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FINDING THE ZEROS OF THE FUNCTIONS TREATED BY q-CALCULUS

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Abstract. We construct q-Taylor formula for the functions of several variables and develop some new methods for solving equations and systems of equations. They are much easier for application than well known ones and very useful when the continuous function does not have fine smooth properties. Especially, we will demonstrate their power in solving the equations where the function is defined by some q-integral. We will discuss the convergence and accuracy of those methods and compare them with well known methods. The conclusions are illustrated by examples.

1. Introduction

We will start with basic notions from q-calculus which can be found, for example, in [3] and [5].

Let $q \in (0, 1)$. A q-natural number $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \qquad n \in \mathbb{N}.$$

Generally, a q-complex number $[a]_q$ is

$$[a]_q := \frac{1 - q^a}{1 - q}, \qquad a \in \mathbb{C}.$$

We define the factorial of the number $[n]_q$ and the q-binomial coefficient by

$$[0]_q! := 1, \qquad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Also, q-Pochammer symbol is defined as follows

$$(z-a)^{(0)} = 1, \qquad (z-a)^{(k)} = \prod_{i=0}^{k-1}(z-aq^i), \quad k \in \mathbb{N}. \tag{1}$$

2. On Partial q-Derivatives and Differentials

Let $f(\vec{x})$, where $\vec{x} = (x_1, x_2, \dots, x_n)$ be a multi-variable real continuous function. We introduce an operator $\varepsilon_{q,i}$, which multiplies the coordinate of the argument, by

$$(\varepsilon_{q,i}f)(\vec{x}) = f(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_n).$$

Furthermore,

$$(\varepsilon_q f)(\vec{x}) := (\varepsilon_{q,1}\cdots\varepsilon_{q,n}f)(\vec{x}) = f(q\vec{x}).$$

We define the partial q-derivative of the function $f(\vec{x})$ with respect to the variable x_i by

$$D_{q,x_i}f(\vec{x}) := \frac{f(\vec{x}) - (\varepsilon_{q,i}f)(\vec{x})}{(1-q)x_i}, \qquad x_i \neq 0,$$
$$D_{q,x_i}f(\vec{x})\big|_{x_i=0} = \lim_{x_i\to 0} D_{q,x_i}f(\vec{x}).$$

In the similar way, high partial q-derivatives are defined as

$$D^n_{q,x_i^n}f(\vec{x}) := D_{q,x_i}\left(D^{n-1}_{q,x_i^{n-1}}f(\vec{x})\right), \quad D^{m+n}_{q,x_i^m,x_j^n}f(\vec{x}) := D_{q,x_i^m}\left(D^n_{q,x_j^n}f(\vec{x})\right).$$

Obviously,

$$D^{m+n}_{q,x_i^m,x_j^n}f(\vec{x}) = D^{m+n}_{q,x_j^n,x_i^m}f(\vec{x}), \qquad i,j = 1,2\dots,n, \quad m,n = 0,1,\dots .$$

Also, for an arbitrary $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we can introduce q-differential by

$$\mathrm{d}_q f(\vec{x},\vec{a}) := (x_1-a_1)D_{q,x_1}f(\vec{a}) + (x_2-a_2)D_{q,x_2}f(\vec{a}) + \cdots + (x_n-a_n)D_{q,x_n}f(\vec{a})$$

and high q-differentials by

$$\begin{aligned}\mathrm{d}_q^k f(\vec{x},\vec{a}) &:= \left(\sum_{i=1}^{n}(x_i-a_i)D_{q,x_i}\right)^{(k)} f(\vec{a}) \\ &= \sum_{\substack{i_1+\cdots+i_n=k \\ i_j\in\mathbb{N}_0}} \frac{[k]_q!}{[i_1]_q![i_2]_q!\cdots[i_n]_q!} D^k_{q,x_i^{i_1},\dots,x_n^{i_n}} f(\vec{a}) \prod_{j=1}^{n}(x_j-a_j)^{(i_j)}.\end{aligned}$$

Notice, that a continuous function $f(\vec{x})$ in a neighborhood, which does not include any point with a zero coordinate, has also continuous partial q-derivatives.

