Seventh International Conference on
Geometry, Integrability and Quantization
June 2–10, 2005, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2005, pp 203–217

MANEV PROBLEM AND ITS REAL FORM DYNAMICS: SUPERINTEGRABILITY AND SYMMETRY ALGEBRAS

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Abstract. The Manev model is known to possess Ermanno–Bernoulli type invariants similar to the Laplace–Runge–Lenz vector of the ordinary Kepler model. If the orbits are bounded these invariants exist only when a certain rationality condition is met and consequently we have superintegrability only on a subset of initial values. On the contrary, real form dynamics of the Manev model is superintegrable for all initial values. Using these additional invariants, we demonstrate here that both Manev model and its real Hamiltonian form have $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ (or $\mathfrak{so}(2,1)$ depending on the value of a parameter in the potential) symmetry algebra in addition to the angular momentum algebra. Thus Kepler and Manev models are shown to have identical symmetry algebras.

1. Introduction

Since Kepler and Newton elliptical trajectories replaced circular ones as an archetype of the (bounded) planetary motion. The advent of Einstein's theory did not produce a new archetype of heavenly motions, apart from the exceptional case of a collapse into the (still hypothetical) black holes. Nevertheless, among the variety of relativistic effects the perihelion shift of inner planets is definitely the best recognizable effect in the Solar system. Maybe it is a time to accept a new archetype of heavenly motions – *precessing ellipse* (or more generally, precessing conics). Apart from relativity there are also some quite classical arguments in its favour: Kepler-type motion is generally not preserved by small perturbations and generally any sort of "real world" interactions like Solar pressure, drag, etc, would destroy "fixed ellipse" motion [15]. If precessing conics give us "the typical" motion of planets it is tempting to ask which central force field produces them. The answer is: the *Manev model* (see [2] for the precise formulation of the statement). Here we already have persistent KAM tori and cylinders for a large class of even non-Hamiltonian perturbations [15] and this is an additional argument in favour of it.

Kepler problem is famous as one of archetypes of superintegrable systems and it is intriguing to ask whether Manev problem shares this property. Recently we reported [14] that indeed it has an additional independent globally defined constant of motion, but not for all initial data. Let us remark that for a generic central potential we could have disjoint set of initial data corresponding to closed orbits but in our case *all* points on certain level sets of the angular momentum lie on closed orbits which are intersections with the level sets of the additional invariant.

Also, it was shown that the real form dynamics of the Manev problem – a closely related dynamical model to be introduced below – is superintegrable for *all* initial data. Real form dynamics of the Manev problem is interesting enough and we describe it briefly at the end of the article.

One may expect that superintegrability could be connected with some hidden symmetry group, or at least some symmetry algebra. In principle, finding of such a connection is not a trivial task. For example, we have the list of natural mechanical superintegrable models with integrals quadratic in momenta in [10] but still very little is known about their symmetry algebras, see e.g. [11]. (The models we will be concerned with fall beyond the scope of this classification and they are not present in the list.)

We have already found in [14] the algebra of Poisson brackets between the angular momentum and the (properly redefined) new first integrals and, unfortunately, it bears no resemblance with the symmetry algebra of the Kepler model. Here we are able to construct new constants of motion which together with a new set of Poisson brackets realize an explicit $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ symmetry algebra of the Manev model. If the values of the parameter B in equation (1) below are large enough (and irrespectively of the sign of the energy) we obtain an $\mathfrak{so}(2,1)$ symmetry algebra. Similarly, we have shown that the real form Manev model also have $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ as symmetry algebras. Thus we see that the Kepler and Manev models actually have identical symmetry algebras. What is different is that Manev model algebras are the same for both negative and positive energies and that they do not lead to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, or $\mathrm{SO}(2,1)$ group action on the phase space.

Recently some connection has been observed between superintegrability and the not so familiar type of dynamical but non-symplectic symmetries [19, 24, 4, 6]. Here we also see the appearance of two dynamical but non-symplectic symmetries thus enlarging the list of symmetry properties of the Manev model and its real form dynamics.

2. The Manev Problem Basics

By **Manev model** [21] we mean here the dynamics given by the Hamiltonian function

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{A}{r} - \frac{B}{r^2} \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and A and B are assumed to be arbitrary real constants whose positive values correspond to attractive forces. The genuine model proposed by George Manev was not invented as an approximation of relativity theory but as a consequence of Max Planck's (more general) action-reaction principle and it was associated with a specific value of the constant $B = \frac{3G}{2c^2}A$. Nevertheless, Manev model offers a surprisingly good practical approximation to Einstein's relativistic dynamics – at least at a Solar system level – capable to describe both the perihelion advance of the inner planets and the Moon's perigee motion. In the last decade it had enjoyed an increased interest either as a very suitable approximation from astronomers' point of view or as a toy model for applying different techniques of the modern mechanics and its dynamics has already been thoroughly analyzed (see e.g. [22, 23, 7, 3, 9]).

