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FROM DEFORMED SURFACES WITH PRESCRIBED QUANTUM PROPERTIES TO NEW TWO-DIMENSIONAL QUANTUM DEVICES

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Abstract. We propose a problem in differential geometry, i.e., which deformed surfaces produce prescribed curvature induced quantum potentials. We solve this inverse problem in the case of surfaces of revolution. We also show that there exist rotational surfaces in the form of a circular strip around the axis of symmetry which allow particles with generic angular momentum to bind. The quantum physics of a collection of circular strips of curved surfaces glued together is discussed in the conclusion in view of the possibility to engineer devices based on thin films.

1. Introduction

It is possible to produce very narrow two-dimensional conducting surfaces which allow electrons to propagate in the channel formed by their boundaries, but require the electron wave function to vanish on these boundaries. In this paper, we study the possibility of creating an effective one dimensional quantum problem by confining a particle to move in a collection of simple rotationally invariant surfaces in the form of ribbons glued together. The interaction between quantum particles and curvature in such a construction induces possible physical applications. Furthermore, curvature leads to surprising effects in quantum systems, for example, in [5] it was shown that a charged quantum particle trapped in a potential of quantum nature due to bending of an elastically deformable thin tube travels without dissipation like a soliton. Surprisingly, the twist of a strip plays a role of a magnetic field and is responsible for the appearance of localized states and an effective transverse electric field thus reminisce the quantum Hall effect [6].

The results of this paper are based on the exploration of the properties of the Schrödinger equation on a submanifold of $\mathbb{R}^3$. Following da Costa [3] an effective potential appears in the Schrödinger equation which has the following form

$$V_s(q_1, q_2) = -\frac{\hbar^2}{2\mu}(M^2 - K) = -\frac{\hbar^2}{8\mu}(k_1 - k_2)^2 \tag{1}$$

where μ is particle's mass, $\hbar$ is the Plank's constant, q_1 and q_2 are the generalized coordinates on the surface, k_1 and k_2 are the principal curvatures of the surface while $M = (k_1 + k_2)/2$ and $K = k_1 k_2$ are the mean and the Gauss curvatures respectively.

The presence of the mean curvature (which cannot be obtained from the metric tensor and its derivatives alone) in (1) results in an important consequence that $V_s(q_1, q_2)$ is not the same for two isometric surfaces and da Costa [3] notes: "thus independent of how small the range of values assumed for q_3 (the third coordinate which measures the distance to the surface along the normal vector), the wave function always moves in three-dimensional portion of space, so that the particle is "aware" of the external properties of the limit surface." The particle is also "aware" of the manner in which it is confined to move to that limit surface [8, 9]. In view of this, solving the inverse problem, we construct a deformed surface which corresponds to an effective free one-dimensional motion. We also report the existence of a circular strip surface creating conditions for a zero angular momentum particle to bind in a harmonic potential.

2. Derivation of da Costa's Quantum Potential Associated with a Constrained Motion on a Surface Immersed in $\mathbb{R}^3$

R. C. T. da Costa's approach towards the quantization of a constrained particle is clearly the most viable since the particle is first thought of as being unconstrained, i.e., described by three Cartesian coordinates of a flat $\mathbb{R}^3$ space (where quantum mechanics is perfectly working theory), but subject to an external potential V_λ, which in a certain suitable limit (that is $\lambda \to \infty$) forces the system to remain on a curved submanifold S of $\mathbb{R}^3$. In order to obtain a meaningful result the particle's wave function is "uniformly compressed" onto a surface thus avoiding the arising of tangential forces which correspond to dissipative constraints in classical mechanics. The resulting Schrödinger equation can be separated into a part which contains the surface variables and independent of the constraining potential V_λ. The most striking feature of the result obtained in this way is the presence of a potential of geometric origin that is not derived from the intrinsic properties of the limit surface alone!

