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A NULL GEODESIC ORBIT SPACE WHOSE NULL ORBITS REQUIRE A REPARAMETRIZATION

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Abstract. We exhibit an example of a homogeneous Lorentzian manifold G/H whose homogeneous geodesics form just the light-cone and a hyperplane in the tangent space. For all light-like homogeneous geodesics, the natural parameter of the orbit is not the affine parameter of the geodesic.

1. Introduction

Homogeneous geodesics (see Definition 1) on homogeneous pseudo-Riemannian manifolds were studied both in physics [13, 14] and in mathematics [1–12, 15]. Penrose limits along null (that is, light-like) homogeneous geodesics are studied in [14]. It is shown there that the Penrose limit of a Lorentzian spacetime along a null homogeneous geodesic is a homogeneous plane wave and the Penrose limit of a reductive homogeneous spacetime along a null homogeneous geodesic is a reductive homogeneous plane wave. Null homogeneous geodesics and n.g.o. spaces (all null geodesics are homogeneous, see Definition 5) were introduced and studied in [13].

Riemannian and pseudo-Riemannian g.o. spaces (that is, spaces whose geodesics are all homogeneous) were studied in [5, 7, 8, 10] and the behaviour of geodesic graphs is investigated in [5]. Pseudo-Riemannian almost g.o. spaces (whose geodesics are almost all homogeneous) were studied in [3,6] and the behaviour of geodesic graphs in this case was investigated in [6]. The fundamental tool to determine homogeneous geodesics is the Geodesic Lemma (see Definition 1). Its generalization to the pseudo-Riemannian framework was proved in [4], and the existence of the two types of null homogeneous geodesics was illustrated in [1, 2, 4].

Up to now, all known examples of n.g.o. spaces are almost g.o. spaces. In this paper, we show an example of a n.g.o. space which is not almost g.o. In contrast to previous examples, in the new one, for all light-like homogeneous geodesics, the parameter of the one-parameter group of isometries (that is, the natural parameter of the orbit) is *not* the affine parameter of the geodesic itself. We also show interesting features of the homogeneous manifold with respect to different groups of isometries.

2. Homogeneous Geodesics

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a **homogeneous pseudo-Riemannian manifold**. Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the *homogeneous space* G/H. In general, there may exist more than one of such groups. For any fixed choice $M = G/H$, G acts effectively on G/H from the left. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *pseudo-Riemannian homogeneous space*.

If the metric g is positive definite, then $(G/H, g)$ is always a **reductive homogeneous space** in the following sense: we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad} : H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. Then, there exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If the metric g is indefinite, the reductive decomposition may not exist. Fixed a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain a scalar product $\langle\, , \rangle$ on $\mathfrak{m}$. This scalar product is obviously $\mathrm{Ad}(H)$-invariant.

The definition of a homogeneous geodesic is well-known in the Riemannian case (see, e.g., [12]). In the pseudo-Riemannian case, the needed generalized version was given in [4].

Definition 1. *Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. The geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be homogeneous if there exists*

1) *a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J*
2) *a vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$.*

*The vector X is then called a **geodesic vector**.*

The basic formula characterizing geodesic vectors in the pseudo-Riemannian case appeared in [14], but without a proof. The correct mathematical formulation with the proof was given in [4].

Lemma 1. *Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \textit{ for all } Z \in \mathfrak{m}, \textit{ where } k \in \mathbb{R} \textit{ is some constant.} \quad (1)$$

Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = \mathrm{e}^{-kt}$ is an affine parameter for the geodesic. The second case can only occur if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

Definition 2. *A pseudo-Riemannian homogeneous space $(G/H, g)$ is called a* ***g.o. space*** *if every geodesic of $(G/H, g)$ is homogeneous. Here "g.o." means "geodesics are orbits".*

It is well known that all *symmetric* spaces and, more generally, all *naturally reductive* homogeneous spaces are g.o. spaces. Some decades ago, it was generally believed that every g.o. space is naturally reductive. The first counter-example came from Kaplan [9]. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called "generalized Heisenberg groups". The extensive study of (Riemannian) g.o. spaces started just with the Kaplan's paper.