Due to the rotational invariance each component of the angular momentum $L = (L_1, L_2, L_3)$

$$L_j = \varepsilon_{jkm} p_k x_m \qquad \text{with} \qquad (x_1, x_2, x_3) = (x, y, z) \tag{2}$$

is an obvious first integral: $\{H, L_j\} = 0$ and so, like the Kepler problem, the Manev model is integrable. The components L_i themselves are not in involution but span an $\mathfrak{so}(3)$ algebra with respect to the Poisson bracket

$$\{L_j, L_k\} = \varepsilon_{jkm} L_m. \tag{3}$$

The dynamics is confined in a plane which we assume to be Oxy and is separable in polar coordinates r and $\theta = \arctan(y/x)$. On the reduced phase space (see e.g. [13] for the generalities of the reduction procedure) obtained by fixing the angular momentum $L_z \equiv L$ to a certain value ℓ the motion is governed by

$$H_{\text{eff}} = \frac{1}{2}\left(p_r^2 + \frac{\ell^2 - 2B}{r^2}\right) - \frac{A}{r}. \tag{4}$$

The dynamics behave like radial motion of Kepler dynamics with angular momentum squared $\ell^2 - 2B$; while the case $2B > \ell^2$ corresponds to overall centripetal effect. On the other hand, the angular equation of motion $\dot{\theta} = \ell/r^2$ is still governed by the "authentic" angular momentum ℓ (and r is as just described). Consequently, the remarkable properties of Kepler dynamics that all negative energy orbits are closed and the frequencies of radial and angular motions coincide (for any initial conditions) are not anymore true. Thus we may have not only purely classical perihelion shifts but also if $2B \geq \ell^2 \neq 0$ we may have collapsing trajectories which are spirals, even though in phase space they are symplectic transformations – while in the Kepler dynamics the only allowed fall down is along straight lines. For this reason the set of initial data leading to collision has a positive measure and this may offer an explanation why collisions in the Solar system are estimated to happen more often than Newton theory predicts [8].

3. The Kepler Problem Invariants

In the case of the **Kepler problem**, corresponding to $B = 0$, we have more first integrals (for details and historical notes see e.g. [16, 17, 25, 5])

$$J_x = p_y L - \frac{A}{r}x, \qquad J_y = -p_x L - \frac{A}{r}y, \qquad \{H_K, \vec{J}\} = 0 \tag{5}$$

where H_K is the Kepler Hamiltonian and J_x and J_y are the components of the **Laplace–Runge–Lenz vector**. They are not independent since

$$J^2 = 2H_K L^2 + A^2. \tag{6}$$

Together with the Hamiltonian and angular momentum they close on an algebra with respect to the Poisson bracket

$$\begin{aligned} \{H_K, L\} &= 0, & \{L, J_x\} &= J_y \\ \{L, J_y\} &= -J_x, & \{J_x, J_y\} &= -2H_K L. \end{aligned} \tag{7}$$

After redefining $\vec{E} = \vec{J}/\sqrt{|-2H_K|}$ we get

$$\{L, E_x\} = E_y, \qquad \{L, E_y\} = -E_x, \qquad \{E_x, E_y\} = -\operatorname{sign}(H_K)L \tag{8}$$

which makes obvious the fact that we have an $\mathfrak{so}(3)$ algebra for negative energies and $\mathfrak{so}(2,1)$ for positive ones. In the case of the three-dimensional Kepler problem the components of the angular momentum give us another copy of $\mathfrak{so}(3)$, see equation (3), so the full symmetry algebra is either $\mathfrak{so}(4)$ or $\mathfrak{so}(3,1)$ depending on the sign of H_K.

According to [17], the first use of these first integrals was made by J. Hermann (known also as J. Ermanno) in 1710 (in order to find *all* possible orbits under an inverse square law force) in the disguise of "Ermanno–Bernoulli" constants

$$J_\pm = J_x \pm \mathrm{i}J_y = \left(\frac{L^2}{r} - A \mp \mathrm{i}Lp_r\right)\mathrm{e}^{\pm\mathrm{i}\theta} \tag{9}$$

which satisfy

$$\{H_K, J_\pm\} = 0, \qquad \{L, J_\pm\} = \pm\mathrm{i}J_\pm, \qquad \{J_+, J_-\} = -4\mathrm{i}H_K L. \tag{10}$$

4. The Manev Problem Invariants and Symmetries

In order to obtain the equation for the trajectories of the Manev model in the case of non-vanishing angular momentum we note that due to $\dot\theta = \ell/r^2$ we have $\mathrm{d}t = r^2\,\mathrm{d}\theta/\ell$. As a result the equation for the radial motion takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\frac{\ell^2}{r} + \frac{\ell^2 - 2B}{\ell^2}\frac{\ell^2}{r} - A = 0 \tag{11}$$

and thus, as in the Kepler model, we could have harmonic oscillations of $1/r$-variable as a function of the "false" time θ.