This potential cannot be obtained in a usual quantization procedure starting from a classical Lagrangian of the already constrained particle $\mathcal{L} = g_{ij}\dot{q}^i\dot{q}^j/2m$ since the Lagrangian depends only on the metric properties of the surface.

The following treatment is also most adequate in describing a "real-world" situation, in which every attempt to reduce the dimensions of the system is strongly oppressed by Heisenberg's uncertainty relations.

Now, consider a particle with mass m permanently attached to a surface with a parametric equation $\vec{r} = \vec{r}(q_1, q_2)$, where $\vec{r}$ is the position vector, with a confining potential $V_\lambda(q_3)$, where λ is a "squeezing" parameter

$$\lim_{\lambda\to\infty} V_\lambda(q_3) = \begin{cases} 0 & q_3 = 0 \\ \infty & q_3 \neq 0. \end{cases} \tag{2}$$

Here q_3 is a coordinate measuring the distance of a point with coordinates (q_1, q_2, q_3) from the surface along the unit normal $\vec{N} \sim \frac{\partial \vec{r}}{\partial q_1} \wedge \frac{\partial \vec{r}}{\partial q_2}$. The coordinate system in $\mathbb{R}^3$ associated with the surface is given by the radius vector

$$\vec{R}(q_1, q_2, q_3) = \vec{r}(q_1, q_2) + q_3\vec{N}(q_1, q_2). \tag{3}$$

If we turn our attention to the Schrödinger equation in the above curvilinear coordinate system we can write

$$-\frac{\hbar^2}{2\mu}\sum_{i,j=1}^{3}\frac{1}{\sqrt{G}}\frac{\partial}{\partial q_i}\left(\sqrt{G}(G^{-1})_{ij}\frac{\partial\Psi}{\partial q_j}\right) + V_\lambda(q_3)\Psi = \mathrm{i}\hbar\frac{\partial\Psi}{\partial t} \tag{4}$$

where $G = \det(G_{ij})$ is the determinant of the metric associated with the coordinate system $\vec{R}$

$$G_{ij} = G_{ji} = \frac{\partial\vec{R}}{\partial q_i}\cdot\frac{\partial\vec{R}}{\partial q_j}, \qquad i, j = 1, 2, 3 \tag{5}$$

where

$$\frac{\partial\vec{R}}{\partial q_3} = \vec{N}, \qquad \frac{\partial\vec{R}}{\partial q_\alpha} = \frac{\partial\vec{r}}{\partial q_\alpha} + q_3\sum_\beta w_{\alpha\beta}\frac{\partial\vec{r}}{\partial q_\beta}, \qquad \alpha, \beta = 1, 2. \tag{6}$$

Here $w_{\alpha\beta}$ are the matrix elements of the Weingarten map of the tangent space in itself. Up to a sign they coincide with the second fundamental form. The invariants of the Weingarten matrix W also up to a sign coincide with the Gaussian ($K = \det W$) and the mean ($M = -1/2\,\mathrm{tr}\,W$) curvatures respectively. A simple calculation yields

$$G^2 = (\det G_{ij})^2 = \left|\frac{\partial\vec{R}}{\partial q_1}\wedge\frac{\partial\vec{R}}{\partial q_2}\right|^2 = \left(1 + q_3\,\mathrm{tr}\,W + q_3^2\det W\right)^2 g^2 \tag{7}$$

where g is the determinant of the metric

$$g_{ij} = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}, \qquad i, j = 1, 2 \tag{8}$$

associated with the surface $\vec{r}(q_1, q_2)$.

From now on we introduce the notation $f(q_1, q_2, q_3) = 1 + q_3 \operatorname{tr} W + q_3^2 \det W$.

