = \frac{\sin \rho_1 s}{\rho_1} c_1 - \frac{\cos \rho_1 s}{\rho_1} c_2 + \frac{\sin \rho_2 s}{\rho_2} c_3 - \frac{\cos \rho_2 s}{\rho_2} c_4$$

where $\rho_{1,2} = \frac{1}{2}(\sqrt{\kappa_1^2 + 4} \pm \kappa_1)$ and c_i are mutually orthogonal vectors in E^8.

Now, assume that M^7 is a three-Sasakian manifold with constant φ_a-sectional curvature equal to one. From the expression of the curvature tensor R^M one obtains that the biharmonic equation $\tau_2(\gamma) = 0$ of a curve $\gamma : I \to M$, parametrized by arc length, which is a non-geodesic Legendre curve for all three contact structures, is equivalent to $\Delta H = H$. By using Theorem 2 we can recover a result from [7].

Theorem 4 ([7]). *Biharmonic Legendre curves in a 7-dimensional three-Sasakian space form, with respect to all three Sasakian structures, are only geodesics.*

Next, we will look for submanifolds with mean curvature vector fields satisfying equation (2). Let (M^m, g) be a Riemannian manifold, ∇ the Levi-Civita connection and N^n a submanifold of M. The **normal connection** $\nabla^\perp$ is given by the equation of Weingarten

$$\nabla_X V = -A_V X + \nabla^\perp_X V$$

for any $X \in TN$ and any section V of the normal bundle $T^\perp N$, where A is the shape operator.

Denote by $\Delta^\perp$ the Laplacian acting on the space of the smooth sections of the normal bundle $T^\perp N$. The operator $\Delta^\perp$ is called the **normal Laplacian** and is given by

$$\Delta^\perp = -\sum_{i=1}^{n} (\nabla^\perp_{e_i} \nabla^\perp_{e_i} - \nabla^\perp_{\nabla^N_{e_i} e_i})$$

where $\{e_i\}$ is a local orthonormal frame on N and ∇^N is the Levi-Civita connection on N.

Let $\gamma : I \to M$ be a Frenet curve of osculating order r. The normal Laplacian is

$$\Delta^\perp = -\nabla^\perp_T \nabla^\perp_T.$$

We have $\nabla_T^{\perp}V = \nabla_T V - g(\nabla_T V, T)T$ for any section V normal to γ. Then, a straightforward computation gives

$$\nabla_T^{\perp}H = \kappa_1' E_2 + \kappa_1\kappa_2 E_3$$

and

$$\Delta^{\perp}H = (-\kappa_1'' + \kappa_1\kappa_2^2)E_2 - (\kappa_1\kappa_2' + 2\kappa_1'\kappa_2)E_3 - \kappa_1\kappa_2\kappa_3 E_4.$$

One obtains also

Proposition 2. *A curve γ satisfies $\Delta^{\perp}H = \lambda H$ if and only if*

$$-\kappa_1'' + \kappa_1\kappa_2^2 = \lambda\kappa_1, \qquad \kappa_1\kappa_2' + 2\kappa_1'\kappa_2 = 0, \qquad \kappa_1\kappa_2\kappa_3 = 0.$$

Proposition 3. *A curve γ satisfies $\Delta^{\perp}H = 0$ if and only if*

$$-\kappa_1'' + \kappa_1\kappa_2^2 = 0, \qquad \kappa_1\kappa_2' + 2\kappa_1'\kappa_2 = 0, \qquad \kappa_1\kappa_2\kappa_3 = 0.$$

Now, assume that M^7 is a three-Sasakian manifold with structure tensors $(\varphi_a, \xi_a, \eta_a)$, $a = 1, 2, 3$, and $\gamma : I \to M^7$ is a Legendre Frenet curve of osculating order r in M^7, with respect to all three Sasakian structures on M^7. It is easy to see that if γ is a geodesic we have $\Delta^{\perp}H = 0$. In the following suppose that γ is not a geodesic. As we have already seen $\kappa_2 \neq 0$ in this case. Then, from Proposition 2, we have that γ satisfies $\Delta^{\perp}H = \lambda H$ if and only if

$$-\kappa_1'' + \kappa_1\kappa_2^2 = \lambda\kappa_1, \qquad \kappa_1\kappa_2' + 2\kappa_1'\kappa_2 = 0, \qquad \kappa_3 = 0.$$

From the last two conditions, since $\nabla_T T = \kappa_1 E_2 = \sum_{a=1}^{3}\lambda_a\varphi_a T$ where λ_a are the relative curvatures, one obtains that λ_a must be constants and $\kappa_2 = 1$.