4.1. The $\ell^2 > 2B > 0$ Case

Denoting

$$\nu^2 = \frac{\ell^2 - 2B}{\ell^2}, \qquad w = \frac{\ell^2}{r} - \frac{\ell^2}{\ell^2 - 2B}A \tag{12}$$

one easily verifies that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(\nu w \pm \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}w\right)\mathrm{e}^{\pm\mathrm{i}\nu\theta}\right] = 0 \tag{13}$$

and since

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{r^2}{\ell}\frac{\mathrm{d}}{\mathrm{d}t} \tag{14}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\nu\frac{\ell^2}{r} - \frac{A}{\nu} \mp \mathrm{i}\ell p_r\right)\mathrm{e}^{\pm\mathrm{i}\nu\theta}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\pm = 0. \tag{15}$$

4.1.1. Compact Motion Case

In the case when $\ell^2 > 2B > 0$, $H < 0$ and $A > 0$ the motion is on a two-dimensional torus. In order to have globally defined constants of motion in this case we have to require that the ν's are rational, i.e.,

$$\nu = \sqrt{\ell^2 - 2B} : \ell = m : k \tag{16}$$

with m and k mutually prime integers. Then due to equation (13)

$$\mathcal{J}_\pm = \mathcal{J}^*_\mp = \left[\frac{m}{k}\frac{\ell^2}{r} - \frac{k}{m}A \mp \mathrm{i}\ell p_r\right]\mathrm{e}^{\pm\mathrm{i}m\theta/k} \tag{17}$$

are conserved by the flow of equation (1) on the surface $L = \ell$ satisfying the rationality condition (16). Thus we have conditional constants of motion which exist only for disjoint but infinite set of values ℓ (c.f. the invariant relations in [18]).

The trajectory in the configuration space is a "rosette" with m petals and this is connected to the fact that $\mathcal{J}_\pm$ are invariant under the action of the cyclic group generated by rotations by angle $2\pi k/m$

$$\theta \to \theta + 2\pi \frac{k}{m} n, \qquad n = 0, 1, \ldots, m-1. \tag{18}$$

While in the Kepler case we could unambiguously attach the Laplace–Runge–Lenz vector to Ermanno–Bernoulli invariants this is not possible now due to this finite symmetry. (It is intuitively clear that if the Laplace–Runge–Lenz vector points to the perihelion of the Kepler ellipse, now we have m petals to choose between.) Anyway, up to this ambiguity, *or* restricting ourselves to one of the m sectors we may note that while the radial/angular components of the Laplace–Runge–Lenz vector take the form

$$J_r = \frac{L^2}{r} - A, \qquad J_\theta = -L p_r \tag{19}$$

one has $\mathcal{J}_r + \mathrm{i}\mathcal{J}_\theta = \left(\nu\frac{\ell^2}{r} - \frac{A}{\nu} - \mathrm{i}\ell p_r\right) \mathrm{e}^{\mathrm{i}(\nu-1)\theta}$ and hence

$$\begin{aligned} \mathcal{J}_r &= \left(\nu\frac{\ell^2}{r} - \frac{A}{\nu}\right)\cos(\nu-1)\theta + \ell p_r \sin(\nu-1)\theta \\ \mathcal{J}_\theta &= -\ell p_r \cos(\nu-1)\theta + \left(\nu\frac{\ell^2}{r} - \frac{A}{\nu}\right)\sin(\nu-1)\theta. \end{aligned} \tag{20}$$

Let us denote by $\sqrt{I\ell}$ and φ the modulus and the phase of the $\nu\frac{\ell^2}{r} - \frac{A}{\nu} \mp \mathrm{i}\ell p_r$ term, and by χ the phase of $\mathcal{J}_-$, i.e.

$$\mathcal{J}_- = \left[\nu\frac{\ell^2}{r} - \frac{A}{\nu} + \mathrm{i}\ell p_r\right] \mathrm{e}^{\mathrm{i}\nu\theta} = \sqrt{I\ell}\,\mathrm{e}^{\mathrm{i}(\varphi-\nu\theta)} = \sqrt{I\ell}\,\mathrm{e}^{\mathrm{i}\chi} \tag{21}$$

with

$$I = \frac{\mathcal{J}_+\mathcal{J}_-}{\ell} = 2H\ell + \frac{A^2}{\nu^2\ell} \tag{22}$$

and let

$$a = \sqrt{I}\,\mathrm{e}^{\mathrm{i}\varphi}, \qquad b = \sqrt{L}\,\mathrm{e}^{\mathrm{i}\theta}. \tag{23}$$

