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ARITHMETIC PROPORTIONAL ELLIPTIC CONFIGURATIONS WITH COMPARATIVELY LARGE NUMBER OF IRREDUCIBLE COMPONENTS

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Abstract. Let T be an arithmetic proportional elliptic configuration on a bi-elliptic surface $A_{\sqrt{-d}}$ with complex multiplication by an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. The present note establishes that if T has s singular points and

$$4s - 5 \leq h \leq 4s$$

irreducible smooth elliptic components, then $d = 3$ and T is $\mathrm{Aut}(A_{\sqrt{-3}})$-equivalent to Hirzebruch's example $T^{(1,4)}_{\sqrt{-3}}$ with a unique singular point and 4 irreducible components.

In [3], it was announced "as a working hypothesis or a philosophy" that … "*up to birational equivalence and compactifications, all complex algebraic surfaces are ball quotients.*" This was proven it for the abelian surfaces. In order to formulate it precisely, one needs the following

Definition 1 (Holzapfel [5]). *A reduced effective divisor T on an abelian surface A is called an intersecting elliptic configuration if all the irreducible components T_i of T are smooth elliptic curves with $s_i := \mathrm{card}(T_i \cap T^{\mathrm{sing}}) \geq 1$, and all the non-void intersections $T_i \cap T_j \neq \emptyset$, $i \neq j$ are transversal.*

Definition 2 (Holzapfel [5]). *An intersecting elliptic configuration $T = T_1 + \cdots + T_h$ on an abelian surface S is proportional if*

$$s_1 + \cdots + s_h = 4s$$

for $s := \mathrm{card}(T^{\mathrm{sing}})$, $s_i := \mathrm{card}(T_i \cap T^{\mathrm{sing}})$.

Theorem 1 (Holzapfel [5]). *An abelian surface A is the minimal model of the toroidal compactification $(\mathbb{B}/\Gamma)'$ of a neat ball quotient $\mathbb{B}/\Gamma$ if and only if $A = E \times E$ is bi-elliptic and there exists a proportional elliptic configuration $T \subset A$.*

This proportionality relation on an abelian surface is extended in [6] to the elliptic fibrations, including the honest elliptic surfaces, the Enriques surfaces and the $K3$-surfaces with a fixed point free involution. The case of general type is straightforward and the treatment of the hyperelliptic surfaces is reduces to the one for the abelian surfaces.

The rest of the present work focuses on the study of the arithmetic proportional elliptic configurations on bi-elliptic surfaces with complex multiplication.

Let us denote by

$$E_{\alpha,\beta} := \{\alpha P, \beta P;\ P \in E_{\sqrt{-d}}\} \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}, \quad E_{\sqrt{-d}} = \mathbb{C}/O_{\sqrt{-1}}$$

the elliptic curves through the origin, whose universal covers are generated by the complex vectors $(\alpha, \beta) \in \mathbb{C}^2$. Put $E_{\alpha,\beta} + (P_1, P_2)$ for the elliptic curves through $(P_1, P_2) \in A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$, whose universal covers are disjoint from the ones of $E_{\alpha,\beta}$.

Here are some examples of arithmetic proportional elliptic configurations.

Proposition 1 (Hirzebruch [1]). *The arithmetic elliptic configuration*

$$T^{(1,4)}_{\sqrt{-3}} = E_{1,0} + E_{0,1} + E_{1,1} + E_{\zeta,1} \subset A_{\sqrt{-3}}, \qquad \zeta = \frac{1-\sqrt{-3}}{2}$$

is proportional. It has a unique singular point (Q_0, Q_0) *where*

$$Q_0 := 0 \left(\operatorname{mod} O_{\sqrt{-3}}\right).$$

Proposition 2 (Holzapfel [2]). *The arithmetic elliptic configuration*

$$T^{(3,6)}_{\sqrt{-3}} = E_{1,1} + E_{\zeta^2,1} + E_{1,\zeta^2} + E_{1,0} + (E_{1,0} + (Q_0, P_0)) + (E_{1,0} + (Q_0, -P_0))$$

on $A_{\sqrt{-3}}$ *is proportional. It has 3 singular points, namely,* (Q_0, Q_0), (P_0, P_0) *and* $(-P_0, -P_0)$ *with* $Q_0 := 0 \left(\operatorname{mod} O_{\sqrt{-3}}\right)$, $P_0 := \frac{1+\zeta}{3} \left(\operatorname{mod} O_{\sqrt{-3}}\right)$.

