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CONNECTIONS ASSOCIATED WITH LINEAR DISTRIBUTIONS

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Abstract. This paper applies differential geometry to multi-dimensional affine space. The three-component distributions of affine space are discussed. Some connections of three-component distributions, which allow to generalize theory of regular and vanishing hyper-zones, zones, hyper-zone distributions, surfaces of full and non-full rank, and tangent equipped surfaces in multidimensional affine spaces are constructed.

1. Introduction

It is proved here that the first kind normal field for the basic distribution determines affine connection on the V-distributions. Another affine connection is defined by an inner invariant way in the second differential neighborhood of forming element of the three-component distribution. In special case studied connections are analog connections for the huperzone distributions of the m-dimensional linear elements. The components of the torsion and curvature tensor of the affine connections are obtained. The results of research can be applied to general theory of distributions in multidimensional spaces and to the theory of connections, which are associated with the multi-component distributions.

The method of my research is based on the differential-geometrical method developed by Laptev [4, 5].

2. Definition of the Three-Component Distribution

Let us consider $(n+1)$-dimensional affine space A_{n+1}, which is taken to a moving frame $R = \{A, \vec{e}_I\}$. Differential equations of the infinitesimal transference of frame R look as follows

$$\mathrm{d}A = \omega^I \vec{e}_I, \qquad \mathrm{d}\vec{e}_I = \omega_I^K \vec{e}_K$$

where ω_I^K, ω^I are invariant forms of an affine group, which satisfy the equations of the structure

$$\mathrm{d}\omega^I = \omega^K \wedge \omega_K^I, \qquad \mathrm{d}\omega_I^K = \omega_I^J \wedge \omega_J^K.$$

Structural forms of current point $X = A + x^I \vec{e}_I$ of space A_{n+1} look as follows

$$\Delta X^I \stackrel{\mathrm{def}}{=} \mathrm{d}x^I + x^K \omega_K^I + \omega^I.$$

Combination of current point X and point of frame A leads to the following equations

$$\Delta X^I = \omega^I.$$

Immobility condition of the point A is written down: $\omega^J = 0$. Frame chosen by this way is called the frame $\widetilde{R}$.

Let the r-dimensional plane Π_r, the m-dimensional plane Π_m, and hyper-plane Π_n in A_{n+1} be given by the following way

$$\Pi_r = \left[A, \vec{L}_p\right], \qquad \Pi_m = \left[A, \vec{M}_a\right], \qquad \Pi_n = \left[A, \vec{T}_\sigma\right]$$

where

$$\vec{L}_p = \vec{e}_p + \Lambda_p^{\hat{u}} \vec{e}_{\hat{u}}, \qquad \vec{M}_a = \vec{e}_a + M_a^{\hat{\alpha}} \vec{e}_{\hat{\alpha}}, \qquad \vec{T}_\sigma = \vec{e}_\sigma + H_\sigma^{n+1} \vec{e}_{n+1}.$$

The $(n+1)$-dimensional manifolds, which are determined by differential equations

$$\Delta \Lambda_p^{\hat{u}} = \Lambda_{pK}^{\hat{u}} \omega^K, \qquad \Delta M_a^{\hat{\alpha}} = M_{aK}^{\hat{\alpha}} \omega^K, \qquad \Delta H_\sigma^{n+1} = H_{\sigma K}^{n+1} \omega^K \tag{1}$$

are called distributions of the first kind accordingly of r-dimensional linear elements (Λ-distribution), m-dimensional linear elements (M-distribution) and hyper-planes (H-distribution).

Let us consider that manifolds (1) are distributions of tangent elements: center A belongs to planes Π_r, Π_m, Π_n.

We demand that in some area of space A_{n+1} for any center A the following condition takes place: $A \in \Pi_r \subset \Pi_m \subset \Pi_n$.

Definition 1. *The triple of distributions of affine space A_{n+1} consisting of basic distribution of the first kind r-dimensional linear elements $\Pi_r = \Lambda$ (Λ-distribution), equipping distribution of the first kind m-dimensional linear elements $\Pi_m = M$ (M-distribution) and equipping distribution of the first kind of hyper-plane elements $\Pi_n = H$ (H-distribution) with relation of the incidence of their corresponding elements in the common center A of the following view: $A \in \Pi_r \subset \Pi_m \subset \Pi_n$ are called $H(M(\Lambda))$-distribution* [1, 8].

Let us make the following canonization of the frame $\widetilde{R}$: we place vectors $\vec{e}_p$ in the plane Π_r, vectors $\vec{e}_i$ – in the plane Π_m, and vectors $\vec{e}_\sigma$ – in the plane Π_n. This frame will be called frame of the zero order R^0. This definition leads to the following equations

$$\Lambda_p^{\hat{u}} = 0, \qquad M_a^{\hat{\alpha}} = 0, \qquad H_\sigma^{n+1} = 0.$$

Within the frame R^0, $H(M(\Lambda))$-distribution is defined by the differential equations

$$\omega_p^{\hat{u}} = \Lambda_{pK}^{\hat{u}}\omega^K, \qquad \omega_i^{\hat{\alpha}} = M_{iK}^{\hat{\alpha}}\omega^K, \qquad \omega_\alpha^{n+1} = H_{\alpha K}^{n+1}\omega^K.$$