One may note also that $r(\theta)$ dynamics could be equally well described by the Hamiltonian

$$H^* = \nu I + L = \nu a^\dagger a + b^\dagger b \tag{24}$$

and the symplectic form

$$\omega = \mathrm{d}I \wedge \mathrm{d}\varphi + \mathrm{d}L \wedge \mathrm{d}\theta = \mathrm{i}\mathrm{d}a \wedge \mathrm{d}a^\dagger + \mathrm{i}\mathrm{d}b \wedge \mathrm{d}b^\dagger \tag{25}$$

or the Poisson brackets

$$\begin{aligned} \{I, \varphi\}^* &= 1, \quad \{L, \theta\}^* = \{L, \theta\} = 1 \\ \{a^\dagger, a\}^* &= \mathrm{i}, \quad \{b^\dagger, b\}^* = \mathrm{i} \end{aligned} \tag{26}$$

with θ taking the role of evolution parameter ("time") due to $\dot{\theta} = \{H^*, \theta\}^* = 1$. Inspired by the harmonic oscillators case (see e.g. [20]) let us introduce the first integrals

$$\begin{aligned} K_0 &= \frac{\nu I + L}{2\nu} = \frac{H^*}{2\nu}, \qquad K_1 = \sqrt{\frac{IL}{\nu}} \sin\chi \\ K_2 &= \sqrt{\frac{IL}{\nu}} \cos\chi, \qquad K_3 = \frac{\nu I - L}{2\nu} \end{aligned} \tag{27}$$

which are not independent as $K_0^2 - (K_1^2 + K_2^2 + K_3^2) = 0$ and thus forming a space of invariants lying on $\mathbb{S}^2$. Up to a coefficient K_0, K_1 and K_2 are actually the new Hamiltonian and $(\mathcal{J}_+ \mp \mathcal{J}_-)$. It is worth noting that they have also the cyclic symmetry (18) and strictly speaking they are properly defined only on the restricted phase space (diffeomorphic to one of the m sectors) obtained by factoring out the corresponding finite group.

It is easy to check that K_1, K_2, K_3 close an $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ algebra under the new Poisson bracket (26)

$$\{K_j, K_k\}^* = \varepsilon_{jkm} K_m \tag{28}$$

and that together with K_0 they form a $\mathfrak{u}(2)$ algebra. As a result the Manev model possesses a symmetry algebra as large as the symmetry algebra of the Kepler model (8). (Of course, we have in addition the $\mathfrak{so}(3)$ angular momentum algebra (3)).

The mere existence of an algebra of well defined first integrals does not presuppose suitable group action on the phase space. Even in the (seemingly trivial) case of two commensurate harmonic oscillators the candidates for group orbits hit singular points which make the unambiguous continuation of the orbits impossible [1]. Here one sees an even more immediate obstacle for the existence of a $\mathrm{SU}(2)$ group action as K_1 and K_2 does not commute with L and hence does not preserve any $L = \ell$ surface.

Dynamical but Non-Symplectic Symmetries

Let us remind that the Hamiltonian vector field for any first integral $\mathfrak{J}$ not only preserves the Hamiltonian and the symplectic form: $X_{\mathfrak{J}}(H^*) = 0 = \mathcal{L}_{X_{\mathfrak{J}}}\omega$ but also it is a dynamical symmetry, i.e., its Lie bracket with the dynamical vector field Γ vanishes

$$[\Gamma, X_{\mathfrak{J}}] = 0 \quad \text{while} \quad \Gamma \lrcorner\, \omega = -\mathrm{d}H^*. \tag{29}$$

As one and the same dynamics could be obtained from different pairs of Hamiltonians plus symplectic forms, it is interesting to know more about the symmetries which preserve the dynamical vector field without requiring the preserving of a certain pair (H, ω). It is worth noting that Manev model *has* such dynamical but non-symplectic symmetries.

By choosing $\mathfrak{J} = \sqrt{L^{\nu-1}}\mathcal{J}_- = a(b^\dagger)^\nu$ one can see that $X_\mathfrak{J}$ splits further into a linear combination of two *new* dynamical symmetries Y, Y' which *do not* preserve the Hamiltonian and the symplectic form

$$\begin{aligned} X_\mathfrak{J} &= -\mathrm{i}Y + \mathrm{i}\nu Y' \\ Y &= \frac{\mathfrak{J}}{a}\frac{\partial}{\partial a^\dagger} = \mathrm{i}\frac{\mathfrak{J}}{a}X_a = (b^\dagger)^\nu \frac{\partial}{\partial a^\dagger} \\ Y' &= \frac{\mathfrak{J}}{b^\dagger}\frac{\partial}{\partial b} = -\mathrm{i}\frac{\mathfrak{J}}{b^\dagger}X_{b^\dagger} = a(b^\dagger)^{\nu-1}\frac{\partial}{\partial b} \end{aligned} \tag{30}$$

as

$$[\Gamma, Y] = \left[\Gamma, \frac{\mathrm{i}\mathfrak{J}}{a}X_a\right] = \Gamma\left(\frac{\mathrm{i}\mathfrak{J}}{a}\right) + \frac{\mathrm{i}\mathfrak{J}}{a}[\Gamma, X_a] = \frac{\nu\mathfrak{J}}{a^2}aX_a - \frac{\nu\mathfrak{J}}{a}X_a = 0 \tag{31}$$

$$Y(H^*) = \nu\mathfrak{J}, \qquad \mathcal{L}_Y\omega = \frac{-\mathrm{i}}{a}\,\mathrm{d}\mathfrak{J}\wedge\mathrm{d}a \tag{32}$$

and similarly for Y'.