Proposition 3 (Holzapfel [4]). *The arithmetic elliptic configuration*

$$T^{(3,6)}_{\sqrt{-1}} = E_{1,0}+E_{0,1}+E_{1,-1+i}+E_{-1-i,1}+(E_{1,-1}+(Q_2, Q_0))+(E_{-i,1}+(Q_2, Q_0))$$

on $A_{\sqrt{-1}}$ *is proportional. It has 3 singular points* (Q_0, Q_0), (Q_2, Q_0) *and* (Q_0, Q_2) *with* $Q_0 := 0 \left(\operatorname{mod} O_{\sqrt{-1}}\right)$, $Q_2 := \frac{1+i}{2} \left(\operatorname{mod} O_{\sqrt{-1}}\right)$.

The uniqueness of these examples within the arithmetic proportional elliptic configurations with at most 3 singular points is established by the following

Proposition 4 ([7]). *If* $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$, $E_{\sqrt{-d}} = \mathbb{C}/O_{\sqrt{-d}}$ *is an arithmetic proportional elliptic configuration with at most 3 singular points then either* T *is* $\operatorname{Aut}(A_{\sqrt{-3}})$*-equivalent to* $T^{(1,4)}_{\sqrt{-3}}$ *or* $T^{(3,6)}_{\sqrt{-3}}$, *or* $T = T^{(3,6)}_{\sqrt{-1}}$ *up to a complex conjugation and an automorphism of* $A_{\sqrt{-1}}$.

In order to get some impression of the proof of Proposition 4, let us cite a lemma of Holzapfel, which is a basic tool for recognizing the proportionality of an intersecting elliptic configuration

Lemma 1 (Holzapfel [2]). *If* $T_1 = E_{\alpha_1,\beta_1} + (R_1, S_1)$ *and* $T_2 = E_{\alpha_2,\beta_2} + (R_2, S_2)$ *are smooth arithmetic elliptic curves on a bi-elliptic surface* $A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$, $E_{\sqrt{-d}} = \mathbb{C}/O_{\sqrt{-d}}$, *then the intersection number*

$$T_1.T_2 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

equals the norm of the determinant of the matrix, formed by the coordinates of the generators of the universal covers of T_1 *and* T_2.

The aforementioned lemma requires the study of some elementary arithmetic in the integers ring $O_{\sqrt{-d}}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. (The notation $O_{\sqrt{-d}}$ reflects the fact that the integers ring of an algebraic number field is always a maximal order.) The invertible elements of $O_{\sqrt{-d}}$ are called units and their multiplicative group is denoted by $O^*_{\sqrt{-d}}$. The article [7] establishes that the difference $a - b$ of units $a, b \in O^*_{\sqrt{-d}}$ is a unit if and only if $d = 3$ and the ratio $\frac{a}{b}$ is a primitive sixth root of unity. Similarly, if $a, b \in O^*_{\sqrt{-d}}$ then the norm $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(a - b) = 2$ if and only if $d = 1$ and $\frac{a}{b}$ is a primitive fourth root of unity. The difference of elements of norm 2 is shown to be never of norm 2. If $a, b \in O^*_{\sqrt{-d}}$ and $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(a - b) = 3$ then $d = 3$ and $\frac{a}{b}$ is a primitive third root of unity. Immediate considerations reveal that $O_{\sqrt{-d}}$ has elements of norm 2 only when $d = 1, 2$ or 7. The article [7] provides complete lists of the elements of norm 2 in the integers ring of an imaginary quadratic number field. In a similar vein are described the elements of $O_{\sqrt{-d}}$ of norm 3, which are shown to occur only for $d = 2, 3$ or 11.

Proposition 4 is a starting point for the results of the present note.

A priori, a proportional elliptic configuration with s singular points has

$$4 \leq h \leq 4s$$

irreducible components. Indeed, $s_i \geq 1$ guarantees that $4s = s_1 + \cdots + s_h \geq h$. On the other hand, $s_i \leq s$ requires $4s = s_1 + \cdots + s_h \leq sh$, whereas $4 \leq h$. The present note focuses on the arithmetic proportional elliptic configurations with comparatively large number of irreducible components.

Bearing in mind that Hirzebruch's example from Proposition 1 has $h = 4s$ and Holzapfel's examples from Propositions 2 and 3 have $h = 4s - 6$, we study the arithmetic proportional elliptic configurations with $4s - 5 \leq h \leq 4s$ irreducible components.