Remark 1. The g.o. property (as well as the natural reductivity) depends on the choice of the group G. In fact, according to Definition 1, the geodesic vector X must belong to the algebra $\mathfrak{g}$ of the group G. There are examples of homogeneous spaces $M = G/H$ which are not naturally reductive with respect to the group G, but they are naturally reductive if we express $M = G'/H'$ for $G \subsetneq G'$ (see [10]). Also, there are examples $M = G/H$ which are not g.o., but they are g.o. when we express $M = G'/H'$ (see [8]). For this reason, it is necessary to make a clear distinction between the properties of a homogeneous pseudo-Riemannian manifold M (that is, properties holding for the representation $M = \tilde{G}/\tilde{H}$, where $\tilde{G} = I_0(M)$) and those of the pseudo-Riemannian homogeneous space G/H.

We now recall one of the techniques used for the characterization of g.o. spaces, based on the concept of "geodesic graph". The original idea (not using such explicit name) comes from Szenthe [15].

Definition 3. *Let $(G/H, g)$ be a pseudo-Riemannian reductive g.o. space with compact isotropy group H and let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A* ***geodesic graph*** *is an $\mathrm{Ad}(H)$-equivariant map*

$\eta : \mathfrak{m} \to \mathfrak{h}$ which is rational on an open dense subset U of $\mathfrak{m}$ and such that $X + \eta(X)$ is a geodesic vector for each $X \in \mathfrak{m}$.

According to [15, Lemma 10], given a reductive g.o. space $(G/H, g)$ as above, there exists at least one geodesic graph. The construction of geodesic graphs is described in details in [10]. The homogeneous space G/H is naturally reductive if there exists a linear geodesic graph. For the examples of geodesic graphs on Riemannian and pseudo-Riemannian g.o. spaces (in both cases with compact isotropy group H) we refer to [5, 8, 10]. In [3, 6], the authors study 6 and 7-dimensional pseudo-Riemannian homogeneous spaces with noncompact isotropy group. It was shown that geodesic graph can be defined on an open dense subset of $\mathfrak{m}$, but not on all $\mathfrak{m}$.

Definition 4. *A pseudo-Riemannian homogeneous space $(G/H, g)$ with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is called an almost g.o. space if a geodesic graph can be defined on the open dense subset $U \subset \mathfrak{m}$, but not on all $\mathfrak{m}$.*

Reformulating the definition of a n.g.o. space given in [13] in terms of geodesic graphs, we have the following

Definition 5. *A pseudo-Riemannian homogeneous space $(G/H, g)$ with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is said to be a* ***n.g.o. space*** *if "a geodesic graph" can be defined on the null cone $N \subset \mathfrak{m}$, but not on all $\mathfrak{m}$. Here "n.g.o." means "null geodesics are orbits".*

Remark 2. As our new example will show, for a n.g.o. space one should not require geodesic graph to be defined on an open dense subset of $\mathfrak{m}$. However, we use here the standard definitions used in the previous papers, because we do not intend to introduce new notations here. The map $\eta : \mathfrak{n} \to \mathfrak{h}$ constructed for the new example satisfies all other properties.

Among the examples of almost g.o. spaces constructed in [3, 6], some are n.g.o. and some are not. Up to now, no examples of a n.g.o. space which is not almost g.o. were known. We recall a conjecture stated in [3].

Conjecture 3. Let G/H be a pseudo-Riemannian g.o. space or almost g.o. space. For all null homogeneous geodesics it holds $k = 0$ in Lemma 1.

In this paper, we are going to construct an example of a n.g.o. space G/H which is not almost g.o. space. More precisely, a geodesic graph can be constructed just on the null cone and on a hyperplane. In contrast with n.g.o. spaces which are almost g.o., in the new example, we have $k \neq 0$ (in Lemma 1) for all light-like homogeneous geodesics.