3. About q-Taylor Formula for a Multi-Variable Function

Now we can discuss one new expansion of a function whose variable is from $\mathbb{R}^n$. First of all, we need the next lemma.

Lemma 1. *It is valid that*

$$D_{q,x}(x-\alpha)^{(n)} = [n]_q(x-\alpha)^{(n-1)}, \qquad x,\alpha \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Proof: For the proof see, for example, Cigler [2]. □

Theorem 1. *Suppose that all partial q-derivatives of $f(x,y)$ exist in some neighborhood of (a,b). Then*

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{D^n_{q,x^i,y^{n-i}} f(a,b)}{[i]_q![n-i]_q!}(x-a)^{(i)}(y-b)^{(n-i)}.$$

Proof: Suppose that the function can be written in the form

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} c_{n,i}(x-a)^{(i)}(y-b)^{(n-i)}.$$

Application of partial q-derivative operators $D_{q,x}$ and $D_{q,y}$ gives us

$$D^{k+m}_{q,x^k,y^m} f(x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} c_{n,i} D^{k+m}_{q,x^k,y^m}(x-a)^{(i)}(y-b)^{(n-i)}.$$

According to the previous lemma, we conclude

$$D^{k+m}_{q,x^k,y^m}(x-a)^{(i)}(y-b)^{(n-i)} = 0, \qquad k > i \ \wedge \ m > n-i.$$

In other cases, we have

$$\begin{aligned} &D^{k+m}_{q,x^k,y^m}(x-a)^{(i)}(y-b)^{(n-i)} \\ &\quad = \left((x-a)^{(i-k)}\prod_{j=1}^{k}[i-j+1]_q\right)\left((y-b)^{(n-i-m)}\prod_{j=1}^{m}[n-i-j+1]_q\right). \end{aligned}$$

The assumed expansion is valid in some neighborhood of (a,b). Putting $x=a$ and $y=b$, all members of the sum vanish, except for $i=k$ and $n-i=m$. Hence,

$$D^{k+m}_{q,x^k,y^m} f(a,b) = c_{k+m,k}\,[k]_q!\,[m]_q!.$$

□

In the same manner, we can prove the analogous theorem for the general case.

Theorem 2. *Suppose that all q-differentials of $f(\vec{x})$ exist in some neighborhood of $\vec{a}$. Then*

$$f(\vec{x}) = \sum_{k=0}^{\infty} \frac{d_q^k f(\vec{x}, \vec{a})}{[k]_q!}.$$

4. On q-Newton–Kantorovich Method

We consider a system of nonlinear equations

$$\vec{f}(\vec{x}) = 0$$

where $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_n(\vec{x}))$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$, $n \in \mathbb{N}$. We will suppose also that this system has an isolated real solution $\vec{\xi}$. Using q-Taylor series of the function $\vec{f}(\vec{x})$ around some point $\vec{x}^{(m)} \approx \vec{\xi}$, we have

$$f_i(\vec{\xi}) \approx f_i(\vec{x}^{(m)}) + \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{x}^{(m)})(\xi_j - x_j^{(m)}), \qquad i = 1, 2, \ldots, n.$$

In a matrix form, we rewrite the above system as

$$\vec{f}(\vec{\xi}) \approx \vec{f}(\vec{x}^{(m)}) + W_q(\vec{x}^{(m)})(\vec{\xi} - \vec{x}^{(m)})$$

where

$$W_q(\vec{x}) = D_q \vec{f}(\vec{x}) = [D_{q,x_j} f_i(\vec{x})]_{n\times n}$$

is the Jacobi matrix of q-partial derivatives. If the matrix W_q is regular, there exists the inverse matrix W_q^{-1}, so that we can formulate q-Newton–Kantorovich method in the form

$$\vec{x}^{(m+1)} = \vec{x}^{(m)} - W_q^{-1}(\vec{x}^{(m)})\vec{f}(\vec{x}^{(m)}).$$

5. On q-Newton Method

If in the previous speculation we take $n = 1$, the system of equations reduces to one equation $f(x) = 0$. A few methods for solving equations of this form were developed in our previous papers [6] and [8].