Proposition 5. *Let* $A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$, $E_{\sqrt{-d}} = \mathbb{C}/O_{\sqrt{-d}}$ *be a bi-elliptic surface with complex multiplication by an imaginary quadratic number field* $\mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N} - \mathbb{N}^2$, *and* $T \subset A_{\sqrt{-d}}$ *be an arithmetic proportional elliptic configuration with* s *singular points and*

$$h \geq 4s - 5$$

irreducible components. Then T *is* $\mathrm{Aut}(A_{\sqrt{-3}})$*-equivalent to Hirzebruch's example* $T^{(1,4)}_{\sqrt{-3}}$ *from Proposition 1.*

We split the proof in several lemmas. Let us start with the following immediate consequence of Lemma 1

Corollary 1. *Suppose that* T_1, T_2 *and* T_3 *are arithmetic smooth elliptic curves on a bi-elliptic surface* $A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$, $E_{\sqrt{-d}} = \mathbb{C}/O_{\sqrt{-d}}$. *Then*

$$T_1.T_3 = 0 \quad \textit{and} \quad T_2.T_3 = 0 \quad \Rightarrow \quad T_1.T_2 = 0.$$

Proof: If $T_j = E_{\alpha_j,\beta_j} + (R_j, S_j)$ then

$$T_1.T_3 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} \det \begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{pmatrix} = 0$$

and

$$T_2.T_3 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} \det \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix} = 0$$

imply that the vector (α_3, β_3) is simultaneously co-linear to (α_1, β_1) and (α_2, β_2). Consequently, (α_1, β_1) and (α_2, β_2) are parallel to each other and

$$T_1.T_2 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = 0.$$

□

Lemma 2. *If* $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$ *is an arithmetic proportional elliptic configuration with* s *singular points and* $h = 4s$ *irreducible components then* T *is* $\mathrm{Aut}(A_{\sqrt{-3}})$*-equivalent to Hirzebruch's example* $T^{(1,4)}_{\sqrt{-3}}$ *from Proposition 1.*

Proof: Let $T \subset A_{\sqrt{-d}}$ be an arithmetic proportional elliptic configuration with s singular points and $4s$ irreducible components. Making use of Proposition 4, one can assume that $s \geq 4$. The proportionality relation $s_1 + \cdots + s_{4s} = 4s$ can hold only with $s_i = 1$, for all $1 \leq i \leq 4s$. In other words, any irreducible component T_i of T passes through a unique singular point of T. Recall also that the multiplicities m_j of all the singular points $p_j \in T^{\mathrm{sing}}$ are to be $m_j \geq 2$. After an eventual permutation of the singular points $p_1, \ldots, p_s$, $s \geq 4$, and the irreducible components $T_1, \ldots, T_{4s}$, $4s > s$ of T, one can assume that $T_1 \cap T^{\mathrm{sing}} =$

$T_2 \cap T^{\rm sing} = \{p_1\}$ and $T_3 \cap T^{\rm sing} = \{p_2\}$. Then $T_1.T_3 = 0$ and $T_2.T_3 = 0$ imply $T_1.T_2 = 0$ by Corollary 1, which is an absurd while $T_1 \cap T_2 = \{p_1\}$. □

Lemma 3. *There is no arithmetic proportional elliptic configuration $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$ with s singular points and $h = 4s - 1$ irreducible components.*

Proof: Let us suppose that $T \subset A_{\sqrt{-d}}$ is an arithmetic proportional elliptic configuration with s singular points and $4s - 1$ irreducible components. According to Proposition 4, one can assume that $s \geq 4$. The proportionality relation $s_1 + \cdots + s_{4s-1} = 4s$ requires $s_1 = 2$, $s_2 = \cdots = s_{4s-1} = 1$, up to a permutation of the irreducible components. Since $4s - 2 > s$ under the assumption $s \geq 4$, there exist $p_k \in T^{\rm sing}$ and $T_i \neq T_j$ with $T_i \cap T^{\rm sing} = T_j \cap T^{\rm sing} = \{p_k\}$ and $i > 1$, $j > 1$. For any other $a > 1$ with $T_a \cap T^{\rm sing} = \{p_b\}$, $b \neq k$, the vanishing of the intersection numbers $T_i.T_a = 0$, $T_j.T_a = 0$ suffices for $T_i.T_j = 0$, which contradicts $T_i \cap T_j = \{p_k\}$. If $T_a \cap T^{\rm sing} = \{p_k\}$ for all $a > 1$ then $T = T_1 + \sum_{a>1} T_a$ has at most 3 singular points, contrary to the assumption $s \geq 4$. □

Lemma 4. *There is no arithmetic proportional elliptic configuration $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$ with s singular points and $h = 4s - 2$ irreducible components.*

Proof: Let us assume that $T \subset A_{\sqrt{-d}}$ is an arithmetic proportional elliptic configuration with s singular points and $4s - 2$ irreducible components. The proportionality relation $s_1 + \cdots + s_{4s-2} = 4s$ for $s_i := \mathrm{card}\,(T_i \cap T^{\rm sing}) \in \mathbb{N}$ occurs either when $s_1 = 3$, $s_2 = \cdots = s_{4s-2} = 1$ or $s_1 = s_2 = 2$, $s_3 = \cdots = s_{4s-2} = 1$, up to a permutation of $T_1, \ldots, T_{4s-2}$. In the presence of Proposition 4, assume that $s \geq 4$.