A partial canonization of zero-order frame R^0 is possible when

$$M_{iq}^{n+1} = 0, \qquad H_{\alpha q}^{n+1} = 0.$$

We will call it the frame of the first order R^1. In the chosen frame R^1, manifold $H(M(\Lambda))$ is determined by the system of differential equations

$$\begin{aligned} \omega_p^{\hat{u}} &= \Lambda_{pK}^{\hat{u}}\omega^K, & \omega_i^{n+1} &= M_{i\hat{u}}^{n+1}\omega^{\hat{u}} \\ \omega_i^{\alpha} &= M_{iK}^{\alpha}\omega^K, & \omega_\alpha^{n+1} &= H_{\alpha\hat{u}}^{n+1}\omega^{\hat{u}} \\ \omega_u^p &= A_{uK}^p\omega^K. \end{aligned} \tag{2}$$

3. Tensor of Ingolonomicity of Basic Distribution

It is easy to show that geometry of three-component distributions can be used for study of geometry of regular and vanishing hyper-zones, zones, hyper-zone distributions, surfaces of full and non full rank, tangent equipped surfaces in affine spaces.

For example, let us suppose that $H(M(\Lambda))$-distribution is holonomic, i.e. the basic distribution is holonomic. The system of differential equations

$$\omega^{\hat{u}} = \Lambda_p^{\hat{u}}\omega^p$$

which is associated with the basic distribution is completely integrable if and only if the tensor of the first order

$$r_{pq}^{\hat{u}} \overset{def}{=} \frac{1}{2}\left(\Lambda_{pq}^{\hat{u}} - \Lambda_{qp}^{\hat{u}}\right)$$

where

$$\nabla r_{pq}^{\hat{u}} = r_{pqK}^{\hat{u}}\omega^K$$

turns into zero.

Tensor $\left\{r_{pq}^{\hat{u}}\right\}$ will be called the tensor of inholonomicity of $H(M(\Lambda))$-distribution.

Basic Λ-distribution determines $(n-r+1)$-parametric assemblage of r-dimensional surfaces V_r.

In case of displacement of center A along fixed surface V_r, differential equations, which determine $H(M(\Lambda))$-distribution relatively to the frame $\widetilde{R}$,

$$\omega^{\hat{u}} = \Lambda_q^{\hat{u}}\omega^q, \qquad \Delta\Lambda_p^{\hat{u}} = \left(\Lambda_{pq}^{\hat{u}} + \Lambda_{p\hat{v}}^{\hat{u}}\Lambda_q^{\hat{v}}\right)\omega^q, \quad \Delta M_i^{\hat{\alpha}} = \left(M_{iq}^{\hat{\alpha}} + M_{i\hat{v}}^{\hat{\alpha}}\Lambda_q^{\hat{v}}\right)\omega^q$$

$$\Delta H_\alpha^{n+1} = \left(H_{\alpha q}^{n+1} + H_{\alpha\hat{v}}^{n+1}\Lambda_q^{\hat{v}}\right)\omega^q$$

are differential equations of r-dimensional zone $V_{r(m)}$ of the order m equipped by field of hyper-planes H. Geometrical object $\{H_{\tau}^{n+1}\}$ (object H) is the fundamental equipping object of this zone $V_{r(m)}$.

Let us note that, relatively to the frame R^0, differential equations of the manifold $H(M(\Lambda))$ have the more simple form

$$\omega^{\hat{u}} = 0, \qquad \omega_p^{\hat{u}} = \Lambda_{pq}^{\hat{u}} \omega^q, \qquad \Lambda_{pq}^{\hat{u}} = \Lambda_{qp}^{\hat{u}} \tag{3}$$

$$\omega_i^{\hat{\alpha}} = M_{iq}^{\hat{\alpha}} \omega^q \tag{4}$$

$$\omega_\alpha^{n+1} = H_{\alpha q}^{n+1} \omega^q \tag{5}$$

where equations (3) and (4) are analogous with equations of zone $V_{r(m)}$, which are discussed in [7] and [6]. Equations (5) characterize equipment of such zone $V_{r(m)}$ by field of hyper-planes H.

On the other hand, equations (3)) and (5) determine hyper-zone H_r in the frame R^0, and equations (4) characterize equipment of hyper-zone H_r by the field of planes M.

Thus, the theory of three-component distribution is generalization of theories of regular hyper-zone H_r and r-dimensional zone $V_{r(m)}$ of the order m of affine space.