The q-derivative of a function $f(x)$ is

$$(D_q f)(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0, \qquad (D_q f)(0) := \lim_{x\to 0}(D_q f)(x) \qquad (2)$$

and high q-derivatives $D_q^0 f := f$, $D_q^n f := D_q(D_q^{n-1} f)$, $n = 1, 2, 3, \ldots$.

From the above definition, it is obvious that a continuous function defined on an interval, which does not include the zero, is continuously q-differentiable.

For deriving the method we need q-Taylor formula for q-differentiable functions [3] and [4]. That is why we will start with q-integral, which is defined by

$$I_q(f) = \int_0^a f(t)\,\mathrm{d}_q(t) := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Notice that (see [5]) it holds

$$I(f) = \int_0^a f(t)\,\mathrm{d}t = \lim_{q\uparrow 1} I_q(f).$$

Also,

$$\int_a^b f(t)\,\mathrm{d}_q(t) := \int_0^b f(t)\,\mathrm{d}_q(t) - \int_0^a f(t)\,\mathrm{d}_q(t).$$

The next is the q-Taylor formula with remainder term

$$f(x) = \sum_{k=0}^{n-1} \frac{(D_q^k f)(a)}{[k]_q!}(x-a)^{(k)} + R_n(f,x,a,q)$$

where

$$R_n(f,x,a,q) = \int_{t=a}^{t=x} \frac{(x-t)^{(n)}}{x-t}\frac{(D_q^n f)(t)}{[n-1]_q!}\,\mathrm{d}_q(t). \tag{3}$$

Suppose that an equation $f(x) = 0$ has the unique isolated solution $x = \xi$. If x_n is an approximation to the exact solution ξ, by using Jackson's q-Taylor formula, we have

$$0 = f(\xi) \approx f(x_n) + (D_q f)(x_n)(\xi - x_n)$$

hence

$$\xi \approx x_n - \frac{f(x_n)}{(D_q f)(x_n)}.$$

So, we can construct the *q-Newton method*

$$x_{n+1} = x_n - \frac{f(x_n)}{(D_q f)(x_n)}.$$

According to (2), we can rearrange the above expression in the form

$$x_{n+1} = x_n \left\{ 1 - \frac{1-q}{1 - \frac{f(qx_n)}{f(x_n)}} \right\}.$$

This method written in the form

$$x_{n+1} = x_n - \frac{x_n - qx_n}{f(x_n) - f(qx_n)} f(x_n)$$

resembles the method of chords (secants).

The next theorem is a q-analogue of the well known statement about convergence (see Bakhvalov [1]).

Theorem 3. *Let the equation $f(x) = 0$ has a unique isolated root $x = \xi$ and $a > 0$, $1 \le p \le 2$. Let the function $f(x)$ satisfies*

i) $|(D_q f)(x)| \ge M_1^{p-1} > 0$

ii) $|f(x) - f(y) - (D_q f)(y)(x-y)| < L^{p-1}|x-y|^p$

where M_1 and L are some positive constants. Then, for all initial values $x_0 \in (\xi - b, \xi + b)$, where $b = \min\{a, M_1/L\}$, the q-Newton method converges to the exact solution of the equation $f(x) = 0$ and