Actually we have obtained

Theorem 5. *A Legendre Frenet curve of osculating order r, $\gamma : I \to M^7$, in a 7-dimensional three-Sasakian manifold, with respect to all three Sasakian structures, verifies $\Delta^{\perp}H = \lambda H$ if and only if either*

i) *γ is a geodesic. In this case $\lambda = 0$*

or

ii) *$\kappa_1 \in \mathbb{R} \setminus \{0\}$ and $\kappa_2 = 1$. In this case $\lambda = 1$.*

4. Integral Surfaces in Three-Sasakian Manifolds

Let us consider the Boothby-Wang fibration $\pi : M^7 \to \bar{M} = M/\xi_3$, where M^7 is a three-Sasakian manifold with structure tensors $(\varphi_a, \xi_a, \eta_a)$, $a = 1, 2, 3$, and of dimension 7, which is strictly regular with respect to the third structure. Let $\bar{\gamma} : I \to \bar{M}$ be a curve parametrized by arc length and the surface $S_{\bar{\gamma}} = \pi^{-1}(\bar{\gamma})$ the inverse image.

In the following we shall look for surfaces $S_{\bar{\gamma}}$ which are integral surfaces of M with respect to the structures $(\varphi_1, \xi_1, \eta_1)$ and $(\varphi_2, \xi_2, \eta_2)$. For such a surface $\{\xi_3, T^H\}$

is a global orthonormal frame field, where $T = \bar{\gamma}'$ and T^H is the horizontal lift of T. The vector fields $\varphi_a T^H$, $a = 1, 2, 3$, are orthogonal to $S_{\bar{\gamma}}$. If $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ we have

$$\nabla_{X^H} Y^H = (\bar{\nabla}_X Y)^H - g(X^H, Y^H)\xi_3$$

for any vector fields $X, Y \in T\bar{M}$ and then

$$\bar{\nabla}_T T = \sum_{a=1}^{3} \lambda_a X_a$$

where λ_a are functions defined along $\bar{\gamma}$ and $X_a^H = \varphi_a T^H$. After a straightforward computation one obtains

$$\nabla^S_{\xi_3} T^H = 0, \qquad \nabla^S_{T^H} \xi_3 = 0, \qquad \nabla^S_{\xi_3}\xi_3 = 0, \qquad \nabla^S_{T^H} T^H = 0$$

$$B(T^H, T^H) = \sum_{a=1}^{3} \lambda_a \varphi_a T^H, \qquad B(T^H, \xi_3) = -\varphi_3 T^H, \qquad B(\xi_3, \xi_3) = 0$$

$$\nabla_{\xi_3}\varphi_1 T^H = \varphi_2 T^H, \qquad \nabla_{\xi_3}\varphi_2 T^H = -\varphi_1 T^H, \qquad \nabla_{\xi_3}\varphi_3 T^H = T^H$$

$$\nabla_{T^H}\varphi_1 T^H = \xi_1 - \lambda_1 T^H + \lambda_2 \varphi_3 T^H - \lambda_3 \varphi_2 T^H$$

$$\nabla_{T^H}\varphi_2 T^H = \xi_2 - \lambda_1 \varphi_3 T^H - \lambda_2 T^H + \lambda_3 \varphi_1 T^H$$

$$\nabla_{T^H}\varphi_3 T^H = \xi_3 + \lambda_1 \varphi_2 T^H - \lambda_2 \varphi_1 T^H - \lambda_3 T^H$$

where ∇^S is the Levi-Civita connection of $S_{\bar{\gamma}}$ and B is the second fundamental form of $S_{\bar{\gamma}}$ in M^7. We observe that $S_{\bar{\gamma}}$ is flat.