4.1.2. Noncompact Motion Case

When $\ell^2 > 2B > 0$ and either $H \geq 0$ or $A \leq 0$ the additional invariant is always globally defined. The Lie group $\mathrm{SO}(3)$ does not act globally on the space of invariants as the $K_3 = \frac{\nu I - \sqrt{|2B|}}{2\nu}$ "parallel" on the sphere $\mathbb{S}^3$ is not accessible by the orbits due to the violation of the condition $\ell^2 > 2B$. The new dynamical symmetries in this case are exactly those just described in equations (30).

4.2. The $0 < \ell^2 \leq 2B$ Case

Similarly, in the case when $0 < \ell^2 < 2B$ we may denote $\frac{2B-\ell^2}{\ell^2} = \upsilon^2$ with υ real and

$$\mathcal{J}_\pm = \left[\upsilon\frac{\ell^2}{r} + \frac{A}{\upsilon} \mp \ell p_r\right]\mathrm{e}^{\pm\upsilon\theta} \equiv \sqrt{I\ell}\,\mathrm{e}^{\mp(\varphi-\upsilon\theta)} \tag{33}$$

with

$$I = \frac{\mathcal{J}_+\mathcal{J}_-}{\ell} = -2H\ell + \frac{A^2}{\upsilon^2\ell} \tag{34}$$

will be first integrals for *any* ℓ. Denoting again $\chi = \varphi - \upsilon\theta$ and assuming the same Poisson brackets as (26) we obtain an $\mathfrak{so}(2,1)$ algebra

$$\{K_1, K_2\}^* = K_3, \qquad \{K_3, K_1\}^* = K_2, \qquad \{K_2, K_3\}^* = -K_1 \tag{35}$$

formed by the first integrals

$$K_1 = \sqrt{\frac{IL}{v}}\sinh\chi, \qquad K_2 = \sqrt{\frac{IL}{v}}\cosh\chi, \qquad K_3 = \frac{vI - L}{2v} \tag{36}$$

which satisfy $K_0^2 + K_1^2 - K_2^2 - K_3^2 = 0$ with

$$K_0 = \frac{vI + L}{2v} \equiv \frac{H^*}{2v} \tag{37}$$

so that this time the space of invariants is lying on a single sheeted hyperboloid and again a certain level set of K_3 corresponding to $\ell^2 = 2B$ obstructs the global action of the group $\mathrm{SO}(2,1)$.

Defining

$$Y = (b^\dagger)^\nu \frac{\partial}{\partial a^\dagger}, \quad Y' = a(b^\dagger)^{\nu-1}\frac{\partial}{\partial b} \quad \text{with} \quad a = \sqrt{I}\,\mathrm{e}^{\mathrm{i}\varphi}, \quad b = \sqrt{L}\,\mathrm{e}^{\mathrm{i}\theta} \tag{38}$$

we easily see that again Y and $Y^\dagger$ are dynamical but non-symplectic symmetries.

Finally, when $\ell^2 = 2B$ we have the first integral

$$j = \ell p_r + A\theta \tag{39}$$

satisfying $\{H, j\} = 0$, $\{L, j\} = A$.

5. Real Form Dynamics

Here we briefly recall the notion of real form (RF) dynamics referring the reader to [12] for more details and a list of examples.

We start with a standard (real) Hamiltonian system $\mathcal{H} \equiv \{\mathcal{M}, \omega, H\}$ with n degrees of freedom and at the present stage we assume that our phase space is just a vector space $\mathcal{M} = \mathbb{R}^{2n}$.

Let us consider its complexification: $\mathcal{H}^{\mathbb{C}} \equiv \left\{\mathcal{M}^{\mathbb{C}}, H^{\mathbb{C}}, \omega^{\mathbb{C}}\right\}$ where $\mathcal{M}^{\mathbb{C}}$ can be viewed as a linear space over the field of complex numbers

$$\mathcal{M}^{\mathbb{C}} = \mathcal{M} \oplus \mathrm{i}\mathcal{M}.$$

In other words the dynamical variables in $\mathcal{M}^{\mathbb{C}}$ now take complex values. We assume that the Hamiltonian H (as well as all other possible first integrals in involution I_k) are *real analytic functions* on $\mathcal{M}$ which can naturally be extended to $\mathcal{M}^{\mathbb{C}}$. We introduce on the phase space $\mathcal{M}$ an involutive, symplectic automorphism $\mathcal{C} : \mathcal{M} \to \mathcal{M}$

$$\mathcal{C}^2 = \mathbb{1}, \qquad \mathcal{C}(\{F, G\}) = \{\mathcal{C}(F), \mathcal{C}(G)\} \tag{40}$$

where with some abuse of terminology we use the same notation for the action of $\mathcal{C}$ on the dual of the phase space.