$$|\xi - x_n| \le \left(\frac{L}{M_1}\right)^{p^n - 1} |\xi - x_0|^{p^n}.$$

Proof: We can write q-Newton method in the form

$$(D_q f)(x_n)(x_{n+1} - x_n) = -f(x_n).$$

From the condition ii) we have

$$|f(\xi) - f(x_n) - (D_q f)(x_n)(\xi - x_n)| < L^{p-1}|\xi - x_n|^p.$$

Hence, using $f(\xi) = 0$, we yield

$$|(D_q f)(x_n)(\xi - x_{n+1})| < L^{p-1}|\xi - x_n|^p.$$

By the condition i) we have

$$|\xi - x_{n+1}| < \frac{L^{p-1}}{|(D_q f)(x_n)|}|\xi - x_n|^p < \left(\frac{L}{M_1}\right)^{p-1} |\xi - x_n|^p.$$

Now, if $x_n \in (\xi - b, \xi + b)$, then

$$|\xi - x_{n+1}| < \left(\frac{L}{M_1}\right)^{p-1} b^p = \left(\frac{L}{M_1}\right)^{p-1} b^{p-1} b \le b.$$

Denote by $c = L/M_1$. Now

$$|\xi - x_{n+1}| < c^{p-1}|\xi - x_n|^p \Rightarrow c|\xi - x_{n+1}| < c^p|\xi - x_n|^p$$

wherefrom we get the final conclusion. □

6. Analysis of the Convergence and Error Estimation

Our purpose is to formulate and prove the theorem for scanning the convergence of an iterative process

$$x_{k+1} = \Phi(x_k), \qquad k = 0, 1, 2, \ldots$$

by q-analysis.

Theorem 4. *Suppose that $\Phi(x)$ is a continuous function on $[a, b]$ $(0 \notin [a, b])$, which satisfies the following conditions:*

i) $\Phi : [a, b] \mapsto [a, b]$

ii) $\forall q \in \left(\frac{\min\{a,b\}}{\max\{a,b\}}, 1\right), \quad \forall x \in (a, b), \quad |(D_q f)(x)| \leq \lambda < 1.$

Then the iterative process $x_{k+1} = \Phi(x_k)$, $k = 0, 1, 2, \ldots$, with initial value $x_0 \in [a, b]$, is converging to the fixed point of $\Phi(x)$, i.e.,

$$\lim_{k\to\infty} x_k = \xi, \qquad \Phi(\xi) = \xi.$$

Proof: Notice that for a continuous function $\Phi(x)$ on $[a, b]$ $(0 \notin [a, b])$, for all x and y such that $a < x < y < b$, it is valid

$$\Phi(y) - \Phi(x) = (D_{x/y}\Phi)(y)(y - x), \qquad \Phi(y) - \Phi(x) = (D_{y/x}\Phi)(x)(y - x).$$

Consider

$$\xi = x_0 + \sum_{k=0}^{\infty}(x_{k+1} - x_k).$$

Let $x_k^{(M)} = \max\{x_k, x_{k-1}\}$, $x_k^{(m)} = \min\{x_k, x_{k-1}\}$ and $q = x_k^{(m)}/x_k^{(M)}$. Now we have

$$\Phi(x_k) - \Phi(x_{k-1}) = (D_q\Phi)(x_k^{(M)})(x_k - x_{k-1}).$$

So, it is valid

$$|x_{k+1} - x_k| = |(D_q\Phi)(x_k^{(M)})|\, |x_k - x_{k-1}| \leq \lambda |x_k - x_{k-1}|.$$

Since $|x_{k+1} - x_k| \leq \lambda^k |x_1 - x_0|$, we get

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| \leq |x_1 - x_0| \sum_{k=0}^{\infty} \lambda^k = \frac{|x_1 - x_0|}{1 - \lambda}.$$

Hence, the series S converges and $\xi = \lim_{n\to\infty} S_n = \lim_{n\to\infty} x_{n+1}$. Since $\Phi(x)$ is a continuous function, we have

$$\xi = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \Phi(x_n) = \Phi(\lim_{n\to\infty} x_n) = \Phi(\xi).$$

$\square$

Definition 1. *An iterative method $x_{n+1} = \Phi(x_n)$, $n = 0, 1, 2, \ldots$, with the fixed point ξ, has $(r; q)$-order of convergence, if there exists $C_r \in \mathbb{R}^+$ such that, for a large enough n, it is valid that*

$$|\xi - x_{n+1}| < C_r |(\xi - x_n)^{(r)}|$$

where the last exponent (r) is defined by (1).

