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FERMIONS AND SUPERSYMMETRY

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Abstract. Fermions and bosons are the fundamental particles of nature, and they are naturally described mathematically by using the methods of supersymmetry. These methods are illustrated here by consideration of a number of physical examples which arise in non-relativistic and relativistic quantum mechanics.

1. Introduction

Matter occurs in nature in two forms: particles of integral spin are *bosons*, particles of half-integral spin are *fermions*. Bosons and fermions obey different statistics: Bosons obey Bose-Einstein statistics, fermions obey Fermi-Dirac statistics. In quantum field theory this difference arises because the bosons obey fundamental equal-time *commutator relations*, whereas fermions obey *anticommutation relations*. At the macroscopic level these differences lead to spectacular effects: Fermi-Einstein statistics and the associated Pauli exclusion principle is responsible for the existence of the shell structure of atoms and nuclei. On the other hand, Bose-Einstein statistics lead to the phenomena of *Bose-Einstein condensation*, which was directly observed for the first time in 2001. The discoverers were awarded the Nobel Prize in Physics in 2001 [4]. The Nobel Prize in 2003 honoured the theorists who succeeded in explaining the phenomena of superconductivity and superfluidity, which are also due to the Bose-Einstein condensation of Cooper pairs [1].

In order to achieve a unified treatment of bosons and fermions in theoretical physics new methods which have their origins in supersymmetric field theories are necessary. While the supersymmetry predicted in field theory for elementary particles has not yet been observed, it has been observed in nuclear physics [11]. It has also become an important tool in quantum mechanics [3], in atomic, condensed matter and statistical physics [12], in the description of gauge theories [2], and in mathematics [10].

While for the description of bosonic physical systems in classical and quantum mechanics real and complex numbers are sufficient, for fermions we must use *Grassman* and *Clifford* variables, respectively. To pass from the classical to the quantum level we use the method of *deformation quantization*, which I have discussed in my lectures for the previous Varna Conference [7]. We shall see in the following that this method leads directly from the *Grassman algebra* which describes fermionic variables at the (pseudo-) classical level to the *Clifford algebra* necessary for the description of fermion variables at the quantum level.

In this presentation I will cover the following topics:

- Pseudoclassical mechanics
- Quantization
- The bosonic oscillator
- The fermionic oscillator
- The supersymmetric oscillator
- Supersymmetric quantum mechanics
- Non-relativistic spin and the Pauli equation
- Relativistic spin and the Dirac equation

2. Pseudoclassical Mechanics

Fermions in nature are directly observed only at the quantum level. Because of the Pauli principle they do not have a classical limit in the same sense that bosons do. Nevertheless, the description of fermion systems at the classical level is necessary in order to achieve a unified conception of matter. The dynamics of a system containing both bosonic and fermionic degrees of freedom is the domain of *pseudoclassical* mechanics.

In pseudoclassical mechanics we deal with systems involving bosonic degrees of freedom, which we describe in terms of ordinary complex variables $\{q^i\}$, and fermionic degrees of freedom, which we describe in terms of Grassman variables $\{\psi^\alpha\}$, and which satisfy anticommutation relations

$$\psi^\alpha \psi^\beta + \psi^\beta \psi^\alpha = 0. \tag{2.1}$$

Thus our system involves variables of different Grassman parity: $\epsilon(q^i) = 0$, $\epsilon(\psi^\alpha) = 1$.

The system is characterized by a Lagrange function $L(q^i, \dot{q}^i, \psi^\alpha, \dot{\psi}^\alpha)$. The index i runs over the range of bosonic degrees of freedom, the index α over the fermionic degrees of freedom. The dot indicates differentiation with respect to time. The conjugate momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad \pi_\alpha = \frac{\partial L}{\partial \dot{\psi}^\alpha}. \tag{2.2}$$

The corresponding Hamilton function is

$$H(q^i, p_i, \psi^\alpha, \pi_\alpha) = \dot{q}^i p_i + \dot{\psi}^\alpha \pi_\alpha - L. \tag{2.3}$$

The Hamilton equations describing the dynamics of the system are

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i}, & \dot{p}_i &= -\frac{\partial H}{\partial q^i}, \\ \dot{\psi}^\alpha &= -\frac{\partial H}{\partial \pi_\alpha}, & \dot{\pi}_\alpha &= -\frac{\partial H}{\partial \psi^\alpha}. \end{aligned} \tag{2.4}$$

These formulae are usefully rewritten in terms of the *super Poisson brackets*. The super Poisson bracket is a $\mathbb{Z}_2$-graded bilinear map with the following properties:

(1) $\{F, G\} = -(-1)^{\epsilon_F \epsilon_G}\{G, F\}$

(2) $\{F, GH\} = \{F, G\}H + (-1)^{\epsilon_F \epsilon_G} G\{F, H\}$

(3) $(-1)^{\epsilon_F \epsilon_H}\{\{F, G\}, H\} + (-1)^{\epsilon_G \epsilon_F}\{\{G, H\}, F\} + (-1)^{\epsilon_H \epsilon_G}\{\{H, F\}, G\}$
$= 0.$

Here F, G are functions on the generalized phase space. Property (1) says that the bracket is graded symmetric, property (2) is the graded Leibnitz rule, and property (3) the graded Jacobi identity.