If $T_1 \cap T^{\rm sing} = \{p_1, p_2, p_3\}$ then up to a permutation of $T_2, \ldots, T_{4s-2}$ one has $T_2 \cap T^{\rm sing} = \{p_4\}$. If there exists $i > 2$ with $T_i \cap T^{\rm sing} = \{p_4\}$ then $T_1.T_2 = 0$ and $T_1.T_i = 0$ force $T_2.T_i = 0$, contrary to $T_2 \cap T_i = \{p_4\}$. Otherwise, $T_i \cap T^{\rm sing} = \{p_{i+2}\}$ for $2 \leq i \leq s-2$ and $T_j \cap T^{\rm sing} \subseteq \{p_1, p_2, p_3\}$ for $s-1 \leq j \leq 4s-2$. Then $T_1.T_2 = 0$ and $T_{s-1}.T_2 = 0$ imply $T_1.T_{s-1} = 0$, while $T_1 \cap T_{s-1} = T_{s-1} \cap T^{\rm sing} \neq \emptyset$.

Suppose that $T_1 \cap T^{\rm sing} = \{p_1, p_2\}$ and there exists $3 \leq j \leq 4$ with $p_j \notin T_1 + T_2$. Then $T_3 \cap T^{\rm sing} = T_4 \cap T^{\rm sing} = \{p_j\}$ after an eventual permutation of $T_3, T_4, \ldots, T_{4s-2}$. As a result, Corollary 1 infers from $T_1.T_3 = 0$ and $T_1.T_4 = 0$ that $T_3.T_4 = 0$, which is not the case of $T_3 \cap T_4 = \{p_j\}$ under study.

For $p_3, p_4 \in T_1 + T_2$ there follows $T_2 \cap T^{\rm sing} = \{p_3, p_4\}$. Since the multiplicity of p_3 with respect to T is at least 2, there exists $T_3 \subset T$ with $T_3 \cap T^{\rm sing} = \{p_3\}$. Applying Corollary 1 to $T_1.T_2 = 0$ and $T_1.T_3 = 0$ one concludes that $T_2.T_3 = 0$, contrary to $T_2 \cap T_3 = \{p_3\}$. □

Lemma 5. *There is no arithmetic proportional elliptic configuration $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$ with s singular points and $h = 4s - 3$ irreducible components.*

Proof: Let $T \subset A_{\sqrt{-d}}$ be an arithmetic proportional elliptic configuration with s singular points and $4s-3$ irreducible components. In this case, the proportionality relation reads as $s_1 + \cdots + s_{4s-3} = 4s$ and holds for

a) $s_1 = 4$, $s_2 = \cdots = s_{4s-3} = 1$
b) $s_1 = 3$, $s_2 = 2$, $s_3 = \cdots = s_{4s-3} = 1$, or
c) $s_1 = s_2 = s_3 = 2$, $s_4 = \cdots = s_{4s-3} = 1$, up to a permutation of $T_1, \ldots, T_{4s-3}$.

According to Proposition 4, there is no loss of generality in assuming $s \geq 4$. Recall that the multiplicities of all the singular points p_i of T are $m_i \geq 2$.

In the case a) let $T_1 \cap T^{\text{sing}} = \{p_1, \ldots, p_4\}$. If $s \geq 5$ then up to permutations of $T_2, \ldots, T_{4s-3}$ and $p_5, \ldots, p_s$, one can assume that $T_2 \cap T^{\text{sing}} = T_3 \cap T^{\text{sing}} = \{p_5\}$. Then $T_1.T_2 = 0$ and $T_1.T_3 = 0$ force $T_2.T_3 = 0$ by Corollary 1, contrary to $T_2 \cap T_3 = \{p_5\}$. For $s = 4$, up to permutations of $T_2, \ldots, T_{13}$ and $p_1, \ldots, p_4$, one has $T_{i+1} \cap T^{\text{sing}} = \{p_i\}$ for $1 \leq i \leq 4$ and $T_6 \cap T^{\text{sing}} = \{p_1\}$. Then $T_2.T_3 = 0$ and $T_6.T_3 = 0$ imply $T_2.T_6 = 0$ by Corollary 1, while $T_2 \cap T_6 = \{p_1\}$.