Further, we will show that the properties stated above are peculiar to the coset space G/H. Because the corresponding homogeneous Lorentzian manifold N

satisfies the property $\nabla R = 0$, it is a symmetric space. Hence, N admits another representation $N = \widetilde{G}/\widetilde{H}$ for $G \subsetneq \widetilde{G}$, where, in particular, all geodesics are homogeneous with respect to the group $\widetilde{G}$. This representation $N = \widetilde{G}/\widetilde{H}$ can be easily found by noticing that N has constant sectional curvature. However, it is interesting to remark that the Lie algebra $\widetilde{\mathfrak{g}}$ of the group $\widetilde{G}$ cannot be obtained by extending the algebra $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ by adding some derivations of $\mathfrak{g}$ which preserve the scalar product on $\mathfrak{n}$. The similar behaviour occurred with the 7-dimensional Riemannian g.o. space in [7].

3. A New Example of a Lorentzian N.G.O. Space

Let us consider a three-dimensional Lorentzian vector space $\mathfrak{n}$ and a pseudo-orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{n}$ with the signature $(1, 1, -1)$. We define the Lie algebra structure on $\mathfrak{n}$ by the relations

$$[E_1, E_2] = 0, \qquad [E_1, E_3] = \alpha E_1 + \beta E_2, \qquad [E_2, E_3] = -\beta E_1 + \alpha E_2 \quad (2)$$

for $\alpha \neq 0$. This algebra is considered in the classification of non-unimodular Lorentzian Lie groups in [2] and denoted by $\mathfrak{g}_5$ there. We denote by N the unique connected and simply connected Lie group corresponding to $\mathfrak{n}$. Now we introduce a linear operator A on $\mathfrak{n}$, which acts on $\mathfrak{n}$ by the relations

$$A(E_1) = E_2, \qquad A(E_2) = -E_1, \qquad A(E_3) = 0. \quad (3)$$

This is the unique operator on $\mathfrak{n}$ which preserves the scalar product and the Lie algebra structure on $\mathfrak{n}$. We put $\mathfrak{h} = \operatorname{span}(A) \simeq \mathfrak{so}(2)$. Then, $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ is a reductive decomposition and the scalar product on $\mathfrak{n}$ induces a left-invariant Lorentzian metric g on $N = G/H = (N \rtimes \mathrm{SO}(2))/\mathrm{SO}(2)$.

We now find the homogeneous geodesics by applying Lemma 1. We write each vector $X \in \mathfrak{n}$ and each vector $\eta(X) \in \mathfrak{h}$ in the form

$$X = \sum_{i=1}^{3} x_i E_i, \qquad \eta(X) = \eta_1 A$$

and we consider the equation (1) in the form

$$\langle [X + \eta(X), Y]_{\mathfrak{m}}, X \rangle = k \langle X, Y \rangle \quad (4)$$

where Y runs over all $\mathfrak{m}$. We have to determine the corresponding $\eta(X)$ to the given X. For $Y \in \mathfrak{m}$, we substitute, step by step, all three elements E_1, E_2, E_3 of the given basis into the formula (4). We obtain the following system of three linear equations for the parameter η_1

$$\begin{aligned} \eta_1 x_2 &= (\alpha x_1 + \beta x_2) x_3 + k x_1 \\ -\eta_1 x_1 &= (-\beta x_1 + \alpha x_2) x_3 + k x_2 \\ 0 &= \alpha (x_1^2 + x_2^2) + k x_3. \end{aligned} \quad (5)$$

If $k = 0$, the last equation in (5) gives $x_1^2 + x_2^2 = 0$, because $\alpha \neq 0$. Hence, $x_1 = x_2 = 0$ and x_3 is arbitrary, that is, system (5) is satisfied for any value of η_1. If $k \neq 0$, we must add the condition $x_1^2 + x_2^2 - x_3^2 = 0$ characterizing the light-like vectors and we obtain

$$k = -\alpha x_3, \qquad \eta_1 = \beta x_3. \tag{6}$$

Thus, a geodesic graph on $\mathfrak{n}$ can be defined only on the hyperplane defined by $x_1 = x_2 = 0$ and on the light-cone. On the given hyperplane it can be defined as the zero map and on the light-cone by the formula (6).

