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LIE SYSTEMS OF DIFFERENTIAL EQUATIONS AND CONNECTIONS IN FIBRE BUNDLES

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Abstract. The study of systems of differential equations admitting a superposition function allowing us to write the general solution in terms of a set of arbitrary, but independent, particular solutions, and some constants determining each solution, can be reduced to that of an equation on a Lie group. It will be shown that all these systems of differential equations can be seen as the systems determining the horizontal curves on an appropriate connection and we will show how the theory of reduction can be used to simplify the problem of finding the general solution of such Lie systems. The theory will be illustrated with several physical applications.

1. Introduction: The Nonlinear Superposition Property

It is well-known that for homogeneous linear differential equation systems of type

$$\frac{\mathrm{d}y^i}{\mathrm{d}x} = \sum_{j=1}^{n} A^i{}_j(x)\, y^j \qquad i = 1, \dots, n \tag{1}$$

the general solution can be written as a linear combination of n independent particular solutions, $y_{(1)}, \dots, y_{(n)}$,

$$y = \Phi(y_{(1)}, \dots, y_{(n)}, k_1, \dots, k_n) = k_1\, y_{(1)} + \cdots + k_n\, y_{(n)}$$

and for each set of initial conditions, the coefficients can be determined. For an inhomogeneous linear system,

$$\frac{\mathrm{d}y^i}{\mathrm{d}x} = \sum_{j=1}^{n} A^i{}_j(x)\, y^j + B^i(x) \qquad i = 1, \dots, n \tag{2}$$

the general solution can be written as an affine function of $n + 1$ independent particular solutions

$$y = \Phi(y_{(1)}, \ldots, y_{(n+1)}, k_1, \ldots, k_n)$$
$$= y_{(1)} + k_1(y_{(2)} - y_{(1)}) + \cdots + k_n(y_{(n+1)} - y_{(1)}).$$

Under a non-linear change of coordinates both systems become non-linear ones. However, the fact that the general solution is expressible in terms of a set of particular solutions is maintained, but the superposition function is no longer linear or affine, respectively. For instance, the general solution of the linear equation $y' = a_1(x)y + a_0(x)$ can be written as a linear combination of two solutions y_1 and y_2, $y = ky_1 + (1-k)y_2$, and it is well-known that the change of variable $u = y^{\frac{1}{1-n}}$ transforms the equation into a Bernoulli equation $u' = a_1(x)u + a_0(x)u^n$. Consequently, the general solution of this last equation can be written as a function of two particular solutions as follows

$$u = [ku_1^{1-n} + (1-k)u_1^{1-n}]^{1/(1-n)}.$$

The very existence of such examples of systems of differential equations admitting a superposition function suggests the problem of characterizing them. That is, to determine what are the systems of differential equations for which a superposition function allowing to express the general solution in terms of m particular solutions does exist. The theorem, giving the answer to this question, is due to Lie [20]

Theorem 1. *Let us consider the system of first order differential equations*

$$\frac{dy^i}{dx} = X^i(y^1, \ldots, y^n, x), \qquad i = 1, \ldots, n. \tag{3}$$

Then, there will be a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ *such that the general solution of the system can be expressed as*

$$y = \Phi(y_{(1)}, \ldots, y_{(m)}; k_1, \ldots, k_n)$$

where

$$\{y_{(j)}\,;\, j = 1, \ldots, m\}$$

is a set of independent particular solutions and $k_1, \ldots, k_n$*, are* n *arbitrary constants, if and only if the system can be written in the form*

$$\frac{dy^i}{dx} = Z_1(x)\xi^{1i}(y) + \cdots + Z_r(x)\xi^{ri}(y)$$

with $Z_1, \ldots, Z_r$*, being* r *functions of only* x*, and* $\xi^{\alpha i}$*,* $\alpha = 1, \ldots, r$*, are functions of the variables* $y = (y^1, \ldots, y^n)$*, such that the* r *vector fields in* $\mathbb{R}^n$ *given by*

$$Y^{(\alpha)} \equiv \sum_{i=1}^{n} \xi^{\alpha i}(y)\frac{\partial}{\partial y^i}, \qquad \alpha = 1, \ldots, r \tag{4}$$

close on a finite-dimensional real Lie algebra. Moreover, r satisfies $r \leq mn$.

From the geometric viewpoint, the system of first order differential equations (3) provides the integral curves of the x-dependent vector field on an n-dimensional manifold M

$$X = \sum_{i=1}^{n} X^i(y, x) \frac{\partial}{\partial y^i}$$

in the same way as it happens for autonomous systems and true vector fields (see, e.g., [7]). The x-dependent vector fields satisfying the hypothesis of the Theorem 1 are those which can be written as a x-dependent linear combination of vector fields,

$$X(y, x) = \sum_{\alpha=1}^{r} Z_\alpha(x) Y^{(\alpha)}(y) \tag{5}$$

with vector fields $Y^{(\alpha)}$ closing on a finite-dimensional real Lie algebra. They will be called Lie systems and have been the subject of a number of works by Anderson, Harnad, Winternitz and collaborators, which deal with the classification of Lie systems and their explicit superposition formulas, as well as with their applications in physics and mathematics [1,4–6,17,18,24,25,29].