Since $\mathcal{C}$ has eigenvalues 1 and -1, it naturally splits $\mathcal{M}$ into two eigenspaces

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_- \tag{41}$$

whose dimensions need not be equal. Due to the fact that $\mathcal{C}$ is symplectic $\mathcal{M}_-$ and $\mathcal{M}_+$ are symplectic subspaces of $\mathcal{M}$ and we will write $\omega = \omega_+ \oplus \omega_-$.

Assuming a symplectic frame adapted to $\mathcal{C}$ we have

$$\omega = \sum_{k=1}^{n_+} \mathrm{d}p_{k+} \wedge \mathrm{d}q_{k+} + \sum_{k=1}^{n_-} \mathrm{d}p_{k-} \wedge \mathrm{d}q_{k-}.$$

The automorphism $\mathcal{C}$ can naturally be extended to $\mathcal{M}^{\mathbb{C}}$ which is splitted also into a direct sum of two eigenspaces

$$\mathcal{M}^{\mathbb{C}} = \mathcal{M}_-^{\mathbb{C}} \oplus \mathcal{M}_+^{\mathbb{C}}.$$

Similarly, the action of the complex conjugation * produces splitting into real and imaginary parts of the corresponding spaces. By construction $\mathcal{C}$ commutes with $*$ and their composition $\widetilde{\mathcal{C}} \equiv \mathcal{C} \circ {}^* = {}^* \circ \mathcal{C}$ is also an involutive symplectic automorphism on $\mathcal{M}^{\mathbb{C}}$ so that we can define $\mathcal{M}_{\mathbb{R}}$ to be the fixed point set of $\widetilde{\mathcal{C}}$, i.e.,

$$\mathcal{M}_{\mathbb{R}} = \mathrm{Re}\, \mathcal{M}_+^{\mathbb{C}} \oplus \mathrm{i}\, \mathrm{Im}\, \mathcal{M}_-^{\mathbb{C}}$$

which is again a symplectic subspace. From now on we will be interested in dynamics on $\mathcal{M}_{\mathbb{R}}$ and its connection to the initial real dynamical system.

In order to construct "real form dynamics" we shall assume that the Hamiltonian is $\mathcal{C}$-invariant, i.e.,

$$\mathcal{C}(H) = H. \tag{42}$$

Then the Hamiltonian on the complexified phase space $H^{\mathbb{C}}$ (being the same analytical function of the complexified variables) will share this property.

The **real form dynamics** may be defined either as:

i) complexified Hamilton equations on $\mathcal{M}^{\mathbb{C}}$ being consistently restricted to $\mathcal{M}_{\mathbb{R}}$. This gives a real vector field tangent to $\mathcal{M}_{\mathbb{R}}$ and satisfying the equations of motion given by the real part of $H^{\mathbb{C}}$ or

ii) dynamics on $\mathcal{M}_{\mathbb{R}}$ defined by the restricted $H^{\mathbb{C}}$ and $\omega^{\mathbb{C}}$ (whose restrictions are real on $\mathcal{M}_{\mathbb{R}}$)

$$\begin{aligned} H|_{\mathcal{M}_{\mathbb{R}}} &= \frac{H + \widetilde{\mathcal{C}}(H)}{2} = \frac{H + \mathcal{C}(H)^*}{2} = \mathrm{Re}\, H^{\mathbb{C}} \\ \omega^{\mathbb{C}}\Big|_{\mathcal{M}_{\mathbb{R}}} &= \mathrm{d}\,\mathrm{Re}\, p_+^{\mathbb{C}} \wedge \mathrm{d}\,\mathrm{Re}\, q_+^{\mathbb{C}} - \mathrm{d}\,\mathrm{Im}\, p_-^{\mathbb{C}} \wedge \mathrm{d}\,\mathrm{Im}\, q_-^{\mathbb{C}}. \end{aligned} \tag{43}$$

Now we have a well defined dynamical system $\mathcal{H}_{\mathbb{R}} = \{\mathcal{M}_{\mathbb{R}}, \omega|_{\mathcal{M}_{\mathbb{R}}}, H|_{\mathcal{M}_{\mathbb{R}}}\}$ with *real Hamiltonian* and *real symplectic form* on a subspace of the complexified phase space.