Due to the structure of the metric components G_{ij} we can break up the Laplacian into two parts: a surface part denoted by $\mathcal{D}(q_1, q_2, q_3)$ and a normal part defined by $i = j = 3$ to obtain

$$-\frac{\hbar^2}{2\mu}\mathcal{D}(q_1, q_2, q_3)\Psi - \frac{\hbar^2}{2\mu}\left[\frac{\partial^2 \Psi}{\partial q_3^2} + \frac{1}{\sqrt{G}}\left(\frac{\partial \sqrt{G}}{\partial q_3}\right)\frac{\partial \Psi}{\partial q_3}\right] + V_\lambda(q_3)\Psi = \mathrm{i}\hbar\frac{\partial \Psi}{\partial t}\cdot$$

Let us work out the form of the normal part

$$-\frac{\hbar^2}{2\mu}\frac{1}{\sqrt{G}}\left(\frac{\partial}{\partial q_3}\sqrt{G}\frac{\partial \Psi}{\partial q_3}\right) = -\frac{\hbar^2}{2\mu}\frac{1}{f}\left(\frac{\partial}{\partial q_3}f\frac{\partial \Psi}{\partial q_3}\right) \tag{9}$$

after introducing

$$\Psi = f^{-1/2}\chi \tag{10}$$

which takes into account the volume element in this coordinate system

$$\mathrm{d}V = f(q_1, q_2, q_3)\mathrm{d}S\mathrm{d}q_3, \qquad \mathrm{d}S = \sqrt{g}\mathrm{d}q_1\mathrm{d}q_2. \tag{11}$$

In terms of χ we obtain

$$\frac{1}{f}\frac{\partial}{\partial q_3}\left(f\frac{\partial}{\partial q_3}\frac{\chi}{\sqrt{f}}\right) = \frac{1}{\sqrt{f}}\left\{\frac{\partial^2 \chi}{\partial q_3^2} + \left[\frac{1}{f^2}\left(\frac{1}{2}\operatorname{tr} W\right)^2 - \frac{\det W}{f}\right]\chi\right\}. \tag{12}$$

We now take the limit $V_\lambda(q_3) = \infty$ as $q_3 \neq 0$ so that the wave function is forced to "see" two steep potential barriers on both sides and its value is significantly different from zero only for infinitesimal range of values around $q_3 = 0$. Thus, we set $q_3 = 0$ in the differential operator to end up with

$$\begin{aligned} &-\frac{\hbar^2}{2\mu}\sum_{i,j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}(g^{-1})_{ij}\frac{\partial \chi}{\partial q_j}\right) - \frac{\hbar^2}{2\mu}\left[\left(\frac{1}{2}\operatorname{tr} W\right)^2 - \det W\right]\chi \\ &-\frac{\hbar^2}{2\mu}\frac{\partial^2 \chi}{\partial q_3^2} + V_\lambda(q_3)\chi = \mathrm{i}\hbar\frac{\partial \chi}{\partial t} \end{aligned} \tag{13}$$

Separating the dependence on the variables $\chi = \chi_{\rm t}(q_1, q_2, t)\chi_{\rm n}(q_3, t)$ we have a set of two equations determining the quantum evolution

$$-\frac{\hbar^2}{2\mu}\sum_{i,j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}(g^{-1})_{ij}\frac{\partial \chi_{\rm t}}{\partial q_j}\right) - \frac{\hbar^2}{2\mu}\left(M^2 - K\right)\chi_{\rm t} = {\rm i}\hbar\frac{\partial \chi_{\rm t}}{\partial t} \tag{14}$$

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2 \chi_{\rm n}}{\partial q_3^2} + V_\lambda(q_3)\chi_{\rm n} = {\rm i}\hbar\frac{\partial \chi_{\rm n}}{\partial t}\,\cdot \tag{15}$$

The normalization condition on $\chi_{\rm t}$ is $\int_S |\chi_{\rm t}|^2\,{\rm d}S = 1$ where ${\rm d}S$ is given by (11) and the normalization condition on $\chi_{\rm n}$ is the usual one-dimensional norm $\int |\chi_{\rm n}|^2\,{\rm d}q_3 = 1$.