4. Tensors of the Ingolonomicity of the Equipping Distributions

Let us consider the system of differential equations

$$\omega_0^{\hat{\alpha}} = M_a^{\hat{\alpha}} \omega^a \tag{6}$$

which is associated with the equipping M-distribution. This system is completely integrable if and only if the tensor of inholonomicity $\left\{r_{ab}^{\hat{\alpha}}\right\}$ of equipping M-distribution

$$r_{ab}^{\hat{\alpha}} = \frac{1}{2}\left(M_{ab}^{\hat{\alpha}} - M_{ba}^{\hat{\alpha}}\right), \qquad \nabla r_{ab}^{\hat{\alpha}} = r_{abk}^{\hat{\alpha}} \omega^k$$

is equal to zero. In this case, system (6) determines $(n-m+1)$-parametric assemblage of the m-dimensional surfaces V_m. In case of displacement of center A along fixed surface V_m, equations that determine $H(M(\Lambda))$-distribution define tangent r-equipped surface $V_{m(r)}$, which is equipped by field of tangent hyper-planes H. Really, from system, which consists of differential equations (6) and equations, which determine $H(M(\Lambda))$-distribution, we can pick out the subsystem

$$\omega_0^{\hat{\alpha}} = M_b^{\hat{\alpha}} \omega_0^b$$

where

$$\Delta M_a^{\hat{\alpha}} = M_{ab}^{\hat{\alpha}} \omega_0^b, \qquad \Delta \Lambda_p^i = \Lambda_{pb}^i \omega_0^b, \qquad M_{[ab]}^{\hat{\alpha}} = 0.$$

This subsystem determines the tangent r-equipped surface $V_{m(r)}$, which is discussed in the paper by Dombrovskyj [3].

On the other hand, in this case $H(M(\Lambda))$-distribution can be interpreted like hyper-zone H_m, which is equipped by field of tangent r-dimensional planes Λ.

Hence, geometry of $H(M(\Lambda))$-distribution of affine space, really, is richer than geometry of tangent r-equipped surfaces and geometry of hyper-zones H_m of affine space, because it consists of constructions, which do not have sense for the latter ones.

Also, geometry of $H(M(\Lambda))$-distribution can be used for study of vanishing hyper-zones and tangent vanishing surfaces.

The system of differential equations

$$\omega_0^{n+1} = H_\tau^{n+1}\omega_0^\tau \tag{7}$$

which is associated with equipping distribution of hyper-planes H (H-distribution) is completely integrable if and only if the tensor of the first order

$$r_{\tau\sigma}^{n+1} = \frac{1}{2}\left(\overline{H}_{\tau\sigma}^{n+1} - \overline{H}_{\tau\sigma}^{n+1}\right), \qquad \nabla r_{\tau\sigma}^{n+1} = r_{\sigma\tau\kappa}^{n+1}\omega^\kappa$$

turns into zero.

On condition that tensor of inholonomicity $\{r_{\tau\sigma}^{n+1}\}$ of equipping H-distribution is equal to zero, the system (7) determines one-parametric assemblage of hyper-surfaces V_n. In case of displacement of center A along fixed hyper-surface, equations, which determine $H(M(\Lambda))$-distribution, represent equations of the hyper-surface, which is equipped by fields of planes Λ and M.

Hence, theory of three-component distribution is also a generalization of theory of hyper-surfaces of affine space.

5. Connection Induced by the Field of the First-Kind Normal

Let us consider basic distribution equipped by the first-kind normals. The first-kind normal is defined by the objects which are determined by the differential equations

$$\nabla\nu^p - \nu^p\omega_{n+1}^{n+1} + \omega_{n+1}^p = \nu_{n+1K}^p\omega^K.$$

Let us make the following canonization of the frame of first-order, where the values $\nu^p = 0$. In the chosen frame we get the conditions

$$\omega_{n+1}^p = v_{n+1K}^p\omega^K. \tag{8}$$

Geometrical sense of this canonization consists of the placement of the vector $\vec{e}_{n+1}$ in the invariant plane – in the first-kind normal of Λ-distribution.

According to conditions (8), it is easy to obtain the system

$$D\omega^K = \omega^J \wedge \omega_J^K, \qquad D\omega^p = \omega^q \wedge \omega_q^p + R^p_{\hat{u}K}\omega^{\hat{u}} \wedge \omega^K$$
$$D\omega_q^p = \omega_q^r \wedge \omega_r^p + \frac{1}{2}R^p_{qKL}\omega^K \wedge \omega^L \tag{9}$$

where the values $R^p_{\hat{u}K}$, R^p_{qKL} are defined by the relations

$$R^p_{\hat{u}K} = \left\{A^p_{[uv]}, A^p_{u\hat{p}}, \nu^p_{n+1p}\right\}$$
$$R^p_{qKL} = 2\left(\Lambda^i_{q[K}A^p_{|i|L]} + \Lambda^\alpha_{q[K}A^p_{|\alpha|L]} + \Lambda^i_{q[K}v^p_{|n+1|L]}\right). \tag{10}$$

Consequently, the system of the forms $\left\{\omega^p, \omega_q^p\right\}$ defines the affine connection Γ on the Λ-distribution, which is induced by the field of the first-kind normals for the Λ-distribution. It is possible to say that this connection is induced by the field of the equipping vector $\vec{\nu}$.