The mean curvature vector field of $S_{\bar{\gamma}}$ is

$$H = \frac{1}{2}(\nabla_{T^H} T^H + \nabla_{\xi_3}\xi_3) = \frac{1}{2}\sum_{a=1}^{3} \lambda_a \varphi_a T^H.$$

Straightforward computations show also that

$$\nabla_{T^H} H = \frac{1}{2}\left(\sum_{a=1}^{3} \lambda'_a \varphi_a T^H + \sum_{a=1}^{3} \lambda_a \xi_a - \sum_{a=1}^{3} \lambda_a^2 T^H\right)$$

$$\nabla^{\perp}_{T^H} H = \frac{1}{2}\left(\sum_{a=1}^{3} \lambda'_a \varphi_a T^H + \lambda_1 \xi_1 + \lambda_2 \xi_2\right)$$

$$\nabla_{\xi_3} H = -\varphi_3 H, \qquad \nabla^{\perp}_{\xi_3} H = \frac{1}{2}(-\lambda_1 \varphi_2 T^H + \lambda_2 \varphi_1 T^H)$$

and

$$\nabla_{T^H}\nabla_{T^H} H = \sum_{a=1}^{3} \lambda'_a \xi_a - \frac{3}{2}\sum_{a=1}^{3} \lambda \lambda'_a T^H$$

$$
\begin{aligned}
&+\frac{1}{2}(\lambda_1'' + \lambda_2'\lambda_3 - \lambda_3'\lambda_2 - \lambda_1(1+\bar{\kappa}_1^2))\varphi_1 T^H \\
&+\frac{1}{2}(\lambda_2'' + \lambda_3'\lambda_1 - \lambda_1'\lambda_3 - \lambda_2(1+\bar{\kappa}_1^2))\varphi_2 T^H \\
&+\frac{1}{2}(\lambda_3'' + \lambda_1'\lambda_2 - \lambda_2'\lambda_1 - \lambda_3(1+\bar{\kappa}_1^2))\varphi_3 T^H
\end{aligned}
$$

$$
\begin{aligned}
\nabla^\perp_{T^H}\nabla^\perp_{T^H} H = \; & \sum_{a=1}^{2}\lambda_a'\xi_a + \frac{1}{2}(\lambda_1'' + \lambda_2'\lambda_3 - \lambda_3'\lambda_2 - \lambda_1)\varphi_1 T^H \\
&+\frac{1}{2}(\lambda_2'' + \lambda_3'\lambda_1 - \lambda_1'\lambda_3 - \lambda_2)\varphi_2 T^H \\
&+\frac{1}{2}(\lambda_3'' + \lambda_1'\lambda_2 - \lambda_2'\lambda_1)\varphi_3 T^H \\
\nabla_{\xi_3}\nabla_{\xi_3} H = \; & -H, \qquad \nabla^\perp_{\xi_3}\nabla^\perp_{\xi_3} H = -\frac{1}{2}(\lambda_1\varphi_1 T^H + \lambda_2\varphi_2 T^H)
\end{aligned}
$$

where $\nabla^\perp$ is the normal connection and $\bar{\kappa}_1 = \sqrt{\sum_{a=1}^{3}\lambda_a^2}$ is the first curvature of the base curve $\bar{\gamma}$.

