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GALTON–WATSON TREE AND BRANCHING LOOPS

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Abstract. We define a kind of branching process on the loop space by using the branching mechanism of a loop of string theory.

1. Introduction

In conformal field theory or in string theory [7, 17] people look at random applications ψ from a Riemann surface Σ into a Riemannian manifold M endowed with the probability measure:

$$\mathrm{d}\mu(\psi) = Z^{-1} \exp[-I(\psi)]\, \mathrm{d}D(\psi) \tag{1}$$

where $\mathrm{d}D(\psi)$ is the formal Lebesgue measure over the set of maps ψ and $I(\psi)$ is the energy of the map ψ. If Σ has boundaries, let us say exit boundaries which are circles S_i^1 and input boundaries which are circles S_i^2, the amplitude related to the measure (1) should realize a map from $\otimes_{output} H$ into $\otimes_{input} H$ where H is an Hilbert space associated to the loop space [42].

In the case where the manifold is the linear space $\mathbb{R}^n$, (1) is a Gaussian measure, which corresponds to the free field measure. Since in two dimension, the Green kernel associated to the Laplacian has a singularity on the diagonal, the random field lives on random distributions [18]. It is difficult to state what is a distribution with values in a curved manifold, because the notion of distribution is linear.

If $\Sigma = [0, 1] \times [0, 1]$, there is another process indexed by Σ with values in $\mathbb{R}$, which is the Brownian sheet and which is continuous. $\frac{\partial^2}{\partial s \partial t}\psi$ is the white noise over $[0, 1] \times [0, 1]$. On Σ, there is a natural order, and it is possible after the work of Cairoli [11] to study the stochastic differential equation in Itô meaning:

$$\delta_{s,t} x_{s,t} = A(x_{s,t}) \delta_{s,t} \psi \tag{2}$$

by using martingale theory, where A is a vector field over $\mathbb{R}$. This gives an example of a non-gaussian random field parametrized by the square. In the Gaussian case, this gives the Brownian motion over the path space. Doss and Dozzi [13] have studied the formal action which is associated to (2), that is they have studied the large deviation theory. Norris [41] has succeeded to give a geometrical meaning to (2) and has constrained $x_{s,t}$ to live over a curved manifold.

But it is difficult to generalize (2) to the case where the world sheet is not the square $[0,1] \times [0,1]$, because (2) uses the multi-parameter martingale theory.

Airault–Malliavin in a series of paper (some of them are published for instance in [1]) have constructed the Brownian motion over a loop group. For that, they use the Brownian motion in a Sobolev space with values in the Lie algebra of the group G. This gives a random field from the cylinder $[0,1] \times S^1$ into G.

Infinite dimensional diffusion processes over infinite dimensional manifolds have a long story initiated by Kuo [21] in 1972. The Russian school has studied infinite dimensional processes over infinite dimensional manifolds [5,12]. Brzezniak–Elworthy [8] have done a general theory of infinite dimensional diffusion processes over infinite dimensional manifolds over $M-2$ Banach spaces. The interest of $M-2$ Banach spaces is that there is a Doob inequality for martingales over them. They apply their theory to the case of the free loop space of a manifold. This produces random cylinders with values in a compact Riemannian manifold, or the Brownian motion with values in the loop space of a Riemannian manifold. The loops are only Hölder.

Brzezniak–Léandre [10] have extended the construction of [8] to the case with Brownian pants. The world sheet has two output boundaries and one input boundary. This gives one application from $E_c \otimes E_c$ into E_c, where E_c is the Banach space of continuous functions over the loop space. This means that the Brownian pants are Feller. This gives an approach to one of Segal's axiom of conformal field theory [42], the Hilbert space of the loop space being replaced by the Banach space of continuous functionals over it. Moreover, the theory is 1+1 dimensional. We refer to [34] for similar statement.

For more general surfaces, in a series of papers Léandre [24,26–33] had considered a 1+2 dimensional theory, for instance the Brownian motion on a torus group, or the Brownian motion on the punctured sphere group. Léandre has adapted to this stochastic situation a lot of classical considerations in mathematical physics.

For instance, in [26] we have averaged over all the metric of the surface, the genus of the surface being fixed and obtain a stochastic analoguous of a string theory.

In [31] or in [33] we have used the Markov property of the nonlinear random field in order to sew together random punctured sphere. The relation with operads is exhibited, as it is classical in mathematical physics [19,22]. In these two papers we sew together **deterministic** punctured spheres in the world sheet.