6. Affine Connection Determined by Inner Way

Let us consider the affine connection determined by the following transformations

$$\widetilde{\omega}_q^p = \omega_q^p + \gamma^p_{qK}\omega^K$$

where

$$D\omega^K = \omega^J \wedge \omega_J^K, \qquad D\omega^p = \omega^q \wedge \widetilde{\omega}_q^p + \frac{1}{2}\widetilde{R}^p_{KL}\omega^K \wedge \omega^L \tag{11}$$
$$D\widetilde{\omega}_q^p = \widetilde{\omega}_q^r \wedge \widetilde{\omega}_r^p + \Delta\gamma^p_{qK} \wedge \omega^K$$

$$\widetilde{R}^p_{KL} = 2\left(\delta^u_{[K}A^p_{|u|L]} + \delta^{n+1}_{[K}v^p_{|n+1|L]} + \delta^q_{[K}\gamma^p_{|q|L]}\right) \tag{12}$$
$$\Delta\gamma^p_{qK} = \nabla\gamma^p_{qK} + \gamma^r_{qK}\gamma^p_{rL}\omega^L + \Lambda^u_{qL}A^p_{uK}\omega^L + \Lambda_{qL}\nu^p_{n+1K}\omega^L.$$

According to differential equations (11) the forms ω^p, $\widetilde{\omega}_q^p$ define the affine connection if and only if the relations

$$\Delta\gamma^p_{qK} = \gamma^p_{qKL}\omega^L \tag{13}$$

hold.

The tensor $\widetilde{R}^p_{KL}$ in equations (11) is the torsion tensor of this affine connection space, and

$$\widetilde{R}^p_{qKL} = 2\gamma^p_{q[KL]}$$

is the curvature tensor.

According to relations (12) and (13) we get the system

$$\nabla\gamma^p_{qs} = \overline{\gamma}^p_{qsK}\omega^K, \qquad \nabla\gamma^p_{qu} = \overline{\gamma}^p_{quK}\omega^K$$
$$\nabla\gamma^p_{qn+1} - \gamma^p_{qn+1}\omega^{n+1}_{n+1} - \gamma^p_{qu}\omega^u_{n+1} = \overline{\gamma}^p_{qn+1K}\omega^K. \tag{14}$$

It is easy to show that the objects constructed below satisfy equations (14). Correspondingly,

$$H^p_{qr} = \Lambda^{ps}\left(\Lambda_{sqr} - \Lambda_{su}\Lambda^u_{qr} - \Lambda_{sn+1}\Lambda_{qr}\right)$$

$$H^p_{qu} = \Lambda^{ps}\left(\Lambda_{squ} - \Lambda_{sn+1}\Lambda_{qu} - \Lambda_{sv}\Lambda^v_{qu} + \Lambda^i_{sq}M_{iu} + \Lambda^\alpha_{sq}H_{\alpha u}\right)$$

$$H^p_{qn+1} = \Lambda^{ps}\left(\Lambda_{sqn+1} - \Lambda_{sn+1}\Lambda_{qn+1} - \Lambda_{sv}\Lambda^v_{qn+1} + \Lambda^i_{sq}M_{in+1} + \Lambda^\alpha_{sq}H_{\alpha n+1}\right)$$

where

$$\nabla_\delta H^p_{qr} = 0, \qquad \nabla_\delta H^p_{qu} = 0, \qquad \nabla_\delta H^p_{qn+1} - H^p_{qn+1}\pi^{n+1}_{n+1} - H^p_{qu}\pi^u_{n+1} = 0.$$

Consequently, in case of invariant normalization of the Λ-distribution, the forms

$$\omega^p \quad \text{and} \quad \tilde{\omega}^p_q = \omega^p_q + H^p_{qK}\omega^K$$

define the affine connection on the Λ-distribution by inner way.

7. Analog of the Affine Connection for the Hyperzone Distributions

Next, we proposed the following transformations

$$\tilde{\tilde{\omega}}^p_q = \omega^p_q + \tilde{\gamma}^p_{Jq}\omega^J$$

where

$$D\omega^K = \omega^J \wedge \omega^K_J, \qquad D\omega^p = \omega^q \wedge \tilde{\tilde{\omega}}^p_q + \frac{1}{2}\tilde{\tilde{R}}^p_{KL}\omega^K \wedge \omega^L$$

$$D\tilde{\tilde{\omega}}^p_q = \tilde{\tilde{\omega}}^r_q \wedge \tilde{\tilde{\omega}}^p_r + \Delta\tilde{\gamma}^p_{qK} \wedge \omega^K$$

$$\tilde{\tilde{R}}^p_{KL} = 2\left(\delta^u_{[K}A^p_{|u|L]} + \delta^{n+1}_{[K}\nu^p_{|n+1|L]} - \delta^q_{[K}\tilde{\gamma}^p_{L]q}\right)$$

$$\Delta\tilde{\gamma}^p_{qK} = \nabla\tilde{\gamma}^p_{Kq} + \tilde{\gamma}^r_{Kq}\tilde{\gamma}^p_{Lr}\omega^L + \Lambda^u_{qL}A^p_{uK}\omega^L + \Lambda_{qL}\nu^p_{n+1K}\omega^L.$$