Similarly, in the case b) with $s \geq 4$, either $T_1 \cap T^{\text{sing}} \supset T_2 \cap T^{\text{sing}}$ and there is a single singular point p_3 of multiplicity 1 with respect to $T_1 + T_2$ or there exist at least 2 singular points of multiplicity 1 with respect to $T_1 + T_2$. In the first case there is $T_3 \subset T$ with $T_3 \cap T^{\text{sing}} = \{p_3\}$ and at least two irreducible components, say $T_4, T_5 \subset T$, with $T_4 \cap T^{\text{sing}} = T_5 \cap T^{\text{sing}} = \{p_4\}$. This case is rejected by the fact that Corollary 1 forces $T_4.T_5 = 0$ from $T_3.T_4 = 0$, $T_3.T_5 = 0$, while $T_4 \cap T_5 = \{p_4\}$. If $s \geq 6$ or $s = 5$ and $(T_1 + T_2) \cap T^{\text{sing}} = \{p_1, \ldots, p_4\}$ then there exists $p_j \notin (T_1 + T_2) \cap T^{\text{sing}}$ with $T_3 \cap T^{\text{sing}} = T_4 \cap T^{\text{sing}} = \{p_j\}$, up to a permutation of $T_3, \ldots, T_{4s-3}$. Then Corollary 1 infers $T_3.T_4 = 0$ from $T_1.T_3 = 0$ and $T_1.T_4 = 0$, which is an absurd. For $s = 5$ and $T_1 \cap T^{\text{sing}} = \{p_1, p_2, p_3\}$, $T_2 \cap T^{\text{sing}} = \{p_4, p_5\}$ there is a permutation of $T_3, \ldots, T_{17}$, such that $T_3 \cap T^{\text{sing}} = T_4 \cap T^{\text{sing}} = \{p_j\}$, $1 \leq j \leq 5$. If $p_j \in T_1$ then $T_2.T_3 = 0$ and $T_2.T_4 = 0$ requires $T_3.T_4 = 0$, which is an absurd. Similarly, for $p_j \in T_2$ there follow $T_1.T_3 = 0$ and $T_1.T_4 = 0$, whereas $T_3.T_4 = 0$, contrary to $T_3 \cap T_4 = \{p_j\}$. For $s = 4$ there is a permutation of $T_3, \ldots, T_{13}$ such that $T_3 \cap T^{\text{sing}} = T_4 \cap T^{\text{sing}} = \{p_j\}$. If $j > 1$ then there is $1 \leq i \leq 2$ such that $T_i.T_3 = 0$ and $T_i.T_4 = 0$. Then by Corollary 1 there follows $T_3.T_4 = 0$, contrary to $T_3 \cap T_4 = \{p_j\}$. If $j = 1$ then up to permutations of the irreducible components $T_5, \ldots, T_{13}$ and the points p_2, p_3, p_4 of multiplicity 1 with respect to $T_1 + T_2$, one can assume that $T_5 \cap T^{\text{sing}} = \{p_2\}$. Then $T_3.T_5 = 0$ and $T_3.T_4 = 0$, which is an absurd.

In the case c) with $s \geq 4$, let us assume that $T_1 \cap T^{\text{sing}} = \{p_1, p_2\}$. Then there exist $T_4, T_5 \subset T$ with $T_4 \cap T^{\text{sing}} = \{p_3\}$ and $T_5 \cap T^{\text{sing}} = \{p_4\}$. If for some $3 \leq j \leq 4$ there holds $p_j \notin T_1 + T_2$ then $T_6 \cap T^{\text{sing}} = \{p_j\}$, up to a permutation of $T_6, \ldots, T_{4s-3}$. As a result, $T_1.T_{j+1} = 0$ and $T_1.T_6 = 0$ require $T_{j+1}.T_6 = 0$ by

Corollary 1, while $T_{j+1} \cap T_6 = \{p_j\}$. In the case of $p_3, p_4 \in T_1 + T_2$ there follows $T_2 \cap T^{\text{sing}} = \{p_3, p_4\}$, so that Corollary 1 requires $T_2.T_4 = 0$ out of $T_1.T_2 = 0$, $T_1.T_4 = 0$. That contradicts $T_2 \cap T_4 = \{p_3\}$. □

Lemma 6. *There is no arithmetic proportional elliptic configuration $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$ with s singular points and $h = 4s - 4$ irreducible components.*

Proof: Let us assume that $T \subset A_{\sqrt{-d}}$ is an arithmetic proportional elliptic configuration with s singular points and $4s - 4$ irreducible components. Without loss of generality, assume that $s \geq 4$. The proportionality relation $s_1 + \cdots + s_{4s-4} = 4s$ splits the considerations into the following subcases:

a) $s_1 = 5$, $s_2 = \cdots = s_{4s-4} = 1$

b) $s_1 = 4$, $s_2 = 2$, $s_3 = \cdots = s_{4s-4} = 1$

c) $s_1 = s_2 = 3$, $s_3 = \cdots = s_{4s-4} = 1$

d) $s_1 = 3$, $s_2 = s_3 = 2$, $s_4 = \cdots = s_{4s-4} = 1$

e) $s_1 = \cdots = s_4 = 2$, $s_5 = \cdots = s_{4s-4} = 1$.