It is noteworthy that the "real form dynamics" corresponding to a Liouville integrable Hamiltonian system is Liouville integrable again [12]. Similarly, the "real form dynamics" corresponding to a superintegrable Hamiltonian system is superintegrable as well. In such a case we have $\kappa \in [n+1, 2n-1]$ independent constants of motion which are no more in involution. It could easily be checked that they will again produce κ independent constants of motion of the RF dynamics.

6. Real Form Dynamics of the Manev Problem

The Manev Hamiltonian and the canonical symplectic form are invariant under the involution $\mathcal{C}$ reflecting the y-degree of freedom

$$\begin{aligned} \mathcal{C}(x) &= x, \qquad & \mathcal{C}(y) &= -y, \qquad & \mathcal{C}(z) &= z \\ \mathcal{C}(p_x) &= p_x, \qquad & \mathcal{C}(p_y) &= -p_y, \qquad & \mathcal{C}(p_z) &= p_z. \end{aligned} \tag{44}$$

Consequently, the "real form dynamics" of Manev model for this choice of involution will be given by

$$\begin{aligned} H_{\mathbb{R}} &= \frac{1}{2}(p_x^2 - p_y^2 + p_z^2) - \frac{A}{\rho} - \frac{B}{\rho^2} \\ \omega_{\mathbb{R}} &= \mathrm{d}p_x \wedge \mathrm{d}x - \mathrm{d}p_y \wedge \mathrm{d}y + \mathrm{d}p_z \wedge \mathrm{d}z \end{aligned} \tag{45}$$

where $\rho = \sqrt{x^2 - y^2 + z^2}$ is the "radius" of the pseudo-sphere. This is not an ordinary central field dynamics but rather an "indefinite metric central field" as $H_{\mathbb{R}}$ depends on indefinite metric distance ρ. The real form Hamiltonian $H_{\mathbb{R}}$ and the appropriate "angular momentum" $\tilde{L}_j$ are still commuting first integrals and the model is integrable. The involution acts on $\tilde{L}_j$ according to: $\mathcal{C}(\tilde{L}_j) = (-1)^j \tilde{L}_j$ and instead of equation (3) we have

$$\{\tilde{L}_j, \tilde{L}_k\} = \varepsilon_{jki}(-1)^{j+k+1}\tilde{L}_i \tag{46}$$

so that the corresponding algebra is $\mathfrak{so}(2,1)$ and this is the real form of $\mathfrak{so}(3)$ obtained with a $\mathcal{C}$-induced Cartan involution.

We shall assume again that the motion is on the Oxy-plane and in order to avoid the question of the behavior of trajectories at the singularities we restrict our attention to $\mathcal{C}$-invariant configuration space

$$\{(x,y,z) \in \mathbb{R}^3;\, x > 0,\; x^2 > y^2,\; z = 0\}.$$

Then the dynamics is separable in pseudo-radial coordinates $\vartheta = \operatorname{artanh}(y/x) \in (-\infty, \infty)$ and $\rho \in (0, \infty)$

$$\begin{aligned} H &= \frac{1}{2}\left(p_\rho^2 - \frac{\pi_\vartheta^2}{\rho^2}\right) - \frac{A}{\rho} - \frac{B}{\rho^2} \\ \omega &= \mathrm{d}p_\rho \wedge \mathrm{d}\rho + \mathrm{d}\pi_\vartheta \wedge \mathrm{d}\vartheta \end{aligned} \tag{47}$$

with $\tilde{L} \equiv \tilde{L}_z = \pi_\vartheta$, hence $\dot{\pi_\vartheta} = 0$ and $\dot{\vartheta} = -\tilde{L}/\rho^2$. Due to the new symplectic form $\tilde{L}$ generates now transformations which preserve ρ.

In order to obtain an equation for the trajectories let us note again that in the case of non-vanishing angular momentum we have $\mathrm{d}t = -\frac{\rho^2}{\ell}\mathrm{d}\vartheta$. As a result the equation for the ρ-motion takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2}\frac{\ell^2}{\rho} - \frac{\ell^2+2B}{\ell^2}\frac{\ell^2}{\rho} - A = 0. \tag{48}$$

Assuming $\ell^2 + 2B \neq 0$ we introduce

$$v^2 = \frac{\ell^2+2B}{\ell^2}, \qquad w = \frac{\ell^2}{\rho} + \frac{\ell^2}{\ell^2+2B}A \tag{49}$$

and obtain an inverted oscillator–type equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2}w - v^2 w = 0. \tag{50}$$

The analysis of the resulting trajectories could be found in [14] and we shall not reproduce it here but we shall concentrate on the symmetry properties of the model.

