Seventh International Conference on
Geometry, Integrability and Quantization
June 2–10, 2005, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2005, pp 218–225

ON A BASIC PROBLEM FOR A SECOND ORDER DIFFERENTIAL EQUATION WITH A DISCONTINUOUS COEFFICIENT AND A SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. In the present paper we investigate the completeness, minimality and basic properties of the eigenfunctions of one discontinuous Sturm–Liouville problem with a spectral parameter in boundary conditions and transmission conditions.

1. Introduction

We consider the discontinuous boundary value problem with a spectral parameter in the boundary conditions for a second order ordinary differential equation:

$$l(u) \equiv -p(x)u'' + q(x)u = \lambda u, \quad x \in [a,c) \cup (c,b] \tag{1}$$

$$\alpha_{11}u(a) - \alpha_{12}u'(a) = \lambda\left(\alpha_{21}u(a) - \alpha_{22}u'(a)\right) \tag{2}$$

$$\beta_{11}u(b) - \beta_{12}u'(b) = \lambda\left(\beta_{21}u(b) - \beta_{22}u'(b)\right) \tag{3}$$

$$u(c+0) - u(c-0) = 0 \tag{4}$$

$$u'(c+0) - u'(c-0) = -\lambda\delta_1 u(c) \tag{5}$$

where $p(x) = \frac{1}{p_1^2}$ for $x \in [a,c)$ and $p(x) = \frac{1}{p_2^2}$ for $x \in (c,b]$, $q(x)$ is a real-valued continuous function on the intervals $[a,c)$ and $(c,b]$ and has a finite limits $q(c \pm 0) = \lim_{x\to c\pm 0} q(x)$; p_i, α_{ij}, β_{ij} $(i,j = 1,2)$ are real constants. We assume also that $\delta_1 > 0$ and

$$\rho_1 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \qquad \rho_2 = \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}.$$

In the present work we investigate the completeness, the minimality and the basic properties of the system of eigenfunctions of the discontinuous boundary value problem (1)–(5). Note that the spectral properties of the boundary-value problem with a discontinuous coefficient have been investigated earlier in [8], [11], [12] and [13]. This problem with physical applications has been discussed in [9]. In [3] the properties of eigenvalues and eigenfunctions of the problem (1)–(5) have been investigated, and asymptotic formulas for the eigenvalues and eigenfunctions have been obtained. We will use these results in our work.

For the ordinary differential equation in the continuous case, the boundary problems regarding the subject of our work with a spectral parameter in the boundary condition have been investigated in many other works (see e.g. [2, 4–7, 10, 14]).

2. Main Results

To investigate the basic property of the problem (1)–(5) we define a special Hilbert space. We denote by $H = L_2[a,b] \oplus \mathbb{C}^3$, the Hilbert space of all elements $\widetilde{u} = (u(x), u_1, u_2, u_3)$ with a scalar product defined by

$$(\widetilde{u},\widetilde{v}) = p_1^2\int_a^c u(x)\overline{v(x)}\mathrm{d}x + p_2^2\int_c^b u(x)\overline{v(x)}\mathrm{d}x + \frac{1}{\rho_1}u_1\overline{v_1} + \frac{1}{\rho_2}u_2\overline{v_2} + \frac{1}{\delta_1}u_3\overline{v_3} \tag{6}$$

where $u(x) \in L_2[a,c) \cup L_2(c,b]$ and $u_1, u_2, u_3 \in \mathbb{C}$. We assume that $\rho_1 > 0$, $\rho_2 > 0$. Regarding the problem (1)–(5), let A be a operator defined by the formula

$$\begin{aligned} A\widetilde{y} = (&-p(x)u'' + q(x)u,\ \alpha_{11}u(a) - \alpha_{12}u'(a),\\ &\beta_{11}u(a) - \beta_{12}u'(a),\ u'(c+0) - u'(c-0)) \end{aligned} \tag{7}$$

on the domain

$$\begin{aligned} D(A) = \{&\widetilde{u} \in H\,;\, \widetilde{u} = (u(x), u_1, u_2, u_3), u(x) \in AC[a,b],\\ &u'(x) \in AC[a,c), u'(x) \in AC(c,b],\ u'(c \pm 0) = \lim_{x\to c\pm 0} u(x),\\ &l(u) \in L_2[a,b],\ u_1 = \alpha_{11}u(a) - \alpha_{12}u'(a),\\ &u_2 = \beta_{11}u(a) - \beta_{12}u'(a),\ u_3 = -\delta_1 u(c)\} \end{aligned}$$

where $AC[a,b]$ denotes the space of all absolutely continuous functions on the interval $[a,b]$.