Both homogeneous and inhomogeneous linear systems (1) and (2) are particular instances of Lie systems. In the first case $m = n$ and the corresponding x-dependent vector field

$$X(y, x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(x) y^j \right) \frac{\partial}{\partial y^i} \tag{6}$$

can be written as a linear combination with x-dependent coefficients $A^i{}_j(x)$ of the n^2 vector fields

$$Y^{ij} = y^j \frac{\partial}{\partial y^i}$$

which close on the $\mathfrak{gl}(n, \mathbb{R})$ algebra, because

$$[Y^{ij}, Y^{kl}] = \left[y^j \frac{\partial}{\partial y^i}, y^l \frac{\partial}{\partial y^k} \right] = \delta^{il} y^j \frac{\partial}{\partial y^k} - \delta^{kj} y^l \frac{\partial}{\partial y^i}$$

i.e.

$$[Y^{ij}, Y^{kl}] = \delta^{il} Y^{kj} - \delta^{kj} Y^{il}$$

to be compared with the commutation relations of the $\mathfrak{gl}(n, \mathbb{R})$ algebra

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

where E_{ij} denotes the matrix with elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

For inhomogeneous systems, the x-dependent vector field is

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(x) y^j + B^i(x) \right) \frac{\partial}{\partial y^i} \tag{7}$$

which is a linear combination with x-dependent coefficients, $A^i{}_j(x)$ and $B^i(x)$,

$$X = \sum_{i,j=1}^{n} A^i{}_j(x) Y^{ij} + \sum_{i=1}^{n} B^i(x) Y_i$$

of the n^2 vector fields Y^{ij} and the n vector fields

$$Y_i = \frac{\partial}{\partial y^i}, \qquad i = 1, \ldots, n.$$

Now, these last vector fields commute among themselves

$$[Y_i, Y_k] = 0, \qquad \forall\, i, k = 1, \ldots, n$$

and

$$[Y^{ij}, Y_k] = -\delta_{kj} Y_i, \qquad \forall\, i, j, k = 1, \ldots, n.$$

Therefore the vector fields $\{Y^{ij}, Y_k\,;\ i, j, k = 1, \ldots, n\}$ generate a Lie algebra isomorphic to the $(n^2 + n)$-dimensional Lie algebra of the affine group.

Another very interesting example for $n = 1$ is the Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = b_2(x) y^2(x) + b_1(x) y(x) + b_0(x) \tag{8}$$

for which $m = 3$ and there is a superposition function coming from the relation

$$\frac{y - y_{(1)}}{y - y_{(2)}} : \frac{y_{(3)} - y_{(1)}}{y_{(3)} - y_{(2)}} = k$$

or in other words,

$$y = \frac{y_{(1)}(y_{(3)} - y_{(2)}) + k y_{(2)}(y_{(1)} - y_{(3)})}{(y_{(3)} - y_{(2)}) + k(y_{(1)} - y_{(3)})}.$$

In this case the x-dependent vector field is $Y = b_0 Y^{(1)} + b_1 Y^{(2)} + b_2 Y^{(3)}$ with the vector fields $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ being given by

$$Y^{(1)} = \frac{\partial}{\partial y}, \qquad Y^{(2)} = y\frac{\partial}{\partial y}, \qquad Y^{(3)} = y^2\frac{\partial}{\partial y} \tag{9}$$

which close on a three-dimensional real Lie algebra, with defining relations

$$[Y^{(1)}, Y^{(2)}] = Y^{(1)}, \qquad [Y^{(1)}, Y^{(3)}] = 2Y^{(2)}, \qquad [Y^{(2)}, Y^{(3)}] = Y^{(3)}$$

i.e. isomorphic to the $\mathfrak{sl}(2, \mathbb{R})$ algebra.

Note that the third vector field, $Y^{(3)}$, is not complete on $\mathbb{R}$, but we can consider the one-point compactification of $\mathbb{R}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and then $Y^{(3)}$ is also complete on $\overline{\mathbb{R}}$ and the flows of such vector fields are, respectively,

$$y \mapsto y + \epsilon, \qquad y \mapsto e^{\epsilon} y, \qquad y \mapsto \frac{y}{1 - y\epsilon}$$

and therefore they can be considered as the fundamental vector fields corresponding to the action of $\mathrm{SL}(2,\mathbb{R})$ on the completed real line $\overline{\mathbb{R}}$, given by

$$\Phi(A,y)=\frac{\alpha y+\beta}{\gamma y+\delta},\qquad \text{if}\quad y\neq -\frac{\delta}{\gamma}$$
$$\Phi\left(A,-\delta/\gamma\right)=\infty,\qquad \Phi(A,\infty)=\frac{\alpha}{\gamma}$$