6.1. Symmetries of the Real Form Manev Model

Proceeding as before, one could easily find the additional first integrals. What is different is that since the motion is never on a two-torus the new integrals are always globally defined for *all* initial data. When $0 \neq \ell^2 > -2B$ they take the form

$$\mathcal{J}_\pm = \left[v\frac{\ell^2}{\rho} + \frac{A}{v} \pm \ell p_\rho\right]\mathrm{e}^{\pm v\vartheta} \equiv \sqrt{I\ell}\,\mathrm{e}^{\mp(\varphi - v\vartheta)} \tag{51}$$

with

$$I = \frac{\mathcal{J}_+\mathcal{J}_-}{\ell} = -2H\ell + \frac{A^2}{v^2\ell}. \tag{52}$$

Denoting again $\chi = \varphi - v\vartheta$ and assuming the same Poisson brackets as (26) we obtain the $\mathfrak{so}(2,1)$ algebra (35) formed by the first integrals

$$K_1 = \sqrt{\frac{IL}{v}}\sinh\chi, \qquad K_2 = \sqrt{\frac{IL}{v}}\cosh\chi, \qquad K_3 = \frac{vI - L}{2v} \tag{53}$$

which satisfy $K_0^2 + K_1^2 - K_2^2 - K_3^2 = 0$ with

$$K_0 = \frac{vI + L}{2v} \equiv \frac{H^*}{2v}. \tag{54}$$

Following the same line of reasoning as at the end of Section 4 we conclude that the global action of $\mathrm{SO}(2,1)$ is obstructed due to the level set $\ell^2 = -2B$.

Defining again

$$Y = (b^\dagger)^\nu \frac{\partial}{\partial a^\dagger}, \quad Y' = a(b^\dagger)^{\nu-1}\frac{\partial}{\partial b} \quad \text{with} \quad a = \sqrt{I}\,\mathrm{e}^{\mathrm{i}\varphi}, \quad b = \sqrt{L}\,\mathrm{e}^{\mathrm{i}\theta} \tag{55}$$

we obtain that Y and Y' are dynamical but non-symplectic symmetries.

In the case when $0 \neq \ell^2 < -2B$ let $\nu^2 = \frac{-(\ell^2+2B)}{\ell^2}$ and we obtain new invariants which are globally defined for *any* ℓ

$$\mathcal{J}_\pm = \mathcal{J}^*_\mp = \left[\nu\frac{\ell^2}{\rho} - \frac{A}{\nu} \pm \mathrm{i}\ell p_\rho\right]\mathrm{e}^{\pm\mathrm{i}\nu\vartheta} \equiv \sqrt{I\ell}\,\mathrm{e}^{\mp\mathrm{i}(\varphi-\upsilon\vartheta)} = \sqrt{I\ell}\,\mathrm{e}^{\mp\mathrm{i}\chi} \tag{56}$$

with

$$I = \frac{\mathcal{J}_+\mathcal{J}_-}{\ell} = 2H\ell + \frac{A^2}{\nu^2\ell}, \qquad \chi = \varphi - \upsilon\vartheta. \tag{57}$$

Again we can introduce the first integrals forming an $\mathfrak{u}(2)$ algebra

$$K_0 = \frac{\nu I + L}{2\nu} \qquad K_1 = \sqrt{\frac{IL}{\nu}}\sin\chi$$
$$K_2 = \sqrt{\frac{IL}{\nu}}\cos\chi, \qquad K_3 = \frac{\nu I - L}{2\nu} \tag{58}$$

which are not independent as $K_0^2 - (K_1^2 + K_2^2 + K_3^2) = 0$ and again we do not have well defined action of $\mathrm{SO}(3)$.

Here again we have the dynamical but non-symplectic symmetries Y and $Y^\dagger$ given by

$$Y = (b^\dagger)^\nu \frac{\partial}{\partial a^\dagger}, \quad Y' = a(b^\dagger)^{\nu-1}\frac{\partial}{\partial b} \quad \text{with} \quad a = \sqrt{I}\mathrm{e}^{\mathrm{i}\varphi}, \quad b = \sqrt{L}\mathrm{e}^{\mathrm{i}\theta}. \tag{59}$$

Finally, when $\ell^2 = -2B$ we have the first integral

$$j = \ell p_\rho - A\vartheta \tag{60}$$

satisfying $\{H, j\} = 0$, $\{\tilde{L}, j\} = -A$.

7. Conclusions

We have shown that Manev model possesses Ermanno–Bernoulli type invariants and symmetry algebras $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ in addition to the angular momentum algebra. These two facts indicate that the Manev model has an exceptional position among the central field theories. It provides a better description of the real motion of the heavenly bodies than Kepler model and in the same time it shares its most celebrated mathematical features: its superintegrability and large symmetry algebras.

Also, we see here an example when the RF dynamics, exotic as it may be, behaves "better" than the original problem remaining always superintegrable.

Acknowledgments

This research was partially supported by PRIN SINTESI. Two of us (AK and VG) are grateful to Physics Department of the University of Salerno for the kind hospitality and acknowledge the partial support by the Italian Istituto Nazionale di Fisica Nucleare and the National Science Foundation of Bulgaria, contract No. F-1410.

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