In the case a), one has $T_1 \cap T^{\text{sing}} = \{p_1, \ldots, p_5\}$ and $T_{j+1} \cap T^{\text{sing}} = \{p_j\}$, $\forall\, 1 \leq j \leq 5$. As far as $h = 4s - 4 \geq 4.4 - 4 = 12$, there exists $1 \leq j_0 \leq 5$ with $T_7 \cap T^{\text{sing}} = \{p_{j_0}\}$ up to a permutation of $T_2, \ldots, T_{4s-4}$. Then for any $i \neq j_0$, $1 \leq i \leq 5$, the vanishing of the intersection numbers $T_{i+1}.T_{j_0+1} = 0$, $T_{i+1}.T_7 = 0$ forces $T_{j_0+1}.T_7 = 0$ by Corollary 1, while $T_{j_0+1} \cap T_7 = \{p_{j_0}\}$.

In the case b), suppose that $T_3 \cap T^{\text{sing}} = T_4 \cap T^{\text{sing}} = \{p_i\}$ and $T_5 \cap T^{\text{sing}} = \{p_j\}$ for some $1 \leq i \neq j \leq s$. Then by Corollary 1, $T_3.T_5 = 0$ and $T_4.T_5 = 0$ suffice for $T_3.T_4 = 0$, while $T_3 \cap T_4 = \{p_i\}$. As far as $h \geq 12$, there always exist at least two irreducible components among $T_3, \ldots, T_{4s-4}$, which pass through one and a same singular point. That reduces the considerations to $T_i \cap T^{\text{sing}} = \{p_j\}$ for some $1 \leq j \leq s$ and $\forall\, 3 \leq i \leq 4s - 4$. As a result, the multiplicities of all p_k with $k \neq j$ have to be at least 2 with respect to $T_1 + T_2$. However, $s_1 = 4$, $s_2 = 2$ allow at most two p_k of multiplicity at least 2 with respect to $T_1 + T_2$ and, therefore, at least one p_m of multiplicity 1 with respect to T.

Similarly, in the case c), the assumption $T_i \cap T^{\text{sing}} = \{p_j\}$ for some $1 \leq j \leq s$ and $\forall\, 3 \leq i \leq 4s - 4$ requires all $p_k \neq p_j$ to be of multiplicity at least 2 in $T_1 + T_2$. That happens exactly when $s = 4$, $T_1 \cap T^{\text{sing}} = T_2 \cap T^{\text{sing}} = \{p_1, p_2, p_3\}$. In order to have a fourth singular point, one has to require $T_i \cap T^{\text{sing}} = \{p_4\}$, $\forall\, 3 \leq i \leq 12$. Now Corollary 1 infers from $T_1.T_3 = 0$ and $T_2.T_3 = 0$ the vanishing of the intersection number $T_1.T_2 = 0$, regardless of $T_1 \cap T_2 = \{p_1, p_2, p_3\}$.

In the case d), if there exist $T_4, T_5, T_6 \subset T$ with $T_4 \cap T^{\text{sing}} = T_5 \cap T^{\text{sing}} = \{p_i\}$ and $T_6 \cap T^{\text{sing}} = \{p_j\}$, $p_i \neq p_j$, then $T_4.T_6 = 0$ and $T_5.T_6 = 0$ lead to $T_4.T_5 = 0$, which is an absurd. Since $(4s-4) - 3 > s$ for $s \geq 4$, there always exist $T_4, T_5 \subset T$ with $T_4 \cap T^{\text{sing}} = T_5 \cap T^{\text{sing}}$. Up to a permutation of $p_1, \ldots, p_s$, that reduces the considerations to $T_i \cap T^{\text{sing}} = \{p_4\}$ for $\forall\, 4 \leq i \leq 4s-4$. If $p_4 \notin T_1 \cap T^{\text{sing}}$ then $s = 4$ and $T_1 \cap T^{\text{sing}} = \{p_1, p_2, p_3\}$, $T_2 \cap T^{\text{sing}} = \{p_1, p_2\}$, $T_3 \cap T^{\text{sing}} = \{p_3, p_4\}$. Now $T_1.T_4 = 0$, $T_2.T_4 = 0$ lead to $T_1.T_2 = 0$, while $T_1 \cap T_2 = T_2 \cap T^{\text{sing}} = \{p_1, p_2\}$. If $p_4 \in T_1 \cap T^{\text{sing}}$ then $T_1 \cap T^{sing} = \{p_1, p_2, p_4\}$ and $p_3 \in T_2 \cap T^{\text{sing}}$, $p_3 \in T_3 \cap T^{\text{sing}}$. Therefore $s = 4$ and $T_2 \cap T^{\text{sing}} = \{p_1, p_3\}$, $T_3 \cap T^{\text{sing}} = \{p_2, p_3\}$. By Corollary 1, $T_2.T_4 = 0$ and $T_3.T_4 = 0$ imply that $T_2.T_3 = 0$, contrary to $T_2 \cap T_3 = \{p_3\}$.