We have proved the next theorem in [6].

Theorem 5. *Let $f(x)$ be a continuous function on $[a,b]$ and $R_n(f,z,c,q)$, $z,c \in (a,b)$, be the remainder term* (3) *in the q-Taylor formula. Then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, can be found $\tau \in (a,b)$ between c and z which satisfies*

$$R_n(f,z,c,q) = \frac{(D_q^n f)(\tau)}{[n]_q!}(z-c)^{(n)}.$$

Now, we are ready to prove the main theorem of this section.

Theorem 6. *Suppose that a function $f(x)$ is continuous on a segment $[a,b]$ and that the equation $f(x) = 0$ has a unique isolated solution $\xi \in (a,b)$. If the conditions*

$$|(D_q f)(x)| \geq M_1, \qquad |(D_q^2 f)(x)| \leq M_2$$

are satisfied for some positive constants M_1 and M_2 and all $x \in (a,b)$, then there exists $\hat{q} \in (0,1)$, such that for all $q \in (\hat{q},1)$, the iterations obtained by q-Newton method satisfy

$$|\xi - x_{k+1}| \leq \frac{M_2}{(1+q)M_1}|(\xi - x_k)^{(2)}|$$

i.e., q-Newton method has $(2;q)$-order of convergence.

Proof: From the formulation of q-Newton method, we have

$$x_{k+1} - \xi = x_k - \xi - \frac{f(x_k)}{(D_q f)(x_k)}$$

hence

$$f(x_k) + (D_q f)(x_k)(\xi - x_k) = (D_q f)(x_k)(\xi - x_{k+1}).$$

By using the q-Taylor formula of order $n = 2$ at the point x_k for $f(\xi)$, we have

$$f(\xi) = f(x_k) + (D_q f)(x_k)(\xi - x_k) + R_2(f,\xi,x_k,q).$$

Since $f(\xi) = 0$, we get

$$(D_q f)(x_k)(\xi - x_{k+1}) = -R_2(f,\xi,x_k,q)$$

i.e.,

$$|\xi - x_{k+1}| = \frac{|R_2(f,\xi,x_k,q)|}{|(D_q f)(x_k)|}.$$

According to Theorem 5, there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$ it can be found $\xi \in (a,b)$ such that

$$R_2(f,\xi,x_k,q) = \frac{(D_q^2 f)(\xi)}{[2]_q}(\xi - x_k)^{(2)}.$$

Now,

$$|\xi - x_{k+1}| = \frac{|(D_q^2 f)(\xi)|}{|(D_q f)(x_k)|}\frac{|(\xi - x_k)^{(2)}|}{1+q}.$$

Using the conditions which the function $f(x)$ and its q-derivatives satisfy, we obtain the statement of the theorem. □

7. The Functions Defined by Infinite Products

For computing the infinite product

$$f(t,q) = \prod_{n=1}^{\infty} (1 - tq^n), \qquad t \in \mathbb{C}, \quad |q| < 1$$

Sokal [7] suggests a quadratically convergent algorithm based on the identity

$$f(t,q) = \sum_{m=0}^{\infty} \frac{(-t)^m q^{m(m+1)/2}}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Here, we are interesting in finding the solutions of the equation

$$F(t) \equiv f(t,q) - a = 0$$

for a fixed value q and a given value $a \in \mathbb{C}$. Let us notice

$$D_{q,t} f(t,q) = \frac{-q}{(1-q)(1-tq)} f(t,q).$$

Applying our q-Newton method, we find iterative process

$$t_{k+1} = t_k + \frac{1-q}{q}(1 - at_k)\left(1 - \frac{a}{f(t_k, q)}\right)$$

which leads us to the solution of the previous equation.