4. The Curvature of N

We now calculate the Riemannian curvature of N. Let ∇ be the pseudo-Riemannian connection and $\widetilde{\nabla}$ the canonical connection of G/H (with respect to the given ad(H)-invariant decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$). We compute the canonical torsion $\widetilde{T}_e$ and the canonical curvature $\widetilde{R}_e$ in the tangent space T_eN by the relations

$$\widetilde{T}_e(X,Y) = -[X,Y]_{\mathfrak{n}} \quad \text{and} \quad \widetilde{R}_e(X,Y) = -[X,Y]_{\mathfrak{h}} \quad \text{for} \quad X,Y \in \mathfrak{n} \tag{7}$$

where $\mathfrak{n}$ is naturally identified with T_eN. In our example we have $\widetilde{R}_e(X,Y) = 0$. Further, the difference tensor $\widetilde{D} = \nabla_e - \widetilde{\nabla}_e$ at the point e can be calculated from the formula

$$2\langle \widetilde{D}_Y X, Z\rangle = \langle \widetilde{T}_e(X,Y), Z\rangle + \langle \widetilde{T}_e(X,Z), Y\rangle + \langle \widetilde{T}_e(Y,Z), X\rangle \tag{8}$$

and the pseudo-Riemannian curvature tensor can be obtained from

$$R_e(X,Y) = \widetilde{R}_e(X,Y) + [\widetilde{D}_X, \widetilde{D}_Y] + \widetilde{D}_{\widetilde{T}_e(X,Y)}. \tag{9}$$

If we denote by A_{kl} and $\bar{A}_{kl}$ the operators on $\mathfrak{n}$ $(= T_eN)$ defined by the relations

$$A_{kl}E_i = \delta_{ik}E_l - \delta_{il}E_k \quad \text{and} \quad \bar{A}_{kl}E_i = \delta_{ik}E_l + \delta_{il}E_k \tag{10}$$

we can express the operators D as

$$D_{E_1} = \alpha\bar{A}_{13}, \qquad D_{E_2} = \alpha\bar{A}_{23}, \qquad D_{E_3} = -\beta A_{12} \tag{11}$$

and the curvature operators as

$$R_e(E_1,E_2) = -\alpha^2 A_{12},\ R_e(E_1,E_3) = -\alpha^2\bar{A}_{13},\ R_e(E_2,E_3) = -\alpha^2\bar{A}_{23}. \tag{12}$$

We easily obtain from (12) that the Ricci operator is diagonal, it has one triple eigenvalue which is equal to $2\alpha^2$ and all sectional curvatures are equal to α^2. Hence, (N,g) is a space of constant curvature, in particular, it is a symmetric space.

More precisely, having supposed N is connected and simply connected, it is a Lorentzian sphere of constant sectional curvature $\alpha^2 > 0$. It is well-known that its isometry group is $\mathrm{O}(1,3)$ and so, N can be realized as $N = \widetilde{G}/\widetilde{H}$, where

$\widetilde{G} = I_0(M)$ is a six-dimensional Lie group (the identity component of $\mathrm{O}(1,3)$) and $\widetilde{H}$ is the identity component of the group $\mathrm{O}(1,2)$. However, $\widetilde{\mathfrak{g}}$ can not be obtained by adding some derivations to $\mathfrak{g}$, hence the Lie group structure on $\widetilde{G}$ is not compatible with the Lie group structure on G (and on N).

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