In canonical coordinates the bracket may be expressed as

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} + (-1)^{\epsilon_F}\left(\frac{\partial F}{\partial \psi^\alpha}\frac{\partial G}{\partial \pi_\alpha} + \frac{\partial F}{\partial \pi_\alpha}\frac{\partial G}{\partial \psi^\alpha}\right). \tag{2.5}$$

The canonical anticommutation relations are then

$$\{q^i, p_j\} = \delta^i_j, \quad \{\psi^\alpha, \pi_\beta\} = -\delta^\alpha_\beta. \tag{2.6}$$

The Hamilton equations become

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\} \tag{2.7}$$

which have the same form as the familiar Hamilton equations for bosonic variables expressed in terms of the ordinary Poisson brackets.

3. Quantization

We shall here use the method of deformation quantization. Since we have discussed this method in last year's meeting, see Reference [7], the description here will be quite brief. Although in contrast to last year we are describing here a more general system involving both bosonic and fermionic degrees of freedom, the generalization is formally quite straightforward. In this method the pointwise multiplication of functions on phase space is replaced by the *star product* of such functions:

$$FG \to F * G, \tag{3.1}$$

where

$$F * G = F \exp\left(\frac{\mathrm{i}\hbar}{2}\alpha^{AB}\overleftarrow{\partial}_{z_A}\vec{\partial}_{z_B}\right) G. \tag{3.2}$$

Here the indices A, B range over the dimension of the generalized phase space, $z_A = \{q^i, p_i, \psi^\alpha, \pi_\alpha\}$ and the α_{AB} are the coefficients of the generalized Poisson structure,

$$\{F, G\} = \alpha^{AB} F\left(\overleftarrow{\partial}_{z_A}\vec{\partial}_{z_B}\right) G. \tag{3.3}$$

The time-development is controlled by the following equation, which corresponds to the time-dependent Schrödinger equation in the ordinary formalism:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Exp}(Ht) = H * \mathrm{Exp}(Ht). \tag{3.4}$$

For time-independent Hamilton functions the solution of this equation is given by the *time-evolution function*

$$\mathrm{Exp}(Ht) = \sum_n \frac{1}{n!}\left(\frac{-\mathrm{i}t}{\hbar}\right)^n H^{n*} \tag{3.5}$$

where $H^{n*} = H * H * \cdots * H$. The time-evolution function admits a *Fourier-Dirichlet* expansion of the form

$$\mathrm{Exp}(Ht) = \sum_E \pi_E \, e^{-\mathrm{i}Et/\hbar}. \tag{3.6}$$

Here the π_E are *projectors* (also called in the literature *Wigner functions*). They satisfy the idempotence and completeness relations

$$\pi_E * \pi_{E'} = \delta_{EE'}\pi_E, \qquad \sum_E \pi_E = 1. \tag{3.7}$$

Corresponding to the time-independent Schrödinger equation we here have

$$H * \pi_E = E\,\pi_E. \tag{3.8}$$

The spectral decomposition of the Hamilton function is

$$H = \sum_E E\,\pi_E. \tag{3.9}$$

3.1. Constraints

For the dynamical systems we are treating here it will be necessary to incorporate some constraints according to Dirac's method [5], which is generalized in a straightforward way to the pseudoclassical context [6]. The constraints will be denoted by

$$\chi_i(q^i, p_i, \psi^\alpha, \pi_\alpha) \approx 0 \tag{3.10}$$

where the index i runs over the number of constraints. *First class* constraints have vanishing brackets among themselves:

$$\{\chi_i, \chi_j\} = 0 \tag{3.11}$$

whereas *second class* constraints have non-vanishing brackets:

$$\{\chi_i, \chi_j\} = C_{ij} \tag{3.12}$$

where the matrix C_{ij} is non-singular. Instead of the ordinary graded brackets we have to use in systems involving second class constraints the *Dirac brackets*, defined as

$$\{F, G\}_D = \{F, G\} - \{F, \chi_i\}C^{ij}\{\chi_j, G\} \tag{3.13}$$

where C^{ij} is the inverse matrix to C_{ij}. After replacing all relevant graded brackets by Dirac brackets the second class constraints are to be imposed as *strong contraints*:

$$\chi_i \equiv 0. \tag{3.14}$$

4. The Bosonic Oscillator

In this section we illustrate the deformation quantization method for the bosonic oscillator. We use the *Moyal* star product:

$$F * G = F \exp\left[\frac{\hbar}{2}(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)\right] G. \tag{4.1}$$

Here a, $\bar{a}$ are the holonomic variables, given in terms of the coordinate and momentum variables by

$$a = \sqrt{\frac{\omega}{2}}\left(q + \mathrm{i}\frac{p}{\omega}\right), \quad \bar{a} = \sqrt{\frac{\omega}{2}}\left(q - \mathrm{i}\frac{p}{\omega}\right). \tag{4.2}$$