Now, there is no problem to use our considerations for solving the systems of the type

$$\vec{H}(f(\vec{x}, q)) = 0.$$

We will demonstrate this in the last section.

8. Zeros of the Functions Defined via q-Integrals

Let us consider the equation

$$F(x) \equiv \int_0^x h(t)\, d_q t - a = x(1-q) \sum_{k=0}^{\infty} h(xq^k) q^k - a = 0$$

where a and q are real numbers and $|q| < 1$.

Since $D_q F(x) \equiv h(x)$, we can apply q-Newton method

$$x_{n+1} = x_n - \frac{F(x_n)}{h(x_n)}, \qquad n = 0, 1, \ldots$$

with some initial value x_0 (for example, $x_0 = a$). Instead of q-integral we evaluate partial sum with a proper exactness. Now,

$$\lim_{n\to\infty} x_n = x.$$

9. Examples

Example 1. Let us consider the following system of nonlinear equations

$$x_1^2 + 7x_2 - x_3^4 = 2, \qquad x_1^2 - 49x_2^2 + x_3^2 = 6, \qquad x_1^2 + 7(x_2 - 1) - x_3^2 = -3.$$

If we use q-method, we yield the next Jacobi matrix

$$W_q = \begin{bmatrix} (1+q)x_1 & 7 & -(1+q)(1+q^2)x_3^3 \\ (1+q)x_1 & -49(1+q)x_2 & (1+q)x_3 \\ (1+q)x_1 & 7 & -(1+q)x_3 \end{bmatrix}.$$

Using $q = 0.9$, we find the solutions $(x_1 = \sqrt{5}, x_2 = 1/7, x_3 = \sqrt{2})$ with accuracy on five decimal digits after $n = 7$ iterations:

$$\vec{x}^{(k)} : \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1.613 \\ 0.705 \\ 2.299 \end{bmatrix}, \begin{bmatrix} 2.199 \\ 0.353 \\ 1.747 \end{bmatrix}, \begin{bmatrix} 2.1794 \\ 0.1937 \\ 1.4633 \end{bmatrix}, \begin{bmatrix} 2.2331 \\ 0.1450 \\ 1.4078 \end{bmatrix} \to \begin{bmatrix} 2.23607 \\ 0.142871 \\ 1.41427 \end{bmatrix}.$$

The next example will show the advantages of q-Newton–Kantorovich method with respect to the classical one.

Example 2. Let us consider next the system of nonlinear equations

$$|x_1^2 - 4| + e^{7x_2 - 36} = 2, \qquad \log_{10}\left(\frac{12x_1^2}{x_2} - 6\right) + x_1^4 = 10.$$

If we use the q-method for $q = 0.9$, this gives the following iterations for the exact solutions $(x_1, x_2) = (\sqrt{3}, 36/7)$:

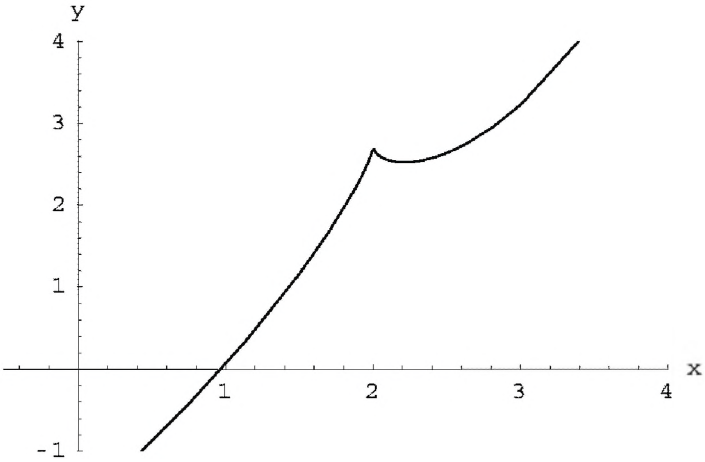
$$\vec{x}^{(k)} : \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1.78067 \\ 5.29844 \end{bmatrix}, \begin{bmatrix} 1.73405 \\ 5.20213 \end{bmatrix}, \begin{bmatrix} 1.73208 \\ 5.15274 \end{bmatrix}, \begin{bmatrix} 1.73205 \\ 5.14302 \end{bmatrix} \to \begin{bmatrix} 1.73205 \\ 5.14286 \end{bmatrix}.$$