3. Rotational Surfaces

Let us take a rotationally invariant surface $\vec{r}(q_1, q_2)$, parameterized in Cartesian coordinates in Monge fashion

$$\vec{r}(q_1, q_2) = \vec{r}(\rho, \phi) = (\rho\cos\phi, \rho\sin\phi, f(\rho)) \tag{16}$$

where $\rho \in [0, \infty)$ and $\phi \in [0, 2\pi]$.

From the first and the second fundamental forms of this surface the following expressions for the principal curvatures are obtained

$$k_1(\rho) = \frac{\ddot{f}(\rho)}{\left(1 + \dot{f}(\rho)^2\right)^{3/2}}, \qquad k_2(\rho) = \frac{\dot{f}(\rho)}{\rho\left(1 + \dot{f}(\rho)^2\right)^{1/2}} \tag{17}$$

where hereafter the dot represents derivative with respect to ρ.

If the surface is not a plane but it is asymptotically planar and cylindrically symmetric then the Schrödinger operator can have at least one isolated eigenvalue of finite multiplicity which guarantees the existence of a geometrically induced bound state [7]. Looking for stationary modes in polar coordinates in which the rotational invariance of the surface (16) is obvious, we separate the variables

$$\chi_{\rm t}(\rho, \phi, t) = \exp\left(-{\rm i}E_m t/\hbar\right)\exp\left({\rm i}m\phi\right)\psi_m(\rho)$$

to end up with a quasi-one-dimensional Sturm-Liouville equation for the ρ-dependent part of the wave function

$$\frac{1}{\rho\sqrt{1 + \dot{f}^2}}\partial_\rho\left(\frac{\rho\partial_\rho\psi_m}{\sqrt{1 + \dot{f}^2}}\right) - \frac{m^2}{\rho^2}\psi_m = -\frac{2\mu E_m}{\hbar^2}\psi_m + \frac{2m}{\hbar^2}V_s(\rho)\psi_m(\rho) \tag{18}$$

where $V_s(\rho)$ is given by (1) with (17). The normalization condition in ρ-space is

$$2\pi\int_0^\infty |\psi_m(\rho)|^2\sqrt{g}\,{\rm d}\rho = 1 \tag{19}$$

with the determinant of the metric g given by

$$g = \rho^2 \left(1 + \dot{f}(\rho)^2\right). \tag{20}$$

Due to the cylindrical symmetry and the conservation of the z-component of the angular momentum, we may reduce the problem to a one-dimensional equation for each angular momentum quantum number m along the Euclidean line length along the geodesic on the surface with fixed ϕ. Introducing the changes [4]

$$x = \int_0^\rho \sqrt{1 + \dot{f}^2(\tilde{\rho})}\, \mathrm{d}\tilde{\rho}, \qquad \psi_m(\rho) = F_m(x)/\sqrt{\rho} \tag{21}$$

we obtain for the function $F_m(x)$ a one-dimensional Schrödinger equation which is the Liouville normal form (see the Appendix) of (18)

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} F_m(x) + \left[W_m(x) - \kappa_m^2\right] F_m(x) = 0. \tag{22}$$

Here we have introduced the wave vector instead of the energy $\kappa_m^2 = 2\mu E_m/\hbar^2$. The geometrical properties of the surface determine the quantum effects through the geometry dependent term $W_m(x)$ in equation (22)

$$W_m\left[x(\rho)\right] = -\frac{1}{4} k_1^2(\rho) + \frac{m^2 - 1/4}{\rho^2} \tag{23}$$

where k_1 is given by (17). The normalization condition in x-space is

$$2\pi \int_0^\infty \left|F_m(x)\right|^2 \, \mathrm{d}x = 1. \tag{24}$$

The term in W_m, proportional to m^2, is the potential that describes the familiar centrifugal force. Less familiar is the negative correction term $-1/4$ which results not from the angular motion but from the radial motion (and can be traced back to the radial derivatives in the Laplacian expressed in the associated with the surface coordinates (16)). This is a coordinate force that comes from the reduction of space from three to two dimensions and was called *quantum anti-centrifugal force* by Cirone *et al* [2] because it possesses binding power due to quantum mechanics. Similar situation for the free radial motion of a particle was noticed in [1]. This contribution is strengthened by the binding curvature induced potential $-k_1^2/4$, a geometric force.