In [3] it has been shown that the eigenvalues of the boundary-value problem (1)–(5) coincide with the zeros of an entire function and form at most countable and bounded below set which is convergent to the infinity at the infinity. We may renumber this set as $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ counted according to their multiplicity. In the other cases, from the properties of the operator A and a J-indefinite metric, the operator A has infinitely many non-real eigenvalues [1].

It is clear that the spectral problem (1)–(5) is equivalent to the eigenvalues problem

$$A\widetilde{u} = \lambda\widetilde{u} \tag{8}$$

for the operator A, and the following lemma regarding this is proved.

Lemma 1. *The eigenvalues of the spectral problem* (1)–(5) *with multiplicity coincide with the eigenvalues of the operator A. For every one chain of the eigenfunctions $y_0, y_1, \ldots, y_h$ corresponding to the eigenvalue λ_0 of the problem* (1)–(5) *is coincide with the eigenfunctions $\widetilde{y_0}, \widetilde{y_1}, \ldots, \widetilde{y_h}$ corresponding to the same eigenvalue λ_0 of the operator A, where*

$$\widetilde{u_k} = (u_k(x)\,;\, \alpha_{21}u(a) - \alpha_{22}u'(a), \beta_{21}u(b) - \beta_{22}u'(b), -\delta_1 u(c))\,, k = 0, 1, \ldots, h.$$

Proof: The proof of the lemma is obtained by directly replacing the values of $A\widetilde{u}$ and $\widetilde{u}$ in the equality (8). Specially, it can be obtained from the general Lemma 1.4 in [14]. □

Lemma 2. *The operator A is selfadjoint in the domain $D(A)$ of the Hilbert space H.*

Proof: Lemma 1.5 in [14] says that the domain $D(A)$ is everywhere dense in the Hilbert space in more general case. According to [13] the operator A is symmetric and has a discrete spectrum. Thus, there exists a number λ such that $R(A - \lambda I) = H$. Hence, the operator A is selfadjoint in H. □

Theorem 1. *The eigenfunctions of the operator A form a orthonormal basis in the space $H = L_2[a, b] \oplus \mathbb{C}^3$.*

Proof: According to [3], the operator A has countable many eigenvalues $\{\lambda_k\}_0^\infty$ each one of them convergent to the infinity at the infinity. Therefore, the operator $A - \lambda I$ has an inverse in the Hilbert space H except for the isolated eigenvalue λ_k. Specially, taking $\lambda = 0$, the bounded operator A^{-1} is defined in H. The selfadjoint operator A^{-1} has at most countable many eigenvalues and each one of them converges to zero at the infinity. So, the operator A^{-1} is a compact operator. By the Hilbert–Schmidt theorem regarding compact operators, we have that the eigenfunctions of the operator A form an orthonormal basis in H. □

Now we investigate the cases $\rho_1 > 0$, $\rho_2 = 0$ or $\rho_1 = 0$, $\rho_2 > 0$. In these cases, only one of these boundary conditions depends on the spectral parameter λ. We consider the case $\rho_1 > 0$, $\rho_2 = 0$. In the Hilbert space $H = L_2[a, b] \oplus \mathbb{C}^2$ we define the operator A_1 by the formula

$$A_1\widetilde{u} = (-p(x)u'' + q(x)u, \alpha_{11}u(a) - \alpha_{12}u'(a), u'(c+0) - u'(c-0)) \quad (9)$$

and its domain

$$\begin{aligned} D(A_1) = \{&\widetilde{u} \in H\,;\, \widetilde{u} = (u(x), u_1, u_2), u(x) \in AC[a, b], \\ &u'(x) \in AC[a, c), \ u'(x) \in AC(c, b], \ u'(c \pm 0) = \lim_{x \to c\pm 0} u(x), \\ &l(u) \in L_2[a, b], \ u_1 = \alpha_{11}u(a) - \alpha_{12}u'(a), \ u_2 = -\delta_1 u(c)\}. \end{aligned}$$

In the Hilbert space $H = L_2[a,b] \oplus \mathbb{C}^2$ we define a scalar product by the formula

$$(\widetilde{u},\widetilde{v}) = p_1^2 \int_a^c u(x)\overline{v(x)}\,\mathrm{d}x + p_2^2 \int_c^b u(x)\overline{v(x)}\,\mathrm{d}x + \frac{1}{\rho_1} u_1\overline{v_1} + \frac{1}{\delta_1} u_2\overline{v_2}$$

for the elements $\widetilde{u} = (u(x), u_1, u_2) \in H$ and $\widetilde{v} = (v(x), v_1, v_2) \in H$. In these cases too, Lemma 1 and Lemma 2 concerning the equality of the eigenvalues problem (8) and the boundary problem

$$-p(x)u'' + q(x)u = \lambda u, \quad x \in [a,c) \cup (c,b] \tag{10}$$

$$\alpha_{11}u(a) - \alpha_{12}u'(a) = \lambda\left(\alpha_{21}u(a) - \alpha_{22}u'(a)\right) \tag{11}$$

$$\beta_{11}u(b) - \beta_{12}u'(b) = 0 \tag{12}$$

$$u(c+0) - u(c-0) = 0 \tag{13}$$

$$u'(c+0) - u'(c-0) = -\lambda\delta_1 u(c) \tag{14}$$

are proved. Repeating the idea of the proof of Theorem 1 we prove the following theorem.