when

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}(2,\mathbb{R}).$$

2. Lie Systems in Lie Groups and its Homogeneous Spaces

We can study the particularly important cases of right-invariant Lie systems in Lie groups and we will show that they play a very relevant role, displaying a kind of universal character. Let G be a r-dimensional connected Lie group and $e\in G$ its neutral element. A basis of T_eG will be denoted $\{a_1,\dots,a_r\}$. The Lie group G acts on itself in a transitive and free way both on the left and on the right by means of left- and right-translations, $L_g:G\to G$, and $R_g:G\to G$, defined, respectively, by $L_g(g')=gg'$ and $R_g(g')=g'g$. The diffeomorphisms L_g and R_g allow us to define the left- and right-invariant vector fields determined by its value in the neutral element $X^L_g=L_{g*e}X_e$ and $X^R_g=R_{g*e}X_e$. We will use the shorter notation $X^L_\alpha=X^L_{a_\alpha}$ and $X^R_\alpha=X^R_{a_\alpha}$, for $\alpha=1,\dots,r$, i.e., $(X^L_\alpha)_g=L_{g*e}(a_\alpha)$, $(X^R_\alpha)_g=R_{g*e}(a_\alpha)$. They generate, respectively, two opposite Lie algebras

$$[X^L_\alpha,X^L_\beta]=\sum_{\gamma=1}^{r}c_{\alpha\beta}{}^{\gamma}X^L_\gamma,\qquad [X^R_\alpha,X^R_\beta]=-\sum_{\gamma=1}^{r}c_{\alpha\beta}{}^{\gamma}X^R_\gamma.$$

Consequently, for every choice of the functions $b_\alpha(x)$, the x-dependent vector field

$$\overline{X}(g,x)=-\sum_{\alpha=1}^{r}b_\alpha(x)X^R_\alpha(g)\tag{10}$$

defines a Lie system in the Lie group G. Actually, this is the most general form of a right-invariant Lie system in the Lie group G. The integral curves $g(x)$ of such x-dependent vector field are the solutions of the system

$$\dot g=-\sum_{\alpha=1}^{r}b_\alpha(x)X^R_\alpha(g).$$

When applying $R_{g^{-1}(x)*g(x)}$ to both sides we obtain equation

$$\dot g(x)g^{-1}(x)=-\sum_{\alpha=1}^{r}b_\alpha(x)a_\alpha\in T_eG.\tag{11}$$

The left hand side is to be understood as $R_{g^{-1}(x)*g(x)}(\dot{g}(x))$. When $G \subset \mathrm{GL}(n, \mathbb{R})$ it reduces to the given expression.

Note that this equation is right-invariant: if $g(x)$ is the solution such that $g(0) = e$, then, for each $g_0 \in G$, $\overline{g}(x) = g(x)g_0$ is the solution such that $\overline{g}(0) = g_0$ and, consequently, the curves solution of the system are obtained from just one solution by right-translations.

It is also noteworthy that the set $\mathcal{G}$ of the curves $\gamma : \mathbb{R} \to G$, $x \mapsto g(x)$, is a group with respect to the point-wise composition law

$$\gamma_2 * \gamma_1 : x \mapsto g_2(x)g_1(x).$$

Let $\Phi : G \times M \to M$ define a left action of the Lie group G on a differentiable manifold M. We will denote: $gu := \Phi_g(u) := \Phi(g, u)$; $\Phi_u(g) := \Phi(g, u)$. Note that the maps Φ_g are diffeomorphisms and that $(\Phi_g)^{-1} = \Phi_{g^{-1}}$. Denote by $\mathfrak{g}$ the Lie algebra of G, i.e., the set of left-invariant vector fields in G. Remark that $\Phi_{u*e} : \mathfrak{g} \cong T_eG \to T_uM$. The map $X : \mathfrak{g} \to \mathfrak{X}(M)$ defined by $a \mapsto X_a(u) = \Phi_{u*e}(-a_e)$ defines a mapping of $\mathfrak{g}$ into $\mathfrak{X}(M)$. We call X_a the fundamental vector field associated to the element a of $\mathfrak{g}$, and is given by

$$(X_af)(u) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(-ta)u)\Big|_{t=0}, \qquad f \in C^\infty(M).$$

Moreover, the minus sign has been introduced for X to be a Lie algebra homomorphism, i.e. $X_{[a,b]} = [X_a, X_b]$. The flow of the vector field $X_a \in \mathfrak{X}(M)$ is $\phi(t, u) = \Phi(\exp(-ta), u)$.