The potential W_m may be repulsive for particles with non-vanishing angular momentum ($m \neq 0$) and no solutions with negative energy may exist.

The effect from different contributions in the potential W_m stands out most clearly for particles with zero angular momentum, that is, $m = 0$. These are attracted to the origin and are found in a ring-shaped region around the axis of symmetry, while all particles with $m \neq 0$ could be repelled from the center. It is clear that curvature not only introduces scale but also breaks the symmetry of the $\mathbb{R}^2$ plane and could act as a selector of particles with different angular momenta.

4. Inverse Problem. Rotational Surfaces Corresponding to Prescribed Quantum Problems

Let us now turn our attention to the inverse problem or equivalently the question "Which rotationally invariant surface leads to an effective geometry induced potential W_m that equals prescribed negative function $-U[x(\rho)]$, where $U \geq 0$ for all ρ?" (negative because we are primary interested in bound states). This is a question of particular interest since for certain classes of negative potentials $-U$ we already know the exact wave functions which can readily be used in revealing the particle's distribution on the surface. The solution of the inverse problem goes through the recognition of its equivalence with the following differential equation (see equations (23) and (17))

$$\frac{1}{4}\frac{\ddot{f}(\rho)^2}{\left(1+\dot{f}(\rho)^2\right)^3} = U(\rho) + \frac{m^2 - 1/4}{\rho^2}\,. \tag{25}$$

From equations (17) and (25) one can easily deduce a condition on $k_1(\rho)$ in order to have a binding potential for a particle with $m \neq 0$, i.e.,

$$k_1^2 > \frac{4m^2 - 1}{\rho^2}\,.$$

For smooth surfaces at $\rho = 0$ this means that there will be just strips where this condition holds. For flat surfaces where $k_1 = 0$ this condition implies that only particles with $m = 0$ bind, a fact previously noticed in [1,2]. Now we try to solve equation (25) using the substitution $\dot{f} = \sinh(\omega)$ (here $\omega = \omega(\rho)$) to end up with the linear equation

$$\frac{1}{2}\frac{\mathrm{d}\tanh(\omega)}{\mathrm{d}\rho} = \pm\sqrt{U(\rho) + \frac{m^2 - 1/4}{\rho^2}} \tag{26}$$

which can be solved yielding a result for $\dot{f}(\rho)$. We integrate to obtain the profile of the surface

$$f(\rho) = \pm\int_{\rho_1}^{\rho} \frac{|A|}{\sqrt{1 - A^2}}\,\mathrm{d}\tilde{\rho} \tag{27}$$

where

$$A(\rho) = \pm 2\int_{\rho_0}^{\rho} \sqrt{U(\tilde{\rho}) + \frac{m^2 - 1/4}{\tilde{\rho}^2}}\,\mathrm{d}\tilde{\rho}. \tag{28}$$

Here ρ_0 and ρ_1 are constants of integration and are to be determined by the boundary conditions due to the behavior of the function U representing the potential we want to model. Since $f(\rho)$ takes only real values (the same is true for all of its derivatives, i.e., $\mathrm{d}^{\mathrm{n}} f/\mathrm{d}\rho^{\mathrm{n}} \in \mathbb{R}$ for $\mathrm{n} = 0, 1, \dots$) as a function describing the profile of a surface we impose $0 < |A| < 1$. The $A = 0$ case is realized by a flat surface. Using the theorem of the mean value in (28) and the above inequality we obtain

$$\rho_0 < \rho < \rho_0 + \frac{1}{2}\left| U(\xi) + \frac{m^2 - 1/4}{\xi^2} \right|^{-1/2} \tag{29}$$

where the point $\xi \in [\rho_0, \rho]$. In that manner we show that for generic angular momentum m and non-zero potential U the corresponding rotational surface creating this potential and allowing a bound state of a quantum particle with $m \neq 0$ exists only in a ribbon, a circular strip around the axis of symmetry.