In the case e) with $T_1 \cap T^{\text{sing}} = \{p_1, p_2\}$, first assume that $p_3 \notin T_2 + T_3 + T_4$. Then there exist $T_5, T_6 \subset T$ with $T_5 \cap T^{\text{sing}} = T_6 \cap T^{\text{sing}} = \{p_3\}$. Now Corollary refParallel infers $T_5.T_6 = 0$ from $T_1.T_5 = 0$, $T_1.T_6 = 0$, which is an absurd in the presence of $T_5 \cap T_6 = \{p_3\}$. From now on, let us suppose that $p_3 \in T_2 + T_3 + T_4$ and without loss of generality, $p_j \in T_2 + T_3 + T_4$, $\forall\, 4 \leq j \leq s$. In particular, that specifies $s \leq 6$. On the other hand, if for some $3 \leq j \leq s$ the point p_j is of multiplicity at least 2 with respect to $T_5 + \cdots + T_{4s-4}$, then up to a permutation of $T_5 + \cdots + T_{4s-4}$, there holds $T_5 \cap T^{\text{sing}} = T_6 \cap T^{\text{sing}} = \{p_j\}$. Then $T_1.T_5 = 0$ and $T_1.T_6 = 0$ force $T_5.T_6 = 0$ by Corollary 1, contrary to $T_5 \cap T_6 = \{p_j\}$. For the rest of the argument, one can anticipate that the multiplicity of p_j with respect to $T_2 + T_3 + T_4$ is at least 1 and the multiplicity of p_j with respect to $T_5 + \cdots + T_{4s-4}$ is at most 1 for all $3 \leq j \leq s$. As far as $s - 2 \leq 4 < 8 \leq (4s-4) - 4$, one can assume that $T_5 \cap T^{\text{sing}} = T_6 \cap T^{\text{sing}} = \{p_1\}$. If there exists $1 \leq i \leq 4s-4$ with $p_1 \notin T_i \cap T^{\text{sing}}$ then $T_5.T_i = 0$ and $T_6.T_i = 0$ implies $T_5.T_6 = 0$ by Corollary 1, which is an absurd in the presence of $T_5 \cap T_6 = \{p_1\}$. Otherwise, $p_1 \in T_i \cap T^{\text{sing}}$, $\forall\, 1 \leq i \leq 4s-4$. In particular, $T_j \cap T^{\text{sing}} = \{p_1\}$, $\forall\, 5 \leq j \leq 4s-4$. As a result, all p_k with $2 \leq k \leq s$ are of multiplicity at least 2 with respect to $T_1 + \cdots + T_4$. That happens either for $T_2 \cap T^{\text{sing}} = \{p_1, p_2\}$ or for $T_2 \cap T^{\text{sing}} = \{p_2, p_3\}$. In both cases, up to a transposition of T_3, T_4, one can assume that $T_3 \cap T^{\text{sing}} = \{p_3, p_4\}$. If $T_2 \cap T^{\text{sing}} = \{p_1, p_2\}$ then $T_4 \cap T^{\text{sing}} = \{p_3, p_4\}$ and $T_1.T_3 = 0$, $T_1.T_4 = 0$ imply that $T_3.T_4 = 0$, contrary to $T_3 \cap T_4 = \{p_3, p_4\}$. If $T_2 \cap T^{\text{sing}} = \{p_2, p_3\}$ then $T_2.T_5 = 0$ and $T_3.T_5 = 0$ forces $T_2.T_3 = 0$, while $T_2 \cap T_3 = \{p_3\}$. □

Lemma 7. *There is no arithmetic proportional elliptic configuration $T \subset A_{\sqrt{-d}} = E_{\sqrt{-d}} \times E_{\sqrt{-d}}$ with s singular points and $h = 4s - 5$ irreducible components.*