The forms ω^p, $\tilde{\tilde{\omega}}^p_q$ define the affine connection if and only if the following relations

$$\Delta\tilde{\gamma}^p_{qK} = \tilde{\gamma}^p_{qKL}\omega^L$$

hold. These relations are equivalent to the following differential equations

$$\nabla\tilde{\gamma}^p_{sq} = \tilde{\tilde{\gamma}}^p_{sqK}\omega^K, \qquad \nabla\tilde{\gamma}^p_{uq} = \tilde{\tilde{\gamma}}^p_{uqK}\omega^K$$

$$\nabla\tilde{\gamma}^p_{n+1q} - \tilde{\gamma}^p_{n+1q}\omega^{n+1}_{n+1} - \tilde{\gamma}^p_{uq}\omega^u_{n+1} = \tilde{\tilde{\gamma}}^p_{n+1qK}\omega^K.$$

The affine connections for the hyperzone distributions in the projective space are discussed by Stoljarov [9]. In case when the relations

$$\omega^p, \qquad \tilde{\tilde{\omega}}^p_q = \omega^p_q + \tilde{\gamma}^p_{Jq}\omega^J \tag{15}$$

with

$$\tilde{\gamma}^p_{sq} = 0, \qquad \tilde{\gamma}^p_{uq} = -A^p_{uq}, \qquad \tilde{\gamma}^p_{n+1q} = -\nu^p_{n+1q} \tag{16}$$

hold, we have the connection which is an analog of the connection obtained by Stoljarov [9].

8. Ricci Tensor of the Affine Connection

The components of torsion tensor and curvature tensor for this connection have the forms

$$\begin{aligned}
\tilde{\tilde{R}}^p_{KL} &= 2\left(\delta^u_{[K}A^p_{|u|L]} + \delta^{n+1}_{[K}\nu^p_{|n+1|L]} + \delta^q_{[K}\delta^u_{L]}A^p_{uq} + \delta^q_{[K}\delta^{n+1}_{L]}\nu^p_{n+1q}\right)\\
\tilde{\tilde{R}}^p_{qKL} &= 2\left(\delta^v_{[K}\delta^u_{L]}A^r_{uq}A^p_{vr} - \delta^{n+1}_{[K}\delta^v_{L]}\nu^r_{n+1q}A^p_{vr}\right.\\
&\quad - \delta^u_{[K}\delta^{n+1}_{L]}A^r_{uq}\nu^p_{n+1r} + A^u_{q[K}A^p_{|u|L]} + \Lambda_{q[K}\nu^p_{|n+1|L]}\\
&\quad + \nu^p_{n+1q}\Lambda_{r[K}\delta^r_{L]} - \delta^i_{[K}\delta^{\hat{u}}_{L]}\nu^p_{n+1q}M_{i\hat{u}} - \delta^\alpha_{[K}\delta^{\hat{\gamma}}_{L]}\nu^p_{n+1q}H_{\alpha\hat{\gamma}}\\
&\quad \left. - A^p_{uq[K}\delta^u_{L]} - \Lambda^u_{r[L}\delta^r_{K]}A^p_{uq} - \Lambda^{\hat{u}}_{q[K}\delta^{n+1}_{L]}\nu^p_{n+1\hat{u}} + \nu^p_{n+1q[L}\delta^{n+1}_{K]}\right).
\end{aligned} \tag{17}$$

The components of curvature tensor $\tilde{\tilde{R}}^p_{qst}$ have more simple form

$$\tilde{\tilde{R}}^p_{qst} = 2\left(A^r_{[t|q|}A^p_{s]r} + \Lambda^u_{q[s}A^p_{|u|t]} + \Lambda_{q[s}\nu^p_{|n+1|t]} + \nu^p_{n+1q}\Lambda_{r[s}\delta^r_{t]} - \Lambda^u_{[st]}A^p_{uq}\right).$$

The values

$$\begin{aligned}
\tilde{\tilde{R}}_{qs} &= \tilde{\tilde{R}}^p_{qsp}\\
&= 2\left(A^r_{[p|q|}A^p_{s]r} + \Lambda^u_{q[s}A^p_{|u|p]} + \Lambda_{q[s}\nu^p_{|n+1|p]} + \nu^p_{n+1q}\Lambda_{r[s}\delta^r_{p]} - \Lambda^u_{[sp]}A^p_{uq}\right)
\end{aligned}$$

form the tensor called Ricci tensor of the affine connection for the normalized Λ-distribution.

9. Another Similar Affine Connection of the Invariant Normalized Λ-distribution

Let us come back to the case when we have the affine connection which is the analog of the connection received by Stoljarov [9].

Another similar affine connection is determined by the transformations

$$\overline{\omega}^p_q = \tilde{\tilde{\omega}}^p_q + \Gamma^p_{qK}\omega^K$$

under the condition that relations (16) hold.