A particularly important case is when we consider a transitive action. Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$, which admits one differentiable structure for which the canonical projection $\tau : G \to G/H$ is a differentiable map and admits local differentiable sections. Then, G acts on G/H by $\lambda(g_2, g_1H) = (g_2g_1)H$ and, moreover, G can be seen as a principal bundle over G/H: $(G, \tau, G/H)$. The right-invariant vector fields X_α^R are τ-projectable [11] and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M

$$\tau_{*g}X_\alpha^R(g) = -X_\alpha(gH).$$

The x-dependent vector field in M projection of (10) will be

$$X(y, x) = \sum_{\alpha=1}^{r} b_\alpha(x)X_\alpha(y) \tag{12}$$

where the vector fields X_α close on a Lie algebra isomorphic to $\mathfrak{g}$, and its integral curves are the solutions of the system of differential equations

$$\dot{y} = \sum_{\alpha=1}^{r} b_\alpha(x)X_\alpha(y).$$

Therefore, a solution of this last system starting from y_0 will be

$$y(x) = \Phi(g(x), y_0)$$

with $g(x)$ being the solution of (11) starting form e. Note that as the vector fields X_α close on a Lie algebra isomorphic to $\mathfrak{g}$, the systems of differential equations we obtain are of the class, characterized by Lie, of systems admitting a superposition function for the general solution in terms of a fundamental set of solutions [10–13, 15, 17, 18, 20]

3. Lie Systems as Connections on Bundles

If G is a connected Lie group, then $\pi_2 : P = G \times \mathbb{R} \to \mathbb{R}$ is a trivial principal G-bundle over the base $\mathbb{R}$. The right-action of G on P is given by $\Phi((g', x), g) = \Phi_g(g', x) = (g'g, x)$.

The remarkable point is that there is a geometric interpretation of Lie systems in Lie groups and homogeneous spaces that provides a useful method of reduction. We shall see that there exists a one-to-one correspondence between Lie systems on Lie groups and connections in the principal bundle $\pi_2 : P = G \times \mathbb{R} \to \mathbb{R}$, and similarly between Lie systems in homogeneous spaces and connections in associated vector bundles.

We first remark that giving a connection in the principal bundle P is equivalent to give a curve in G. For instance, one such that $g(0) = e$. This curve provides us first a section for π_2, $\sigma(x) = (g(x), x)$, and then a family of sections which are right-translated from such a section: $\{\sigma_{g_0}(x) = \sigma(x)g_0 ;\ g_0 \in G\}$. The tangent vectors to such family of sections span the horizontal spaces in each point and the vertical and horizontal spaces in a point of P are given, respectively, by $VP_{(g_0,x)} = \langle (X^R_\alpha(g_0), 0)\rangle$ and $HP_{(g_0,x)} = \langle (R_{g_0*e}(\dot g(x)\, g^{-1}(x)), 1)\rangle$. This choice of horizontal subspaces is right-invariant and, conversely, any principal connection in P will be determined by horizontal spaces $HP_{(e,x)} = \langle (a(x), 1)\rangle$, where, for each t, $a(x) \in T_eG$ is given $a(x) = -\sum_{\alpha=1}^r b_\alpha(x) a_\alpha$, with $b_\alpha(x)$ being arbitrary functions of t, and then $HP_{(g_0,x)} = \langle (R_{g_0*e}(a(x)), 1)\rangle$. Therefore, we see that there is a one-to-one correspondence between principal connections in the principal bundle $\pi_2 : P = G \times \mathbb{R} \to \mathbb{R}$ and Lie systems in the Lie group G: The curve $x \mapsto g(x)$ is a solution of the Lie system if and only if the section $x \mapsto (g(x), x)$ is an integral 1-dimensional surface of the corresponding distribution in P.

A (transitive) left-action $\Psi : G \times M \to M$ defines an associated bundle E with base $\mathbb{R}$ and typical fibre M. The total space of the bundle is the set of orbits of the right-action of G on $P \times M$ given by $(u, y)g = (\Phi(u, g), \Psi(g^{-1}, y))$, and the projection is $\pi_E[u, y] = \pi_2(u)$, where $[u, y]$ denotes the equivalence class of $(u, y) \in P \times M$.

A connection in the principal bundle translates into a connection in the associated bundle E, and so the horizontal curves will then be $[\tilde{\gamma}(x), y]$, where $\tilde{\gamma}(x)$ is an horizontal curve in P. More explicitly, the horizontal curves in the associated bundle are

$$[(g(x)g_0, x), y_0] = [\Phi((e, x), g(x)g_0), y_0] = [(e, x), \Psi(g(x), \Psi(g_0, y_0))].$$

Remark that E is equivalent to the product $E = M \times \mathbb{R}$, i.e., $[(e, x), y]$ corresponds to (y, x). With this identification, the horizontal curves here considered correspond to the integral curves starting from the points $\Psi(g_0, y_0)$ of the associated Lie system in M with respect to the action of G on M given by Ψ.

The simplest case is when M is a vector space V, and a linear representation of G on V is considered, the associated bundle being then a vector bundle and the corresponding Lie system being a linear system. This means that a linear system of differential equations can be seen as defining the horizontal curves corresponding to a connection in an associated vector bundle as pointed out by [22]. See also [2] for the case of Schrödinger equation. Note that different linear representations will give rise to different associated vector bundles and correspondingly different systems of differential equations.

Actions on more general differentiable manifolds can also be considered. We mentioned before the case of the action of $\mathrm{SL}(2, \mathbb{R})$ on the compactified real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ giving rise to Riccati equations.