Theorem 2. *The eigenfunctions of the operator A_1 form an orthonormal basis in the Hilbert space $H = L_2[a,b] \oplus \mathbb{C}^2$.*

In the case $\rho_1 = 0$, $\rho_2 > 0$ we have the boundary-value problem

$$\begin{aligned}
-p(x)u' + q(x)u &= \lambda u, \quad x \in [a,c) \cup (c,b] \\
\alpha_{11}u(a) - \alpha_{12}u'(a) &= 0 \\
\beta_{11}u(b) - \beta_{12}u'(b) &= \lambda\left(\beta_{21}u(a) - \beta_{22}u'(a)\right) \\
u(c+0) - u(c-0) &= 0 \\
u'(c+0) - u'(c-0) &= -\lambda\delta_1 u(c).
\end{aligned}$$

Corollary 1. *For the case $\rho_1 > 0$, $\rho_2 > 0$, the remainder system of eigenfunctions $\{u_n(x)\}_0^\infty$ of the boundary problem (1)–(5) obtained by omitting three elements from them is a complete and minimal system in $L_2[a,b]$.*

Proof: By Theorem 1, the system of all eigenfunctions $\widetilde{u_k}(x) = \{u_k(x), a, b, c\}$ $(a,b,c \in \mathbb{C})$ of the boundary problem (1)–(5) forms a basis in $H = L_2[a,b] \oplus \mathbb{C}^3$. Hence, the system of the eigenfunctions $\{\widetilde{u}_n(x)\}_0^\infty$ is complete and minimal in H. We denote by P the orthogonal projection defined by the formula $P\widetilde{u}_k(x) = u_n(x)$. Then, of course, $\operatorname{codim} P = 3$. According to Lemma 2.1 in [14], the complementary system in $\{P\widetilde{u}_k(x)\}_0^\infty = \{u_n(x)\}_0^\infty$ obtained by omitting three elements from $\{u_n(x)\}_0^\infty$ is a complete and minimal system in $L_2[a,b]$. Hence, the complementary system of eigenfunctions $\{u_n(x)\}_0^\infty$ of the boundary problem (1)–(5) obtained by omitting three elements from $\{u_n(x)\}_0^\infty$ is a complete and minimal system in $L_2[a,b]$. □

In the similar way we obtain the following result.

Corollary 2. *In the cases $\rho_1 > 0$, $\rho_2 = 0$ or $\rho_1 = 0$, $\rho_2 > 0$, the complementary systems of eigenfunctions $\{u_n(x)\}_0^\infty$ of the boundary problem* (10)–(14) *obtained by omitting two elements from them are complete and minimal systems in $L_2[a,b]$.*

To investigate the cases $\rho_1 > 0$, $\rho_2 < 0$ or $\rho_1 < 0$, $\rho_2 > 0$ we assume that the operator A is defined by the formula (7) on the domain $D(A)$ and introduce the operator J

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \operatorname{sgn}\rho_1 & 0 & 0 \\ 0 & 0 & \operatorname{sgn}\rho_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is selfadjoint and has a bounded inverse operator in $H = L_2[a,b] \oplus \mathbb{C}^3$. In this case, the boundary problem (1)–(5) is equivalent to the eigenvalue problem (8) or the eigenvalue problem for the operators pencil

$$(B - \lambda J)\widetilde{u} = 0 \tag{15}$$

in the space H. In fact, the operator $G = B - \lambda J$ is bounded and has an inverse operator. Applying J to the self side of (8) we obtain that (8) is equivalent to (15). In the case $\rho_1 > 0$, $\rho_2 < 0$ the scalar product in $H = L_2[a,b] \oplus \mathbb{C}^3$ is defined by the equality

$$(\widetilde{u},\widetilde{v}) = p_1^2 \int_a^c u(x)\overline{v(x)}\,\mathrm{d}x + p_2^2 \int_c^b u(x)\overline{v(x)}\,\mathrm{d}x + \frac{1}{\rho_1}u_1\overline{v_1} - \frac{1}{\rho_2}u_2\overline{v_2} + \frac{1}{\delta_1}u_3\overline{v_3}. \tag{16}$$

where $u(x) \in L_2[a,c) \cup L_2(c,b]$ and $u_1, u_2, u_3 \in \mathbb{C}$.