The archetype of an operad is the set of trees. It is classical that we can put measures on the set of trees, for instance Galton–Watson measures. So in this paper, we would like to get a random world sheet, as it is done in string theory, but in a little bit different set-up: instead to choose the geometry at random of a given surface whose genus is fixed, we choose the topology at random (a tree at random), but when we fix the topology, we do not perform any randomization of the considered geometry of the surface. Our goal is to perform a 1+1 dimensional theory (that is a kind of branching process on the loop space) instead of a 1+2 dimensional theory as it was done in [31, 33].

But let us recall that the classical branching mechanism for loops initiated in physics in the so-called dual resonances models (see [38] or [43]), is a branching mechanism which is different than traditional branching mechanism of the theory of branching process [4, 36]. In the first part, we repeat the consideration of [10] of construction of the elementary branching mechanism of a loop in two loops in a 1+1 dimensional theory. In the second part, by using a certain Galton–Watson process, we iterate this branching mechanism, and since we get in the theory of Galton–Watson process a time, we deduced a random tree labelled by the loop space, which satisfies a certain Markov property.

The reader interested by the relation about analysis over loop space and mathematical physics can see the survey of Albeverio [2] and the two surveys [23, 25].

2. The Elementary Branching Mechanism

We recall briefly the construction of the Brownian pants of Brzezniak–Léandre [10].

We consider a compact Riemannian manifold M of dimension d embedded in $\mathbb{R}^n$ isometrically. If $x \in M$, $\Pi(x)$ is the orthogonal projection from $\mathbb{R}^n$ into $T_x(M)$. It can be extended to a map from $\mathbb{R}^n$ into the linear applications over $\mathbb{R}^n$, which is smooth and have bounded derivatives of all orders. We introduce the Hilbert Sobolev space $\mathbb{H} = H^{1,2}(S^1, \mathbb{R}^n)$ of the set of loops in $\mathbb{R}^n$ such that

$$\int_0^1 |\gamma(s)|^2 \,\mathrm{d}s + \int_0^1 |\gamma'(s)|^2 \,\mathrm{d}s = \|\gamma\|^2 < \infty.$$

Let $B_t(\cdot)$ be the Brownian motion with values in $\mathbb{H}$.

We can construct it as follows. Let Ψ_s be the linear map from $\mathbb{H}$ into $\mathbb{R}^n$ defined as follows: $\Psi_s(\gamma(\cdot)) = \gamma(s)$. Since $\mathbb{H}$ is an Hilbert space and since Ψ_s is continuous, we get

$$\Psi_s(\gamma(\cdot)) = \int_0^1 \langle \gamma(u), \alpha_s(u)\rangle \,\mathrm{d}u + \int_0^1 \langle \gamma'(u), \alpha'_s(u)\rangle \,\mathrm{d}u$$

where $t \to B_t(s)$ is a Brownian motion with covariance $\|\alpha_s(\cdot)\|^2$. (We did as we were working on $\mathbb{R}$ in order to simplify the notation, but it is easy to reduce our study to the case of $\mathbb{R}$ by looking the coordinates of Ψ_s.) Moreover, if $s \neq s'$, Ψ_s is independent of $\Psi_{s'}$ as a linear map. This shows us that $\alpha_s(\cdot)$ and $\alpha_{s'}(\cdot)$ are independents and that the couple $t \to (B_t(s), B_t(s'))$ realized a non degenerated

Brownian motion over $\mathbb{R}^n \times \mathbb{R}^n$, although $t \to B_t(s)$ and $t \to B_t(s')$ are not independent. Moreover, the covariance matrix of $B_t(s)$ and $B_t(s')$ are not degenerated. In others words, we can write

$$B_t(s') = \alpha_1(s, s')B_t(s) + \alpha_2(s, s')B_t(s, s') \tag{3}$$

where $B_t(s, s')$ is independent of $B_t(s)$ and where the two constants in the decomposition (3) are not equal to 0.

The family of Stratonovitch equations

$$\mathrm{d}_t x_t(s) = \Pi(x_t(s))\,\mathrm{d}_t B_t(s), \qquad x_0(s) = x$$

has a meaning. It constitutes a family of Brownian motions over the manifold over M parametrized by the circle. (In this work, s will denote the internal time of the loop and t the propagation time of the loop)

Let $s_1 < s_2$ be two times. We constrain the elliptic diffusion $t \to (x_t(s_1), x_t(s_2))$ to be equals at y at time 1.