It is well-known that there is an action of the group of automorphisms of a principal bundle on the sets of its connections. In our case it will be shown that it leads to a reduction procedure and a Wei-Norman method for dealing with Lie systems. More explicitly, if $\Psi : P \to P$ is an automorphism of a principal G-bundle, and H is a right-invariant distribution on P, then we can define a new right-invariant distribution on P by $K_{\Psi(u)} = \Psi_{*u}(H_u)$. But Ψ is a fibre-preserving map over the identity, and then the image under Ψ of vertical vectors are vertical vectors, and, therefore, if the horizontal distribution H defines a connection in P, then the new distribution K will define a new connection on P. In this way we define an action of the group of automorphisms of the principal G-bundle on the set of its principal connections.

In the particular case of $P = G \times \mathbb{R}$ considered before, the G-bundle automorphisms will be given by the maps $\Psi(g, x) = (\psi(g, x), x)$ where ψ is such that $\psi(g, x)g' = \psi(gg', x)$, from which we obtain that $\psi(g_0, x) = \psi(e, x)g_0$. Therefore, every automorphism Ψ is determined by a curve $\gamma : \mathbb{R} \to G$, $g(x) = \psi(e, x)$ by means of $\Psi(g_0, x) = (g(x)g_0, x)$. Conversely, any curve γ in G given by $x \mapsto g(x)$ defines an automorphism of the G-bundle $P = G \times \mathbb{R}$: $\Psi(g_0, x) =$

$(g(x)g_0, x)$. In this way we can identify the group of automorphisms of the G-bundle with the above mentioned group $\mathcal{G}$ of curves in G, and, with this identification, the action of the automorphism is just left multiplication by $g(x)$ of the G-component.

We can make use of this action of the group $\mathcal{G}$ on the set of Lie systems in order to relate a given problem with other similar but maybe simpler Lie systems obtained by such automorphisms. More specifically, we first remark that given an equation on a Lie group like (11), it may happen that the only non-vanishing coefficients are those of a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the equation reduces to a simpler one on a subgroup, involving less coordinates. Then, starting with an equation like (11) our aim is to choose a curve $g'(x)$ defining an automorphism in such a way that the new equation,

$$R_{\overline{g}(x)^{-1}*\overline{g}(x)}(\dot{\overline{g}}(x)) = R_{g'^{-1}(t)*g'(x)}(\dot{g}'(x)) - \sum_{\alpha=1}^{r} b_\alpha(t)\,\mathrm{Ad}(g'(x))a_\alpha \tag{13}$$

which is an equation similar to the original one but with a different right hand side, be simpler in the preceding sense. The fundamental result, whose proof can be found in [11], is that the knowledge of a particular solution of the associated Lie system in the homogeneous space G/H allows us to reduce the problem to an analogous one but in the subgroup H.

If $\Psi : G \times M \to M$ is a transitive action of G on a homogeneous space M, which can be identified with the set G/H of left-cosets by choosing a point at M, then the horizontal curve $y(x)$ starting from the point $y_0 = y(0)$ and the horizontal curve $\overline{y}(x)$ starting from $\Phi(g'(0), y_0)$, associated with the connections defined by $g(x)$ and $\overline{g}(x) = g'(x)g(x)$, respectively, are related by

$$\overline{y}(x) = \Psi(\overline{g}(x), y_0) = \Psi(g'(x)g(x), y_0) = \Psi(g'(x), y(x)).$$

Therefore, the action of the group of curves in G on the set of connections translates to the homogeneous space and gives an action on the corresponding set of associated Lie systems.

The main result establishing how the reduction can be carried out when a particular solution of the corresponding Lie system in a homogeneous space is known is [11]

Theorem 2. *Each solution of* (11) *on the group G can be written in the form $g(x) = g_1(x)h(x)$, where $g_1(x)$ is a curve on G projecting onto a solution $\tilde{g}_1(x)$ for the left action λ on the homogeneous space G/H and $h(x)$ is a solution of an equation but for the subgroup H, given explicitly by*

$$(\dot{h}h^{-1})(x) = -\,\mathrm{Ad}(g_1^{-1}(x))\left(\sum_{\alpha=1}^{r} b_\alpha(x)a_\alpha + (\dot{g}_1 g_1^{-1})(x)\right) \in T_eH.$$

Before ending this section we will mention a method of dealing with equation (11) (or the corresponding equation in the subgroup once the reduction procedure has been carried out). This method is a generalization of the procedure proposed by Wei and Norman [27,28] for finding the time evolution operator for a linear system of type $\mathrm{d}U(x)/\mathrm{d}x = H(x)U(x)$, with $U(0) = I$, see also [12]. We will only give here the recipe of how to proceed; the proof can be found, for instance, in [9,11,15].