The Laplacian and the normal Laplacian are given by

$$
\Delta = -(\nabla_{T^H}\nabla_{T^H} + \nabla_{\xi_3}\nabla_{\xi_3}) \qquad \text{and} \qquad \Delta^\perp = -(\nabla^\perp_{T^H}\nabla^\perp_{T^H} + \nabla^\perp_{\xi_3}\nabla^\perp_{\xi_3})
$$

respectively. Their explicit forms are

$$
\begin{aligned}
\Delta H = \; & -\sum_{a=1}^{3}\lambda_a'\xi_a + \frac{3}{2}\sum_{a=1}^{3}\lambda\lambda_a' T^H \\
&-\frac{1}{2}(\lambda_1'' + \lambda_2'\lambda_3 - \lambda_3'\lambda_2 - \lambda_1(2+\bar{\kappa}_1^2))\varphi_1 T^H \\
&-\frac{1}{2}(\lambda_2'' + \lambda_3'\lambda_1 - \lambda_1'\lambda_3 - \lambda_2(2+\bar{\kappa}_1^2))\varphi_2 T^H \\
&-\frac{1}{2}(\lambda_3'' + \lambda_1'\lambda_2 - \lambda_2'\lambda_1 - \lambda_3(2+\bar{\kappa}_1^2))\varphi_3 T^H
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta^\perp H = \; & -\sum_{a=1}^{2}\lambda_a'\xi_a - \frac{1}{2}(\lambda_1'' + \lambda_2'\lambda_3 - \lambda_3'\lambda_2 - 2\lambda_1)\varphi_1 T^H \\
&-\frac{1}{2}(\lambda_2'' + \lambda_3'\lambda_1 - \lambda_1'\lambda_3 - 2\lambda_2)\varphi_2 T^H \\
&-\frac{1}{2}(\lambda_3'' + \lambda_1'\lambda_2 - \lambda_2'\lambda_1)\varphi_3 T^H.
\end{aligned}
$$

Now, we can state

Theorem 6. *A surface $S_{\bar{\gamma}}$ in a three-Sasakian strictly regular (with respect to the third structure) manifold M^7, which is an integral surface with respect to the first two Sasakian structures, satisfies $\Delta H = \lambda H$ if and only if either*

i) *$S_{\bar{\gamma}}$ is a minimal surface ($\lambda = 0$)*

or

ii) *λ_a are constants such that $\sum_{a=1}^{3} \lambda_a^2 \neq 0$. In this case $\lambda = 2 + \sum_{a=1}^{3} \lambda_a^2 = 2 + \bar{\kappa}_1^2$, where $\bar{\kappa}_1$ is the first curvature of the base curve $\bar{\gamma}$.*

Corollary 2. *The surface $S_{\bar{\gamma}}$ has harmonic mean curvature if and only if it is a minimal surface.*

Theorem 7. *A surface $S_{\bar{\gamma}}$ in a three-Sasakian strictly regular (with respect to the third structure) manifold M^7, which is an integral surface with respect to the first two Sasakian structures, satisfies $\Delta^{\perp} H = \lambda H$ if and only if $\lambda_1 = \lambda_2 = 0$ and λ_3 has one of the following expressions*

i) $\lambda_3 = as + b,\ a, b \in \mathbb{R},\ \lambda = 0$

ii) $\lambda_3 = a\cos(\sqrt{\lambda}s) + b\sin(\sqrt{\lambda}s),\ a, b \in \mathbb{R},\ \lambda > 0$

iii) $\lambda_3 = a\mathrm{e}^{\sqrt{-\lambda}s} + b\mathrm{e}^{\sqrt{-\lambda}s},\ a, b \in \mathbb{R},\ \lambda < 0.$

Assume that M^7 is a three-Sasakian space form with constant φ_3-curvature one. Then, from the expression of the curvature tensor it is easy to obtain that the biharmonic equation of a surface $S_{\bar{\gamma}}$ as above is equivalent to $\Delta H = 2H$. Thus, from Theorem 6, we have

Proposition 4. *In a 7-dimensional three-Sasakian space form there are not proper-biharmonic surfaces $S_{\bar{\gamma}}$ which are integral submanifolds with respect to two of the Sasakian structures.*

Next, in order to find examples of surfaces $S_{\bar{\gamma}}$ satisfying equation $\Delta H = \lambda H$ we will use again the 7-sphere endowed with its canonical three-Sasakian structure. One obtains