It is easy to obtain the following differential equations

$$D\omega^K = \omega^J \wedge \omega^K_J, \qquad D\omega^p = \omega^q \wedge \overline{\omega}^p_q + \frac{1}{2}\overline{R}^p_{KL}\omega^K \wedge \omega^L$$

$$D\overline{\omega}^p_q = \overline{\omega}^r_q \wedge \overline{\omega}^p_r + \Delta\Gamma^p_{qK} \wedge \omega^K$$

where

$$\overline{R}^p_{KL} = \tilde{\tilde{R}}^p_{KL} - 2\Gamma^p_{[KL]}, \qquad \Gamma^p_{\hat{u}L} = 0$$

$$\Delta\Gamma^p_{qK} = \nabla\Gamma^p_{qK} + \left(\Gamma^p_{rK}A^r_{uq} - \Gamma^r_{qK}A^p_{ur}\right)\omega^u$$
$$+ \left(\Gamma^p_{rK}\nu^r_{n+1q} - \Gamma^r_{qK}\nu^p_{n+1r}\right)\omega^{n+1} + \left(\frac{1}{2}\tilde{\tilde{R}}^p_{qLK} - \Gamma^r_{qL}\Gamma^p_{rK}\right)\omega^L.$$

The forms ω^p, $\overline{\omega}^p_q$ define the affine connection if and only if the following relations

$$\begin{gathered}\nabla\Gamma^p_{qs} = \tilde{\Gamma}^p_{qsK}\omega^K, \qquad \nabla\Gamma^p_{qu} = \tilde{\Gamma}^p_{quK}\omega^K \\ \nabla\Gamma^p_{qn+1} - \Gamma^p_{qu}\omega^u_{n+1} - \Gamma^p_{qn+1}\omega^{n+1}_{n+1} = \tilde{\Gamma}^p_{qn+1K}\omega^K\end{gathered} \tag{18}$$

hold.

It is easy to show that the objects (scopes)

$$\Gamma^p_{qs} = H^p_{qs}, \qquad \Gamma^p_{qu} = H^p_{qu}, \qquad \Gamma^p_{qn+1} = H^p_{qn+1}$$

satisfy the differential equations (18) correspondingly, and in this case the forms

$$\omega^p, \qquad \overline{\omega}^p_q = \tilde{\tilde{\omega}}^p_q + H^p_{qK}\omega^K$$

define the affine connection on the Λ-distribution by inner way.

The torsion tensor of this affine connection space is the tensor $\overline{R}^p_{KL}$, and the curvature tensor is the tensor

$$\overline{R}^p_{qKL} = 2\Gamma^p_{q[KL]}.$$

10. Projective Connection Associated with Basic Distribution

It is considered the space of the projective connection $P_{n+1,r}$, which has been determined by the following way: affine space A_{n+1} is $(n+1)$-dimensional base of this space, and the r-dimensional planes Π_r of the basic Λ-distribution are layers of this space [2].

The projective connection Γ of space $P_{n+1,r}$ is determined by the system of forms

$$\theta^p = \omega^p - \Gamma^p_{oK}\omega^K, \qquad \theta^p_q = \omega^p_q - \Gamma^p_{qK}\omega^K.$$

The transformed forms θ^p, θ^p_q satisfy the following structural equations

$$D\theta^p = \theta^q \wedge \theta^p_q + \omega^K \wedge \Delta\Gamma^p_{oK}, \qquad D\theta^p_q = \theta^r_q \wedge \theta^p_r + \omega^K \wedge \Delta\Gamma^p_{qK}$$

where

$$\Delta\Gamma^p_{oK} = \nabla\Gamma^p_{oK} + \delta^{n+1}_K\omega^p_{n+1} - \Gamma^p_{qK}\omega^q - \Gamma^q_{oK}\Gamma^p_{qJ}\omega^J - A^p_{uK}\omega^u$$
$$\Delta\Gamma^p_{qK} = \nabla\Gamma^p_{qK} + \Lambda_{qK}\omega^p_{n+1} + \left(\Lambda^u_{qK}A^p_{uJ} - \Gamma^r_{qK}\Gamma^p_{rJ}\right)\omega^J.$$

The forms $\Delta\Gamma^p_{oK}$, $\Delta\Gamma^p_{qK}$, ω^K on $H(M(\Lambda))$-distribution constitute a completely integrable system and determine the field of the geometrical object $\{\Gamma^p_{oK}, \Gamma^p_{qK}\}$ over the initial base (ω^K_o). This geometrical object will be called the object of the projective connection of the space $P_{n+1,r}$.

To determine the projective connection in the layers of the space $P_{n+1,r}$ by the forms θ^p, θ^p_q, it is necessary and sufficient to determine the field of the object of the connection Γ.

Structural equations for layer forms θ^p, θ^p_q look as follows

$$D\theta^p = \theta^q \wedge \theta^p_q + \frac{1}{2}R^p_{oKL}\omega^K \wedge \omega^L, \qquad D\theta^p_q = \theta^r_q \wedge \theta^p_r + \frac{1}{2}R^p_{qKL}\omega^K \wedge \omega^L$$

where $\{R^p_{oKL}, R^p_{qKL}\}$ is the torsion-curvature tensor of the projective connection Γ of this space $P_{n+1,r}$

$$R^p_{oKL} = 2\Gamma^p_{o[KL]}, \qquad R^p_{qKL} = 2\Gamma^p_{q[KL]}.$$

11. The Scope of Object of the Projective Connection Γ by the Fundamental Objects of the $H(M(\Lambda))$-distribution

The projective connection Γ is constructed, which is defined by the three-component distribution $H(M(\Lambda))$ by inner way, i.e., the scope of the object of the projective connection Γ is constructed by the fundamental objects of the $H(M(\Lambda))$-distribution.