The classical Newton-Kantorovich method with initial values $x_1 = 2$, $x_2 = 5$ can not be used in this case because the partial derivative of the first function with respect to the first variable does not exist.

Example 3. Let us consider the equation

$$f(x) \equiv \sqrt[3]{x^3 - 9x^2 + 24x - 20} + e^{x/2} = 0.$$

The function $f(x)$ is not differentiable at the point $x = 2$. However, it is not a problem for our q-Newton method. Really, starting with the initial value $x_0 = 2$, we find the solution with six exact digits after five iterations (see Figure 1).



Iteration	Value
0	2.000000
1	1.592916
2	1.043515
3	0.970143
4	0.969425
5	0.969426

Figure 1. The function is not differentiable in the initial point, but it does not have influence to convergence.

Example 4. Advantages of q-Newton method with respect to the classical Newton method can be seen in the case of the equations with multiple zeros.

So, for solving the equation

$$f(x) \equiv x^6 - 5x^5 + 8.25x^4 - 10x^3 + 13.5x^2 - 5x + 6.25 = 0, \qquad x_0 = 2$$

the classical Newton method must be changed by the special Newton method for multiple zeros ($\xi = 2.5$ is a double root). But q-Newton method has large enough intervals of convergence, what can be seen in Figure 2.

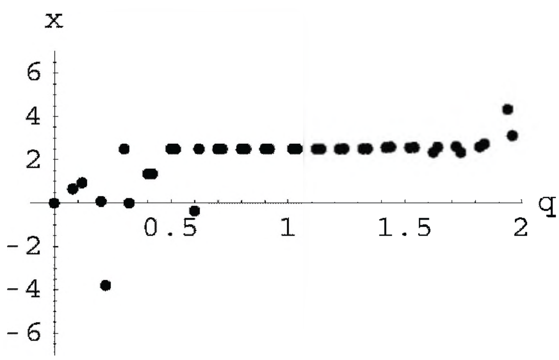


Figure 2. Solving of the equation with multiple roots. The values of the iterations from $n = 100$ to $n = 140$.

Example 5. For a given q, $0 < |q| < 1$, the solutions x and y of the system with some infinite products

$$\prod_{n=1}^{\infty}(1-xq^n)\Big/\prod_{n=1}^{\infty}(1-yq^n) = 1/2$$
$$\prod_{n=1}^{\infty}(1-xq^n) + \exp\left(\prod_{n=1}^{\infty}(1-yq^n)\right) = 5$$

can be also found by this method.

For example, for $q = 3/4$, we have the solutions $x = 0.104199\ldots$ and $y = -0.127765\ldots$. (Another approach to this system is to solve the system introducing new notation for products, and then, to find x and y by our q-method from the fifth section.)

Example 6. Let us consider the equation

$$\int_0^x t^{5/2}\,\mathrm{d}_{3/4}t = \frac{64\sqrt[4]{8}}{128-27\sqrt{3}}.$$

Applying q-Newton method with initial value equal to the number on the right side, we get

k	0	1	2	3	$\cdots$	10
x_k	1	1.45871	1.39966	1.41999	$\cdots$	1.41421

Really, the exact solution is $x = \sqrt{2}$.

Remark. All presented examples were evaluated by the MATHEMATICA® software package.

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