The generalization of the Wei–Norman method consists on writing the solution $g(x)$ of (11) in terms of its second kind canonical coordinates w.r.t. a basis of the Lie algebra $\mathfrak{g}$, $\{a_1, \ldots, a_r\}$, for each value of x, i.e.,

$$g(x) = \prod_{\alpha=1}^{r} \exp(-v_\alpha(x)a_\alpha) = \exp(-v_1(x)a_1)\cdots\exp(-v_r(x)a_r). \tag{14}$$

A straightforward generalization of (13) for a product of l instead of two elements is that if $g(x) = g_1(x)g_2(x)\cdots g_l(x)$ then (see [15] for a proof)

$$R_{g(x)^{-1}*g(x)}(\dot{g}(x)) = \sum_{i=1}^{l}\left(\prod_{j<i}\mathrm{Ad}(g_j(x))\right)\left\{R_{g_i(x)^{-1}*g_i(x)}(\dot{g}_i(x))\right\}.$$

This relation allows us to transform the equation (11) into a system of differential equations for the unknown functions $v_\alpha(x)$, and the curve $g(x)$ we are looking for is the one given by the solution of this last system determined by the initial conditions $v_\alpha(0) = 0$ for all $\alpha = 1, \ldots, r$. The minus signs in the exponentials have been introduced for computational convenience. Now, it can be shown that using the expression (14) and after some mathematical manipulations, equation (11) becomes the fundamental expression of the Wei–Norman method [15]

$$\sum_{\alpha=1}^{r}\dot{v}_\alpha\left(\prod_{\beta<\alpha}\exp(-v_\beta(x)\,\mathrm{ad}(a_\beta))\right)a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(x)a_\alpha \tag{15}$$

with $v_\alpha(0) = 0$, $\alpha = 1, \ldots, r$. The resulting differential equation system for the functions $v_\alpha(x)$ is integrable by quadratures if the Lie algebra is solvable [27,28], and in particular, for nilpotent Lie algebras.

As a simple but illustrative instance one can consider the affine group in one dimension, $\mathcal{A}_1$, i.e. the set of transformations of the real line $\bar{y} = \alpha_1 y + \alpha_0$, with $\alpha_1 \neq 0$ and α_0 being real numbers. The corresponding differential equation is the inhomogeneous linear first order equation $\dot{y} = b_1(x)y + b_0$, and an appropriate use of Wei–Norman method gives rise to the explicit solution which involves two quadratures

$$y(x) = \mathrm{e}^{\int_0^x \mathrm{d}t\, b_1(t)}\left\{y_0 + \int_0^x \mathrm{d}t\, b_0(t)\,\mathrm{e}^{-\int_0^t \mathrm{d}t'\, b_1(t')}\right\}.$$

As another academical but more interesting example, from the physical point of view, illustrating the possible applications of the theory that has recently been studied using the theory of Lie systems is the motion of a classical particle under the action of a linear potential [15].

4. The General Riccati Equation

Each Riccati equation (8) can be considered as a Lie system in a homogeneous space, $\overline{\mathbb{R}}$, for the Lie group $\mathrm{SL}(2,\mathbb{R})$, as stated in Section 1. Once a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ has been chosen, for instance

$$M_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

the equation (8) can be considered as a curve in $\mathbb{R}^3$: $b(x) = (b_2(x), b_1(x), b_0(x))$. It defines a principal connection of the corresponding principal bundle $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}$ and of the associated bundle with fibre $\overline{\mathbb{R}}$. Any automorphism of the G-bundle, i.e. a curve in $\mathrm{SL}(2,\mathbb{R})$ given by

$$A = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix} \in \mathrm{Map}\,(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$$

transforms the principal connection into a new one and same for the connection in the associated bundle. In this way the automorphism gives rise to a new Riccati equation in which the coefficients are related to the original ones as follows [13]

$$\begin{aligned}
\bar{b}_2 &= \delta^2 b_2 - \delta\gamma b_1 + \gamma^2 b_0 + \gamma\dot{\delta} - \delta\dot{\gamma} \\
\bar{b}_1 &= -2\beta\delta b_2 + (\alpha\delta + \beta\gamma) b_1 - 2\alpha\gamma b_0 + \delta\dot{\alpha} - \alpha\dot{\delta} + \beta\dot{\gamma} - \gamma\dot{\beta} \\
\bar{b}_0 &= \beta^2 b_2 - \alpha\beta b_1 + \alpha^2 b_0 + \alpha\dot{\beta} - \beta\dot{\alpha}
\end{aligned}$$

This defines an affine action of $\mathcal{G}$ on the set of Riccati equations and we can use this action either for obtaining solutions of a Riccati equation by transforming the original equation in a simpler one in the same orbit, and finding first its solutions, or also for establishing some solubility criteria [13].