Proof: Suppose that there exists an arithmetic proportional elliptic configuration $T \subset A_{\sqrt{-d}}$ with s singular points and $4s - 5$ irreducible components. The proportionality relation $s_1 + \cdots + s_{4s-5} = 4s$ splits the considerations in the following subcases:

a) $s_1 = 6$, $s_2 = \cdots = s_{4s-5} = 1$

b) $s_1 = 5$, $s_2 = 2$, $s_3 = \cdots = s_{4s-5} = 1$

c) $s_1 = 4$, $s_2 = 3$, $s_3 = \cdots = s_{4s-5} = 1$

d) $s_1 = 4$, $s_2 = s_3 = 2$, $s_4 = \dots = s_{4s-5} = 1$

e) $s_1 = s_2 = 3$, $s_3 = 2$, $s_4 = \dots = s_{4s-5} = 1$

f) $s_1 = 3$, $s_2 = s_3 = s_4 = 2$, $s_5 = \dots = s_{4s-5} = 1$

g) $s_1 = \dots = s_5 = 2$, $s_6 = \dots = s_{4s-5} = 1$.

Without loss of generality, assume that $s \geq 4$, whereas $(4s-5)-5 > s$. In all the cases that provides the presence of $p_i \in T^{\rm sing}$ with $T_{4s-6} \cap T^{\rm sing} = T_{4s-5} \cap T^{\rm sing} = \{p_i\}$, up to a permutation of the irreducible components of T with a single singular point.

Suppose that there exists $T_k \subset T$ with $T_k \cap T^{\rm sing} = \{p_j\}$ for some $p_j \neq p_i$. Then $T_k.T_{4s-6} = 0$ and $T_k.T_{4s-5} = 0$ suffice for $T_{4s-6}.T_{4s-5} = 0$, according to Corollary 1. That contradicts $T_{4s-6} \cap T^{\rm sing} = T_{4s-5} \cap T^{\rm sing} = \{p_i\}$. From now on, we assume the coincidence of all $T_j \cap T^{\rm sing} = \{p_i\}$ of cardinality 1.

If there is an irreducible component $T_k \subset T$ with $\mathrm{card}(T_k \cap T^{\rm sing}) \geq 2$ and $p_i \notin T_k$ then $T_k.T_{4s-6} = 0$ and $T_k.T_{4s-5} = 0$. Now Corollary 1 implies that $T_{4s-6}.T_{4s-5} = 0$, which contradicts $T_{4s-6} \cap T_{4s-5} = \{p_i\}$.

The rest of the speculations suppose that $p_i \in T_j \cap T^{\rm sing}$, $\forall\, 1 \leq j \leq 4s-5$.

In the case a), the singular points $p_j \neq p_i$ of T_1 are of multiplicity 1, which is an absurd.

In the case b), suppose that $T_1 \cap T^{\rm sing} = \{p_i, p_{j_1}, \dots, p_{j_4}\}$ for some $j_k \neq i$. Then p_i and at most one p_{j_1} can have multiplicity ≥ 2, so that T is not an arithmetic proportional elliptic configuration.

In the case c), let $T_1 \cap T^{\rm sing} = \{p_i, p_{j_1}, p_{j_2}, p_{j_3}\}$. At most three of these points, say p_i, p_{j_s} and p_{j_t} belong to $T_2 \cap T^{\rm sing}$, so that there remains at least one point of $T^{\rm sing}$ of multiplicity 1.

In the case d), let $T_1 \cap T^{\rm sing} = \{p_i, p_{j_1}, p_{j_2}, p_{j_3}\}$. Then $T_2 \cap T^{\rm sing} = \{p_i, p_{j_1}\}$ and $T_3 \cap T^{\rm sing} = \{p_i, p_{j_2}\}$, up to a permutation of $p_{j_1}, p_{j_2}, p_{j_3}$. In either case there remains a point p_{j_3} of multiplicity 1 with respect to T.

In the case e), let us put $T_3 \cap T^{\rm sing} = \{p_i, p_j\}$. Then up to a transposition of T_1 and T_2, one has $T_1 \cap T^{\rm sing} = \{p_i, p_j, p_k\}$ with $p_i, p_k \in T_2 \cap T^{\rm sing}$. Then the fourth singular point of T cannot be of multiplicity > 1.

In the case f), the elliptic configuration $T_1 + T_2 + T_3$ has to contain at most two different singular points except p_i. That requires $s \leq 3$, while we work under the assumption $s \geq 4$.

In the case g), one has $T_j \cap T^{\rm sing} = \{p_i, p_j\}$ for $1 \leq j \leq 5$, $p_j \neq p_i$, and $T_k \cap T^{\rm sing} = \{p_i\}$, $\forall\, 6 \leq k \leq 4s-5$. In order to have multiplicities at least 2, one can have at most 2 different singular points of T, except p_i. However, then $s \leq 3$, contrary to $s \geq 4$. □

That concludes the proof of Proposition 5.

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