Lemma 3. *The operator A is J-selfadjoint in the space H.*

Proof: In [14] by Lemma 1.5, in more general case it has shown that the domain $D(A)$ is everywhere dense in space H. From the equalities (7) and (16) and applying two times the integration by parts as in Theorem 1, we have that

$$(B\widetilde{f},\widetilde{g}) = (\widetilde{f},B\widetilde{g})$$

for $\widetilde{f},\widetilde{g} \in D(A)$ and $B = JA$. Then the operator A is J-symmetric in the space H. It has been obtained in Section IV of [3], that the eigenvalues of the boundary problem (1)–(5) are zeros of an entire function and form a bounded set. In the considered case in the similar way it can be proved that the operator A has a discrete spectrum. Taking into consideration that the operator B is symmetric we have that the operator JA is selfadjoint. □

Theorem 3. *The eigenfunctions of the operator A form a Riesz basis in the Hilbert space $H = L_2[a,b] \oplus \mathbb{C}^3$.*

Proof: According to Lemma 3 the operator $A = J^{-1}B$ is J-selfadjoint in H. Taking into consideration the ideas of Theorem 1 and according to the Azizov–Iokhvidov theorem in Section IV of [1], we obtain that the eigenfunctions of the operator A form a Riesz basis in the Hilbert space $H = L_2[a,b] \oplus \mathbb{C}^3$. □

Following the similar method one can prove the same result for the case $\rho_1 < 0$, $\rho_2 > 0$. Corollary 1 is true for these cases, too.

Now we consider the cases $\rho_1 < 0$, $\rho_2 = 0$ or $\rho_1 = 0$, $\rho_2 < 0$. For the case $\rho_1 < 0$, $\rho_2 = 0$ the scalar product in $H = L_2[a,b] \oplus \mathbb{C}^2$ is defined by the equality

$$(\widetilde{u}, \widetilde{v}) = p_1^2 \int_a^c u(x)\overline{v(x)}\,\mathrm{d}x + p_2^2 \int_c^b u(x)\overline{v(x)}\,\mathrm{d}x - \frac{1}{\rho_1} u_1 \overline{v_1} + \frac{1}{\delta_1} u_2 \overline{v_2}$$

and we assume that the operator A_1 is defined by the equality (9) in the domain $D(A_1)$. Let the operator J_1 be

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sgn}\rho_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is selfadjoint and has a bounded inverse operator in $H = L_2[a,b] \oplus \mathbb{C}^2$. In this case, too, it can be shown that Lemma 1 and Lemma 3 regarding the equality of the eigenvalues problem (8) and the boundary problem (10)–(14) are proved. Repeating the proof of Theorem 3 for this case we have the next theorem.

Theorem 4. *The eigenfunctions of the operator A_1 forms a Riesz basis in the space $H = L_2[a,b] \oplus \mathbb{C}^2$.*

By analogy, the same result can be obtained for the case $\rho_1 = 0$, $\rho_2 < 0$.

For the cases $\rho_1 < 0$, $\rho_2 = 0$ and $\rho_1 = 0$, $\rho_2 < 0$ Corollary 2 proves.

In the case $\rho_1 < 0$, $\rho_2 < 0$ we define the scalar product in $H = L_2[a,b] \oplus \mathbb{C}^3$ by the equality

$$(\widetilde{u}, \widetilde{v}) = p_1^2 \int_a^c u(x)\overline{v(x)}\,\mathrm{d}x + p_2^2 \int_c^b u(x)\overline{v(x)}\,\mathrm{d}x - \frac{1}{\rho_1} u_1 \overline{v_1} - \frac{1}{\rho_2} u_2 \overline{v_2} + \frac{1}{\delta_1} u_3 \overline{v_3}$$

and we assume that the operator A is defined in the domain $D(A)$. Let the operator J_2 be

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is selfadjoint and has a bounded inverse operator in $H = L_2[a,b] \oplus \mathbb{C}^3$. In the considered case, the boundary problem (1)–(5) is equivalent to the eigenvalues problem (8) or the eigenvalues problem for the operators pencil (15) in the space

H, where $B = J_2 A$. In the similar way of the other cases we obtain the following results.

Theorem 5. *In the case $\rho_1 < 0$, $\rho_2 < 0$ the eigenfunctions of the operator A form a Riesz basis in the Hilbert space $H = L_2[a, b] \oplus \mathbb{C}^3$.*

Using this theorem and the proof of Corollary 1 we have the next result.

Corollary 3. *In the case $\rho_1 < 0$, $\rho_2 < 0$, the complementary system of eigenfunctions of the boundary problem* (1)–(5) *obtained by omitting two elements from them is a complete and minimal system in $L_2[a, b]$.*

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