Let us also note that for $m \neq 0$ the bump (a macroscopic structure) has a magnetic moment, i.e., the macroscopic deformation of the surface acquires quantum number. Indeed the probability density current $\vec{J}$ ($\operatorname{div} \vec{J} = 0$) associated with the wave function χ_{t} is given by

$$\vec{J} = (J_\phi, J_\rho, J_z) = \frac{\hbar}{m}\left(m\frac{|\psi_m|^2}{\rho}, \mathrm{Re}\, \frac{\psi_m^* \partial_\rho \psi_m}{\mathrm{i}\sqrt{1 + \dot{f}^2}}, 0 \right)$$

where J_ϕ, J_ρ and J_z are the components of the vector $\vec{J}$ in the orthonormal right-handed triad $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$, has non-zero and quantized with m circulation along the circumference of the bump.

Now let us give a couple of examples.

Example 1. Free motion, $U = 0$. From (28) and (27) it follows that

$$f_m^{\mathrm{free}}(\rho) = \pm\rho_0 \int_{\rho_1/\rho_0}^{\rho/\rho_0} \frac{\sqrt{|4m^2 - 1|}\,|\ln(\tilde{\rho})|}{\sqrt{1 - (4m^2 - 1)\ln^2(\tilde{\rho})}}\, \mathrm{d}\tilde{\rho}. \tag{30}$$

Here ρ_0 determines the characteristic scale of the surface on which we consider this free quantum problem. Its magnitude can be chosen in $[\mu m]$ or $[nm]$ depending on the scale that we want to model. If we consider the shape of a surface allowing a free motion of a particle with zero angular momentum, that is $m = 0$, we can integrate (30) with $\rho_1 = \rho_0$ to produce Fig. (1). That surface is asymptotically flat as it tends to a cone $f_m^{\mathrm{free}}(\rho) \to \rho$ as $\rho \to \infty$.

Next we consider the strip of a surface allowing a free motion of a particle with non-zero angular momentum, that is $m \neq 0$. We can impose also the condition $1 > (4m^2 - 1)\ln^2(\rho/\rho_0)$ to find the extensions of the strip

$$\rho_0 < \rho < \rho_0 \exp\left[(4m^2 - 1)^{-1/2}\right]. \tag{31}$$

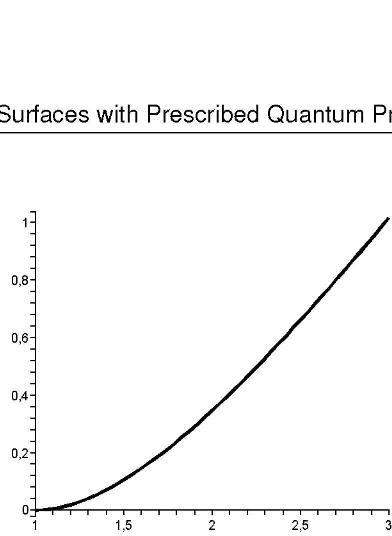

Figure 1. The surface $f(\rho)/\rho_0$ (vertical axis) with a cusp on which a particle with $\mathrm{m} = 0$ moves freely. On the horizontal axis is plotted ρ/ρ_0. At infinity that surface tends to a cone.