Theorem 8. *The parametric equation of a surface $S_{\bar{\gamma}}$ as above in $(\mathbb{S}^7, \varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, with mean curvature vector field H satisfying $\Delta H = \lambda H$, $\lambda > 2$, thought as a surfaces in $(\mathbb{R}^8, \langle\, ,\rangle)$, is*

$$x = x(u, v) = \sqrt{\frac{B}{A+B}}\cos(Au \pm v)e_1 \mp \sqrt{\frac{B}{A+B}}\sin(Au \pm v)\mathcal{K}e_1$$
$$+ \sqrt{\frac{A}{A+B}}\cos(Bu \pm v)e_3 \mp \sqrt{\frac{A}{A+B}}\sin(Bu \pm v)\mathcal{K}e_3$$

where e_1, e_3 are unit constant vector fields and $\{e_1, e_3, \mathcal{I}e_1, \mathcal{I}e_3, \mathcal{J}e_1, \mathcal{J}e_3, \mathcal{K}e_1, \mathcal{K}e_3\}$ is an orthonormal basis in the Euclidean space $(\mathbb{R}^8, \langle\, ,\rangle)$, and A, B are *given by*

$$A = \sqrt{\frac{\lambda - \sqrt{\lambda^2 - 4}}{2}}, \qquad B = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 4}}{2}}. \tag{6}$$

Proof: We consider the Boothby-Wang fibration $\pi : (\mathbb{S}^7, g) \to \mathbb{CP}^3$, where $\mathbb{CP}^3$ is the complex projective space with constant sectional holomorphic curvature 4. Let us denote by $\bar{\nabla}$, ∇ and $\tilde{\nabla}$ the Levi-Civita connections on $\mathbb{CP}^3$, $(\mathbb{S}^7, g)$ and $(\mathbb{R}^8, \langle\, , \rangle)$, respectively.

Let $S_{\bar{\gamma}}$ be an integral surface with respect to the first two Sasakian structures in $(\mathbb{S}^7, g)$ with $\Delta H = \lambda H$, where $\bar{\gamma} : I \to \mathbb{CP}^3$ is the base curve parametrized by arc length. Using Theorem 6, the first curvature of $\bar{\gamma}$ is given by $\bar{\kappa}_1^2 = \sum_{a=1}^3 \lambda_a^2$ with $\lambda_a = \mathrm{constant}$, $a = 1, 2, 3$ and $\lambda = 2 + \bar{\kappa}_1^2$, where $\bar{\nabla}_T T = \sum_{a=1}^3 \lambda_a X_a$, $T = \bar{\gamma}'$ and $X_a = \varphi_a T^H$. Assume that $\bar{\gamma}$ is a Frenet curve of osculating order r, with Frenet frame field $\{T, \bar{E}_2, \ldots, \bar{E}_r\}$.

We recall that $[\xi_3, T^H] = 0$, therefore we can choose a local chart $x = x(u, v)$ such that $T^H = x_u$ and $\xi_3 = x_v$.

From the equation of Gauss and Frenet equations we get

$$\tilde{\nabla}_{T^H} T^H = \nabla_{T^H} T^H - \langle T^H, T^H \rangle x = (\bar{\nabla}_T T)^H - x = \bar{\kappa}_1 \bar{E}_2^H - x$$

$$\tilde{\nabla}_{T^H} \tilde{\nabla}_{T^H} T^H = -(1 + \bar{\kappa}_1^2) T^H + \bar{\kappa}_1^2 \bar{\kappa}_2^2 \bar{E}_3^H + \lambda_3 \xi_3$$

and finally

$$\tilde{\nabla}_{T^H} \tilde{\nabla}_{T^H} \tilde{\nabla}_{T^H} T^H = -(2 + \bar{\kappa}_1^2) \tilde{\nabla}_{T^H} T^H - x.$$