It is easy to show that the components of the object of the projective connection $\{\Gamma^p_{oK}, \Gamma^p_{qK}\}$ are realized by the following way

$$\Gamma^p_{oq} = 0, \qquad \Gamma^p_{ou} = -b^{pq}_u \ell_q, \qquad \Gamma^p_{on+1} = a^u b^{pq}_u \ell_q + \nu^p$$
$$\Gamma^p_{qK} = \Lambda_{qK}\left(a^u b^{pt}_u \ell_t + \nu^p\right) - \Lambda^u_{qK} b^{pr}_u \ell_r.$$

Thus, it is proved that the forms θ^p, θ^p_q:

$$\theta^p = \omega^p + b^{pq}_u \ell_q \omega^u - (a^u b^{pq}_u \ell_q + \nu^p)\,\omega^{n+1}$$
$$\theta^p_q = \omega^p_q - \left(\Lambda_{qK} a^u b^{pq}_u \ell_q + \Lambda_{qK}\nu^p - \Lambda^u_{qK} b^{pr}_u \ell_r\right)\omega^K$$

which define the layer of the space of the projective connection $P_{n+1,r}$, are determined on the $H(M(\Lambda))$-distribution by inner way and associated with the basic Λ-distribution where the objects $\{h^p\}$ and $\{\lambda^p\}$, which have been constructed before, can be taken as the object $\{\nu^p\}$.

12. The Reflection Determining Projective Connection on the Distribution Received by Method of Projection

It is proved that the constructed projective connection Γ belongs to the type of the projective connections which are defined by the way of projection.

Really, the plane

$$\left[\vec{A}(u, \mathrm{d}u), \vec{e}_p(u, \mathrm{d}u)\right] = \Pi_r(u, \mathrm{d}u)$$

is the image of the current plane

$$\Pi_r(u + \mathrm{d}u) \equiv \left[\vec{A}(u + \mathrm{d}u), \vec{e}_p(u + \mathrm{d}u)\right]$$

of the basic Λ-distribution in case of mapping

$$\begin{aligned} \vec{A}(u + \mathrm{d}u) &\to \vec{A}(u, \mathrm{d}u) = \vec{A}(u) + \omega^J \vec{e}_J(u) + \text{higher order forms} \\ \vec{e}_p(u + \mathrm{d}u) &\to \vec{e}_p(u, \mathrm{d}u) = \vec{e}_p(u) + \omega_p^J \vec{e}_J(u) + \text{higher order forms} \end{aligned} \tag{19}$$

which determines the connection.

Let us project the image $\Pi(u, \mathrm{d}u)$ of neighboring plane $\Pi(u + \mathrm{d}u)$ onto the current plane of the Λ-distribution $\Pi(u)$, taking the equipping plane $K_{n-r+1}(A)$ as the projecting center.

The invariant plane $K_{n-r+1}(A)$ is defined by vectors

$$\vec{K}_{n+1} = \vec{e}_{n+1} + (a^u b_u^{pq} \ell_q + \nu^p)\vec{e}_p, \qquad \vec{K}_u = \vec{e}_u - b_u^{pq} \ell_q \vec{e}_p.$$

This projection determines the reflection

$$\begin{aligned} \vec{A}(u, \mathrm{d}u) \to \vec{\hat{A}}(u, \mathrm{d}u) &= \vec{A}(u, \mathrm{d}u) + \ell^{n+1} \vec{K}_{n+1} + \ell^u \vec{K}_u \\ &= \vec{A}(u) + \omega^J \vec{e}_J + \ell^{n+1}(\vec{e}_{n+1} + H_{n+1}^p \vec{e}_p) + \ell^u(\vec{e}_u + H_u^p \vec{e}_p) \\ &= \vec{A}(u) + (\omega^p + \ell^{n+1} H_{n+1}^p + \ell^u H_u^p)\vec{e}_p \\ &\quad + (\omega^u + \ell^u)\vec{e}_u + (\omega^{n+1} + \ell^{n+1})\vec{e}_{n+1} \\ \vec{e}_p(u, \mathrm{d}u) \to \vec{\hat{e}}_p(u, \mathrm{d}u) &= \vec{e}_p(u, \mathrm{d}u) + \ell_p^{n+1} \vec{K}_{n+1} + \ell_p^v \vec{K}_v \\ &= \vec{e}_p + \omega_p^J \vec{e}_J + \ell_p^{n+1}(\vec{e}_{n+1} + H_{n+1}^q \vec{e}_q) + \ell_p^v(\vec{e}_v + H_v^q \vec{e}_q \\ &= \vec{e}_p(u) + (\omega_p^q + \ell_p^{n+1} H_{n+1}^q + \ell_p^v H_v^q)\vec{e}_q \\ &\quad + (\omega_p^u + \ell_p^u)\vec{e}_u + (\omega_p^{n+1} + \ell_p^{n+1})\vec{e}_{n+1}. \end{aligned} \tag{20}$$

Thus, superposition of reflections (19) and (20) gives the reflection, which determine projective connection on the $H(M(\Lambda))$-distribution received by method of projection

$$\begin{aligned} \vec{A}(u, \mathrm{d}u) &\to \vec{\hat{A}}(u, \mathrm{d}u) = \vec{A}(u) + \theta^p \vec{e}_p \\ \vec{e}_p(u, \mathrm{d}u) &\to \vec{\hat{e}}_p(u, \mathrm{d}u) = \vec{e}_p(u) + \theta_p^q \vec{e}_q. \end{aligned}$$

Here the forms θ^p, θ^p_q

$$\theta^p = \omega^p - H^p_{n+1}\omega^{n+1} - H^p_u\omega^u, \qquad \theta^p_q = \omega^p_q - H^p_{n+1}\omega^{n+1}_q - H^p_v\omega^v_q$$

determine the main part of the received reflection and are the forms of projective connection Γ on the $H(M(\Lambda))$-distribution, which was determined by the projecting method.