For instance, if we choose $\beta = \gamma = 0$ and $\delta = \alpha^{-1}$, then we see that $\bar{b}_1 = 0$ if and only if $b_1 = -2\dot{\alpha}/\alpha$, i.e.

$$\alpha = \exp\left[-\frac{1}{2}\int_0^x b_1(t)\ \mathrm{d}t\right].$$

In other words, the change $x' = \mathrm{e}^{-\phi} x$ with $\phi = \int_0^x b_1(t)\ \mathrm{d}t$, leads to $\bar{b}_1 = 0$, $\bar{b}_2 = b_2\,\mathrm{e}^{\phi}$ and $\bar{b}_0 = a_0\,\mathrm{e}^{-\phi}$, which is the property **3-1-3.a.i** of the book by Murphy [23]. In fact, under such a transformation

$$\bar{b}_2 = \alpha^{-2} b_2, \qquad \bar{b}_1 = b_1 + 2\frac{\dot{\alpha}}{\alpha}, \qquad \bar{b}_0 = \alpha^2 b_0$$

and therefore with the above choice for α we see that $\bar{b}_1 = 0$.

If, instead, $\alpha = \delta = 1$ and $\gamma = 0$, the function β can be chosen in such a way that $\bar{b}_1 = 0$ if and only if $\beta = b_1/(2b_2)$, and then

$$\bar{b}_0 = b_0 + \dot{\beta} - \frac{b_1^2}{4b_2}, \qquad \bar{b}_2 = b_2$$

which is the property **3-1-3.a.ii** of [23].

Finally, the original equation would be reduced to one with $\bar{b}_0 = 0$ if and only if there exist functions $\alpha(x)$ and $\beta(x)$ such that

$$\beta^2 b_2 - \alpha\beta b_1 + \alpha^2 b_0 + \alpha\dot{\beta} - \beta\dot{\alpha} = 0.$$

This relation was considered by Strelchenya [26], even if written in a slightly modified way, to be an integrability criterium.

Note that when dividing the preceding expression by α^2 we see that $y_1 = -\beta/\alpha$ is a solution of the original Riccati equation, and conversely, if a particular solution is known, y_1, then the matrix

$$A(x) = \begin{pmatrix} 1 & -y_1 \\ 0 & 1 \end{pmatrix}$$

will transform the equation into a new one with $\bar{b}_0 = 0$, $\bar{b}_2 = b_2$ and $\bar{b}_1 = b_1 + 2y_1 b_2$, which can be easily integrated by two quadratures. Consequently, the criterium given by Strelchenya is nothing but the well-known fact that once a particular solution is known the original Riccati equation can be reduced to a Bernoulli one and therefore the general solution can easily be found. However, in our opinion the previous theory gives a very appropriate group theoretical explanation of the convenience of the associated change of variables. Note that the inverse matrix of $A(x)$ is playing the role of $g_1(x)$ in Theorem 2, and the isotopy group of $0 \in \mathbb{R}$ is the group generated by M_1 and M_2, which is isomorphic to the usual affine group in one dimension, generated by M_0 and M_1. The latter is the isotopy group of $\infty \in \overline{\mathbb{R}}$ which is the image of the point $0 \in \mathbb{R}$ under the transformation defined by the matrix $M_0 + M_2 \in \mathrm{SL}(2, \mathbb{R})$. This is the reason why the intermediate Bernoulli equation becomes a inhomogeneous linear equation under the change of coordinates $y \mapsto 1/y$.

It was also shown in [13] that the knowledge of a second solution allows us to reduce the original Riccati equation to a homogeneous linear differential equation (i.e. to the subgroup corresponding to the generator M_1) and, therefore, to just one quadrature. Moreover, if a third solution is known, the general solution can be written without any quadrature by means of the above mentioned superposition rule.

5. Applications of Lie Systems in Supersymmetric Quantum Mechanics

The simplest example and in some sense a prototype of Lie systems is that of linear systems, but, as indicated in the preceding section, Riccati equation is also an interesting example for $n = 1$ [13]. Both, non-autonomous linear systems and Riccati equation appear very often in Physics. For instance, linear systems appear in the time evolution of classical time-dependent harmonic oscillators and related problems. But also in Quantum Mechanics one often considers finite-dimensional Hilbert spaces, for instance when only the internal degrees of freedom are taken into account, and then time-dependent Schrödinger equation reduces to a linear system. As an instance, Barata used a particular solution of a complex Riccati equation in order to determine the dynamical evolution of a two level system [3], i.e. described by a Hamiltonian $H = \epsilon\sigma_3 - f(t)\sigma_1$. Such problem was analyzed from the perspective of Lie systems in [10], where it was proved that the relevant group in both problems is $\mathrm{SL}(2,\mathbb{R})$.

The importance of Lie systems in Supersymmetric Quantum mechanics is based on the fact that Riccati equation can be considered as a Lie system with group $\mathrm{SL}(2,\mathbb{R})$. Recall, for instance, that the condition for the determination of the **super-potential** W in the factorization of a Hamiltonian H in such a way that

$$H - c = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + W\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W\right)$$

is a Riccati equation. Moreover, a similar equation plays a relevant role in the search for shape invariant potentials using the so called **Infeld–Hull factorization method** [19] (see also [14] for a modern approach).