Since the problem is defined on a strip, we can quantize it in the usual finite volume method, which would lead to standing wave solutions on the surface. Their energy is

$$E_n^m(\rho_0) = \frac{2\pi^2\hbar^2 n^2}{m\rho_0^2\left[\mathrm{e}^{(4m^2-1)^{-1/2}} - 1\right]^2}\,. \tag{32}$$

Example 2. Harmonic oscillator potential $U = \omega^2\rho^2$. From (28) it follows that

$$A_m^{\mathrm{harm}}(\rho) = \pm 2\int_{\rho_0}^{\rho}\sqrt{\omega^2\tilde{\rho}^2 + \frac{m^2 - 1/4}{\tilde{\rho}^2}}\,\mathrm{d}\tilde{\rho}. \tag{33}$$

Here we consider only the $m = 0$ case. It yields for (33) the result

$$A_0^{\mathrm{harm}}(\rho) = \pm\frac{1}{2}\left\{\sqrt{4\omega^2\rho^4 - 1} + \arctan\left[\left(4\omega^2\rho^4 - 1\right)^{-1/2}\right] - \frac{\pi}{2}\right\} \tag{34}$$

where we have set the value of $\rho_0 = 1/\sqrt{2|\omega|}$. From the obvious requirement $0 < |A_0^{\mathrm{harm}}| < 1$ we obtain the following estimate

$$\rho < \frac{\left(1 + (2+\varepsilon)^2\right)^{1/4}}{\sqrt{2|\omega|}} \approx \frac{5^{1/4} + \varepsilon/5^{3/4}}{\sqrt{2|\omega|}} \tag{35}$$

where $\varepsilon = \pi/2 - \arctan\left(4\omega^2\rho^4 - 1\right)^{-1/2} \ll 1$. Thus, we find an expression for the extensions of the circular strip of a rotational surface creating harmonic potential and allowing a bound state of a particle with vanishing angular momentum in a harmonic oscillator potential

$$\frac{1}{\sqrt{2|\omega|}} \le \rho < \left(5^{1/4} + \frac{\varepsilon}{5^{3/4}}\right)\frac{1}{\sqrt{2|\omega|}}\,. \tag{36}$$

5. Conclusion

In conclusion we speculate that a classical object (the ribbon) exhibits quantum characteristics (the magnetic moment due to non-vanishing quantized probability current circulation along the circumference) acquired due to curvature. The foundation of another speculation, namely, a collection of circular strips of curved rotational surfaces glued together may lead to a new quantum device is to be found in equation (22) which is the Schrödinger equation in terms of the Euclidean line length on the surface. By appropriately choosing the curved ribbons we may, at least in theory, model a desired potential (23) which would be a piece-wise function. At the places where the pieces are glued (that is at certain $\rho = \rho_j$) we have to require continuity of the wave function and its derivative which is exactly the condition which reproduces the energy eigenvalues E_j and the corresponding eigenvectors $|j\rangle$. There is no a priori restriction on their number N. Thus, in theory, we reproduce a N-level system on which transitions between levels can be felt by the distribution of the particles on the surface. An N-level quantum system is clearly a quantum device which sometimes is referred to as q-bit.

Appendix: Basic Facts on the Sturm-Liouville Equation

The Sturm-Liouville equation is an equation cast in the form

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(p(z)\frac{\mathrm{d}\Psi}{\mathrm{d}z}\right) + q(z)\Psi(z) = \varepsilon w(z)\Psi(z)$$

where $\varepsilon = 2\mu E/\hbar^2$. Introducing a new variable x as

$$x = \int^z \mathrm{d}\tilde{z}\sqrt{\frac{w(\tilde{z})}{p(\tilde{z})}}$$

the above equation acquires a simpler form

$$-\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2} + W(x)Y(x) = \varepsilon Y(x)$$

where

$$Y(x) = \left(p(z)w(z)\right)^{1/4}\Psi$$

and

$$W(x) = \frac{q(z)}{w(z)} + \frac{1}{\left(p(z)w(z)\right)^{1/4}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(p(z)w(z)\right)^{1/4}.$$

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