The components of the torsion-curvature tensor of the space $P_{n+1,r}$ in the structural equations look as follows

$$\begin{aligned}
R^p_{qKJ} = {} & 2(\Lambda_{q[KJ]}(a^u b^{pq}_u \ell_q + \nu^p) + \Lambda_{q[K}a^u_{J]}b^{pq}_u\ell_q + \Lambda_{q[K}b^{pq}_{|u|J]}a^u\ell_q \\
& + \Lambda_{q[K}\ell_{|q|J]}a^u b^{pq}_u + \Lambda_{q[K}\nu^p_{J]} - \Lambda^u_{q[KJ]}b^{pr}_u\ell_r - \Lambda^u_{q[K}b^{pr}_{|u|J]}\ell_r \\
& - \Lambda^u_{q[K}\ell_{|r|J]}b^{pr}_u - \Lambda_{q[K}\Lambda_{|r|J]}a^u b^{rt}_u\ell_t a^v b^{ps}_v\ell_s - \Lambda_{q[K}\Lambda_{|r|J]}\nu^r a^v b^{pt}_v\ell_t \\
& + \Lambda^u_{q[K}\Lambda_{|r|J]}b^{rt}_u\ell_t a^v b^{ps}_v\ell_s - \Lambda_{q[K}\Lambda_{|r|J]}a^u b^{rt}_u\ell_t\nu^p - \Lambda_{q[K}\Lambda_{|r|J]}\nu^r\nu^p \\
& + \Lambda^u_{q[K}\Lambda_{|r|J]}b^{rt}_u\ell_t\nu^p + \Lambda_{q[K}\Lambda^v_{|r|J]}a^u b^{rt}_u\ell_t b^{ps}_v\ell_s \\
& + \Lambda_{q[K}\Lambda^v_{|r|J]}\nu^r b^{ps}_v\ell_s - \Lambda^u_{q[K}\Lambda^v_{|r|J]}b^{rt}_u\ell_t b^{ps}_v\ell_s) \\
R^p_{oKL} = {} & 2\Gamma^p_{o[KL]}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^p_{ouJ} = {} & - b^{pq}_{uJ}\ell_q - b^{pq}_u\ell_{qJ} - \delta^q_J\Lambda_{qu}a^v b^{pt}_v\ell_t - \delta^q_J\Lambda_{qu}\nu^p + \delta^q_J\Lambda^v_{qu}b^{pr}_v\ell_r \\
& + \Lambda_{qJ}b^{qr}_u a^v\ell_r b^{pt}_v\ell_t + \nu^p\ell_r b^{qr}_u\Lambda_{qJ} - b^{qr}_u\Lambda^v_{qJ}\ell_r b^{pt}_v\ell_t - A^p_{vu}\delta^v_J \\
\Gamma^p_{oqL} = {} & - \delta^r_L\Lambda_{rq}a^u b^{pt}_u\ell_t - \delta^r_L\Lambda_{rq}\nu^p + \delta^r_L\Lambda^u_{rq}b^{pt}_u\ell_t - A^p_{uq}\delta^u_L \\
\Gamma^p_{on+1J} = {} & a^u_J b^{pq}_u\ell_q + \nu^p_J + a^u b^{pq}_{uJ}\ell_q + a^u b^{pq}_u\ell_{qJ} + a^u b^{pq}_u\ell_{qJ} \\
& - \delta^q_J(\Lambda_{qn+1}a^u b^{pt}_u\ell_t + \nu^p\Lambda_{qn+1} - \Lambda^u_{qn+1}b^{pr}_u\ell_r) \\
& - a^u b^{qr}_u\ell_r\Lambda_{qJ}a^v b^{pt}_v\ell_t - \delta^u_J A^p_{un+1} - \nu^q\Lambda_{qJ}a^u b^{pt}_u\ell_t \\
& - a^u b^{qr}_u\ell_r\Lambda_{qJ}\nu^p - \nu^q\Lambda_{qJ}\nu^p + a^u b^{qr}_u\ell_r\Lambda^v_{qJ}b^{pt}_v\ell_t + \nu^q\Lambda^u_{qJ}b^{pt}_u\ell_t.
\end{aligned}$$

The torsion-curvature tensor determines the projective connection space.

Thus, it is proved that projective connection Γ for the three-component distribution is defined by inner way in the differential neighborhood of second order and belongs to the type of the projective connections which are defined by the way of the projection.

The results of research can be applied to the general theory of distributions in multidimensional spaces and to the theory of connections, which are associated with the multi-component distributions.

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