Let us recall that if we consider an elliptic diffusion $\widetilde{y}_t(\widetilde{x})$ over a compact manifold $\widetilde{M}$, it has an heat kernel $q_t(\widetilde{x}, \widetilde{y})$ satisfying the estimate

$$|\operatorname{grad} \log q_t(\widetilde{x}, \widetilde{y})| \leq C/t\widetilde{d}(\widetilde{x}, \widetilde{y})$$

for the associated Riemannian metric and the natural Riemannian distance $\widetilde{d}$ associated to the elliptic diffusion if $\widetilde{x}$ and $\widetilde{y}$ are close [6, 39]. Let us recall that if the stochastic differential equation of the elliptic diffusion is given by

$$\mathrm{d}\widetilde{y}_t(\widetilde{x}) = \sum \widetilde{X}_i(\widetilde{y}_t(\widetilde{x}))\,\mathrm{d}\widetilde{w}_t^i + \widetilde{X}_0(\widetilde{y}_t(\widetilde{x}))\,\mathrm{d}t \tag{4}$$

over the compact manifold, the bridge between $\widetilde{x}$ and $\widetilde{y}$ satisfies to the following stochastic differential equation (in Stratonovitch sense)

$$\mathrm{d}\widetilde{y}_t(\widetilde{x}, \widetilde{y}) = \sum \widetilde{X}_i(\widetilde{y}_t(\widetilde{x}, \widetilde{y}))\,(\mathrm{d}\widetilde{w}_t^i + \beta_t^i\,\mathrm{d}t) + \widetilde{X}_0(\widetilde{y}_t(\widetilde{x}, \widetilde{y}))\,\mathrm{d}t \tag{5}$$

where $\beta_t^i = \langle \widetilde{X}_i(\widetilde{y}_t(\widetilde{x}, \widetilde{y})), \operatorname{grad} \log q_{1-t}(\widetilde{y}_t(\widetilde{x}, \widetilde{y}), \widetilde{y})\rangle$. This means that we transform $\mathrm{d}\widetilde{w}_t^i$ into $\mathrm{d}\widetilde{w}_t^i + \beta_t^i\,\mathrm{d}t$ in the equation (5) [6, 39]. By the previous estimate of the gradient of the logarithm of the heat-kernel, we have

$$E\left[\int_0^1 |\beta_t^i|\,\mathrm{d}t\right] < \infty.$$

Let us recall briefly Brzezniak–Elworthy theory [8].

Let $W_{\theta,p}$ the Sobolev–Slobodetski space of maps γ from S^1 into $\mathbb{R}^n$ such that

$$\left(\int_{S^1} |\gamma(s)|^p\,\mathrm{d}s + \int_{S^1 \times S^1} \frac{|\gamma(s) - \gamma(t)|^p}{|s - t|^{1+\theta p}}\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p} = \|\gamma\|_{\theta,p} < \infty.$$

The Brownian motion with values in $\mathbb{H}$ takes in fact its values in $W_{\theta,p}$ for some p and some θ. Moreover, $W_{\theta,p}$ is a M-type 2 Banach space, where there is a nice stochastic integration theory [40]. $\mathbb{H}$ is continuously embedded in some $W_{\theta,p}$. Let $\mathbb{H}^1$ the finite dimensional sub-Hilbert space of $\mathbb{H}$ spanned by $\alpha_{s_1}(\cdot)$ and $\alpha_{s_2}(\cdot)$. Let $\pi(\alpha_s)$ be the Hilbert projection of α_s into $\mathbb{H}^1$. $s \to \pi(\alpha_s)$ belongs to $W_{\theta,p}$ for some θ and some p.

Let Ξ be the Nemytski map

$$\gamma(\cdot) \to \{s \to \Pi(\gamma(s))\}.$$

As it was shown in [8], Ξ is Lipschitz and Fréchet smooth with linear growth on $W_{\theta,p}$.

We want to solve the stochastic differential Stratonovitch equation starting from a given element $\gamma(\cdot)$ of $W_{\theta,p}$ on $W_{\theta,p}$

$$\mathrm{d}X_t = \Xi(X_t)\,\mathrm{d}B_t(\cdot) + \Xi(X_t)\langle \pi(\alpha_.), \beta_t\rangle\,\mathrm{d}t. \tag{6}$$

Since $\Xi\langle \pi(\alpha_.), \beta_t\rangle$ is smooth on $W_{\theta,p}$, (6) has a unique solution on $W_{\theta,p}$ [8] up to a stopping blowing time τ. Let us show that $\tau = 1$. Let O_n be the event where $\int_0^1 |\beta_4|\,\mathrm{d}t \le n$. Over O_n we have

$$\sup_{s\le t} \|X_s\|^2_{\theta,p}$$
$$\le C\left(\|\gamma\|^2_{\theta,p} + \sup_{s\le t}\left\|\int_0^s \Xi(X_u)\,\mathrm{d}B_u(\cdot)\right\|^2_{\theta,p} + \int_0^t \sup_{u\le s}\|X_u\|^2_{\theta,p}|\beta_u|\,\mathrm{d}u\right).$$