The fact we want now to stress is that Lie reduction theory for dilation symmetry of linear second order differential equations produces a Riccati equation. Actually dilations are symmetries of such equations,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}z}{\mathrm{d}x} + c(x)z = 0$$

and we can use the Lie recipe to reduce the problem to a first order differential equation which turns out to be a Riccati equation. In fact, the dilation vector field is $X = z\partial/\partial z$ and Lie prescription amounts to change the variable z to a new one $u = \varphi(z)$ such that X becomes the translation operator in u, i.e. $X = \partial/\partial u$. The condition $Xu = 1$ leads to $u = \log z$, i.e. $z = \mathrm{e}^u$, and then, $\mathrm{d}z/\mathrm{d}x = \mathrm{e}^u$, $\mathrm{d}z/\mathrm{d}x = z\,\mathrm{d}u/\mathrm{d}x$, and the differential equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + c(x) = 0$$

where the unknown function u appears only through its derivatives: the order is lowered by introducing the change of variable $w = \mathrm{d}u/\mathrm{d}x$ and we get a Riccati equation $w' = -c(x) - b(x)w - w^2$. If w satisfies such Riccati equation, then $z(x) = \exp\left(\int^x w(\zeta)\, \mathrm{d}\zeta\right)$ satisfies the given SODE. In the particular case of the time independent Schrödinger equation,

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (V(x) - E)\psi = 0$$

$b = 0$ and the reduction leads to the first order equation for $\phi = \psi'/\psi$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = -\phi^2 + (V - E).$$

This relation of some Riccati equations with stationary Schrödinger equation, just those equations (8) for which $b_2(x) = -1$, $b_1(x) = 0$ and $b_0(x)$ an arbitrary function which will correspond to $b_0(x) = V(x) - E$, suggests us to make use of this affine action of $\mathcal{G}$ on the set of Riccati equations for relating spectral problems of two Hamiltonians whose associated Riccati equations are connected by such $\mathcal{G}$-action as it was done in [8].

For instance, if $y(x)$ is a solution of the equation (8) and we transform it by means of $\overline{y}(x) = \Phi(A(x), y(x))$, then $\overline{y}(x)$ will be a solution of the Riccati equation with new coefficient functions $\overline{b}_0$, $\overline{b}_1$ and $\overline{b}_2$. By means of this technique it has been proved in [8] the following result.

Theorem 3 (Finite difference Bäcklund algorithm [16,21]). *Let $w_k(x)$, $w_l(x)$ be two solutions of the Riccati equations $w' + w^2 = V(x) - \epsilon_k$ and $w' + w^2 = V(x) - \epsilon_l$, respectively, where $\epsilon_k < \epsilon_l$. Then the function $w_{kl}(x)$ defined by*

$$w_{kl}(x) = -w_k(x) - \frac{\epsilon_k - \epsilon_l}{w_k(x) - w_l(x)} \tag{16}$$

is a solution of the Riccati equation $w' + w^2 = V(x) - 2w_k'(x) - \epsilon_l$.

The proof consists on transforming the function $w_l(x)$ solution of the Riccati equation $w' + w^2 = V(x) - \epsilon_l$ by means of the element of $\mathcal{G}$ given by

$$A(x) = \frac{1}{\sqrt{a}} \begin{pmatrix} h(x) & -h^2(x) + a \\ -1 & h(x) \end{pmatrix} \tag{17}$$

and then we see that if $a = \epsilon_l - \epsilon_k$ and the function $h(x)$ satisfies the Riccati equation $w^2 + w' = V(x) - \epsilon_k$, and we rename it as $h(x) = w_k(x)$, the new coefficients reduce to $\overline{b}_2(x) = -1$, $\overline{b}_1(x) = 0$ and $\overline{b}_0(x) = V(x) - 2w_k'(x) - \epsilon_l$.

Let us note that in [16] the proof of the Theorem 3 was just sketched. In addition, there exists an alternative proof; see, e.g., Mielnik *et al* [21].

The result of this theorem admits a generalization whose proof is a bit more cumbersome and which was also given in [8]:

Theorem 4. *Let $w(x)$ be a solution of the Riccati equation*

$$w' + w^2 = V(x) - \epsilon \tag{18}$$

for some function $V(x)$ and some constant ϵ, and $\gamma(x)$ a never vanishing differentiable function defined on the domain of $V(x)$. If $v(x)$ is a solution of the Riccati equation

$$v' + v^2 = V(x) + \frac{1}{\gamma^2(x)} - \epsilon \tag{19}$$

such that is defined in the same domain as $w(x)$ and $w(x) - v(x)$ does not vanish, then the function $\overline{w}(x)$ defined by

$$\overline{w}(x) = -v(x) - \frac{1/\gamma^2(x)}{w(x) - v(x)} + \frac{\gamma'(x)}{\gamma(x)} \tag{20}$$

is a solution of the Riccati equation

$$\overline{w}' + \overline{w}^2 = V(x) - 2\left(\frac{\gamma'}{\gamma}v + v'\right) + \frac{\gamma''}{\gamma} - \epsilon. \tag{21}$$

Theorem 4 has a counterpart for linear second-order differential equations of Schrödinger type, which are of direct interest in physical applications. Some applications of these results can be found in [8].

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