That means

$$x_{uuuu} + \lambda x_{uu} + x = 0. \tag{7}$$

Again using the Gauss equation we have

$$\tilde{\nabla}_{\xi_3} \xi_3 = \nabla_{\xi_3} \xi_3 - x = -x.$$

Hence

$$x_{vv} + x = 0. \tag{8}$$

Solving the equation (7) and replacing into (8), we get

$$x = x(u, v) = \cos(Au \pm v) c_1 + \sin(Au \pm v) c_2 + \cos(Bu \pm v) c_3 + \sin(Bu \pm v) c_4$$

where A, B are given by (6) and $\{c_i\}$ are constant vectors in $\mathbb{R}^8$. Since

$$\langle x, x \rangle = 1,\ \langle x, x_u \rangle = 0,\ \langle x_u, x_u \rangle = 1,\ \langle x, x_{uu} \rangle = -1,\ \langle x_u, x_{uu} \rangle = 0$$

$$\langle x, x_{uuu} \rangle = 0,\ \langle x_{uu}, x_{uu} \rangle = 1 + \bar{\kappa}_1^2,\ \langle x_{uu}, x_{uuu} \rangle = 0,\ \langle x_u, x_{uuu} \rangle = -1 - \bar{\kappa}_1^2$$

$$\langle x_{uuu}, x_{uuu} \rangle = 1 + 3\bar{\kappa}_1^2 + \bar{\kappa}_1^4$$

one obtains for $(u, v) = (0, 0)$

$$c_{11} + 2c_{13} + c_{33} = 1 \tag{9}$$

$$Ac_{12} + Bc_{14} + Ac_{23} + Bc_{34} = 0 \tag{10}$$
$$A^2c_{22} + 2ABc_{24} + B^2c_{44} = 1 \tag{11}$$
$$A^2c_{11} + (A^2 + B^2)c_{13} + B^2c_{33} = 1 \tag{12}$$
$$A^3c_{12} + AB^2c_{23} + A^2Bc_{14} + B^3c_{34} = 0 \tag{13}$$
$$A^3c_{12} + A^3c_{23} + B^3c_{14} + B^3c_{34} = 0 \tag{14}$$
$$A^4c_{11} + 2A^2B^2c_{13} + B^4c_{33} = 1 + \bar{\kappa}_1^2 \tag{15}$$
$$A^5c_{12} + A^3B^2c_{23} + A^2B^3c_{14} + B^5c_{34} = 0 \tag{16}$$
$$A^4c_{22} + (AB^3 + A^3B)c_{24} + B^4c_{44} = 1 + \bar{\kappa}_1^2 \tag{17}$$
$$A^6c_{22} + 2A^3B^3c_{24} + B^6c_{44} = 1 + 3\bar{\kappa}_1^2 + \bar{\kappa}_1^4 \tag{18}$$

where $c_{ij} = \langle c_i, c_j \rangle$.
From (10), (13), (14) and (16) we have $c_{12} = c_{14} = c_{23} = c_{34} = 0$. From (9), (12) and (15) it follows that $c_{11} = \frac{B}{A+B}$, $c_{13} = 0$, $c_{33} = \frac{A}{A+B}$. Finally, from (11), (17) and (18) one obtains $c_{22} = \frac{B}{A+B}$, $c_{24} = 0$, $c_{44} = \frac{A}{A+B}$. Thus, $\{c_i\}$ are orthogonal vectors in E^8 with $|c_1| = |c_2| = \sqrt{\frac{B}{A+B}}$ and $|c_3| = |c_4| = \sqrt{\frac{A}{A+B}}$. Hence $c_1 = \sqrt{\frac{B}{A+B}}e_1$, $c_2 = \sqrt{\frac{B}{A+B}}e_2$, $c_3 = \sqrt{\frac{A}{A+B}}e_3$, $c_4 = \sqrt{\frac{A}{A+B}}e_4$, where $\{e_i\}$ are mutually orthogonal unit constant vectors in $\mathbb{R}^8$. Imposing that $x_v = \xi_3 = -\mathcal{K}x$ and that $S_{\bar{\gamma}}$ be a integral submanifold with respect to the first two Sasakian structures of $\mathbb{S}^7$ one obtains the conclusion of the theorem. □

Remark 3. If we take any of the Sasakian structures instead of the third when consider the Boothby-Wang fibration for a three-Sasakian manifold M^7, all results in this section remain valid.

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