By Gronwall lemma we deduce that on O_n

$$\sup_{s\le t}\|X_s\|^2_{\theta,p} \le C_n\left(\|\gamma\|^2_{\theta,p} + \sup_{s\le t}\left\|\int_0^s \Xi(X_u)\,\mathrm{d}B_u(\cdot)\right\|^2_{\theta,p}\right)$$

where C_n depends only on n.

Since $W_{\theta,p}$ is a M-type 2 Banach space, we deduce since Ξ has linear growth that on O_n

$$\sup_{s\le 1} E[\|X_s\|^2_{\theta,p}] < \infty$$

by using Gronwall lemma [6, (2.15)].

If we start from a loop $\gamma \in W_{\theta,p}$ in M, we deduce that $X_1(\gamma)$ is a random loop in M which belongs almost surely to $W_{\theta,p}$. Moreover, almost surely, $X_1(\gamma)(s_1) = X_1(\gamma)(s_2) = y$ by (5). The loop $s \to X_1(\gamma)(s)$ is split in two loops.

3. Galton–Watson Trees

Let Y be a binary Galton–Watson tree. The probability that a vertex has one child is $p_1 > 0$ and that it has two children is $p_2 > 0$. Moreover, we suppose that $p_1 + p_2 = 1$. At the step n, we consider the labelled exit vertices $\alpha_i(n)$. Each label $\alpha_i(n)$ are indexed in increasing order and the set of children is ranged in increasing order. If two vertices at step $n+1$ are coming from two different parents, the order of the two exit vertices is the same than the order of their parents. We get a random interval of N, and if at time n, there is no vertex at the site i, we will say that $\alpha_i(n) = \infty$. We get a set of random variable $X_i(n)$, $i \in N$, where $X_i(n)$ is the label of the parent of i (if $\alpha_i(n) = \infty$, we put $X_i(n) = \infty$). Moreover, $X_i(n) \leq X_{i+1}(n)$.

Let us define our random tree with values at $W_{\theta,p}$. At the root, we start from a loop in M belonging to $W_{\theta,p}$:

- Either the root has two children. We consider $s_1 = 0$, $s_2 = 1/2$ and the given loop at the root. We consider the branching mechanism of the starting loop in two loops given at the first part. We get two loops $\gamma_{\alpha_1(1)}$ and $\gamma_{\alpha_2(1)}$.
- Or the root has only one child. We consider with the notation of the previous part the equation:

$$\mathrm{d}X_t = \Xi(X_t)\,\mathrm{d}B_t(\cdot) \tag{7}$$

  starting from the initial loop. X_1 is a random loop belonging to $W_{\theta,p}$ denoted by $\gamma_{\alpha_1(1)}$.

Let us iterate the procedure. Let us suppose that at step n, we get I vertices $\alpha_i(n)$ associated to the random loops $\gamma_{\alpha_i(n)}$. Either, $\alpha_i(n)$ has two children, and we consider the Branching mechanism given before associated to a leading infinite dimensional Brownian motion independent of the others considered. Or $\alpha_i(n)$ has one child, and we perform the transformation (3.1). All the leading Brownian motions considered are independents. If $\alpha_i(n) = \infty$, we put $\gamma_{\alpha_i(n)} = \emptyset$.

We get by this procedure a set of random loops $\gamma_{\alpha_i(n+1)}$ belonging to $W_{\theta,p}$. (We omit to describe the different rescaling which occur when a loop branches in two loops.)

We complete all the σ-algebras which are considered. Let Pa_n be the σ-algebra spanned by the $X_i(j)$ and $\gamma_{\alpha_i(j)}$ for $j \leq n$. Let Pr_n be the σ-algebra spanned by the $X_i(n)$ and $\gamma_{\alpha_i(n)}$. And let F_n be the σ-algebra spanned by the $X_i(j)$ and $\gamma_{\alpha_i(j)}$, $j > n$. Since we work on a Galton–Watson tree and since the leading flat infinite dimensional Brownian motions are all independents, we get

Theorem 1. *Let Ψ be a random variable which belongs to L^1 and which is F_n-measurable. Then almost surely,*

$$E[\psi \mid Pa_n] = E[\psi \mid Pr_n].$$

It is a kind of Markov property for our random tree.

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