They satisfy

$$a * \bar{a} = a\bar{a} + \hbar/2, \quad \bar{a} * a = a\bar{a} - \hbar/2. \tag{4.3}$$

The star commutator is defined as

$$[F, G]_* = F * G - G * F. \tag{4.4}$$

The star commutators for a and $\bar{a}$ are reminiscent of the commutator relations in ordinary quantum mechanics:

$$[a, a]_* = [\bar{a}, \bar{a}]_* = 0, \quad [a, \bar{a}]_* = \hbar. \tag{4.5}$$

The Lagrange function for the simple harmonic oscillator is

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) \tag{4.6}$$

while the canonical momentum is $p = \dot{q}$, and the Hamilton function is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{4.7}$$

The time-evolution function is

$$\mathrm{Exp}(tH) = \frac{1}{\cos\left(\frac{\omega t}{2}\right)} \exp\left[\left(\frac{2H}{\mathrm{i}\hbar\omega}\right) \tan\left(\frac{\omega t}{2}\right)\right]. \tag{4.8}$$

It has the Fourier-Dirichlet expansion

$$\mathrm{Exp}(Ht) = \sum_n e^{-\mathrm{i}(n+1/2)\omega t} \pi_n \tag{4.9}$$

with the projectors

$$\pi_0 = 2e^{-2a\bar{a}/\hbar}, \qquad \pi_n = \frac{1}{\hbar^n n!} \bar{a}^n * \pi_0 * a^n \tag{4.10}$$

for the ground state and the n-th excited state, respectively. The equivalent of the Schrödinger equation is

$$H * \pi_n = (n + 1/2)\hbar\omega\pi_n. \tag{4.11}$$

The energy eigenvalues are

$$E_n = (n + 1/2)\hbar\omega. \tag{4.12}$$

We also use the *normal* star product:

$$F * G = F \exp\left(\hbar\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}\right) G. \tag{4.13}$$

In terms of this product the variables a, $\bar{a}$ satisfy

$$a * \bar{a} = a\bar{a} + \hbar, \quad \bar{a} * a = a\bar{a}, \quad [a, \bar{a}]_* = \hbar. \tag{4.14}$$

The time-evolution function using this product is

$$\mathrm{Exp}(Ht) = e^{-\frac{H}{\hbar\omega}} \exp\left[\left(\frac{H}{\hbar\omega}\right) e^{-\mathrm{i}\omega t}\right]. \tag{4.15}$$

The Fourier-Dirichlet expansion is

$$\mathrm{Exp}(Ht) = \sum_n e^{-\mathrm{i}n\omega t} \pi_n \tag{4.16}$$

from which we easily read off the projectors:

$$\pi_0 = e^{-\frac{H}{\hbar\omega}}, \quad \pi_n = \frac{1}{\hbar^n n!}\bar{a}^n * \pi_0 * a^n. \tag{4.17}$$

The equivalent to the Schrödinger equation is

$$H * \pi_n = n\hbar\pi_n \tag{4.18}$$

and the energy eigenvalues are

$$E_n = n\hbar\omega. \tag{4.19}$$

5. The Fermionic Oscillator

We need two Grassman variables to describe the simplest fermionic system. The appropriate Lagrange function is

$$L = \frac{\mathrm{i}}{2}\left(\psi^1\dot{\psi}^1 + \psi^2\dot{\psi}^2\right) + \mathrm{i}\omega\psi^1\psi^2. \tag{5.1}$$

The conjugate momenta are

$$\pi_\alpha = -\frac{\mathrm{i}}{2}\delta_{\alpha\beta}\psi^\beta, \tag{5.2}$$

where $\alpha, \beta = 1, 2$. We obviously have the following constraints between the phase space variables:

$$\chi_\alpha = \pi_\alpha + \frac{\mathrm{i}}{2}\delta_{\alpha\beta}\psi^\beta \approx 0. \tag{5.3}$$

The Hamilton function is

$$H = \dot{\psi}^\alpha\pi_\alpha - L = -\mathrm{i}\omega\psi^1\psi^2. \tag{5.4}$$

The constraints are second class:

$$\{\chi_\alpha, \chi_\beta\} = -\mathrm{i}\delta_{\alpha\beta}. \tag{5.5}$$

The Dirac brackets are

$$\{F, G\}_D = F\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\psi^\alpha}\frac{\overrightarrow{\partial}}{\partial\pi_\alpha} - \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\pi_\alpha}\frac{\overrightarrow{\partial}}{\partial\psi^\alpha} + \mathrm{i}\frac{\overleftarrow{\partial}}{\partial\psi^\alpha}\frac{\overrightarrow{\partial}}{\partial\psi^\alpha} - \frac{\mathrm{i}}{4}\frac{\overleftarrow{\partial}}{\partial\pi_\alpha}\frac{\overrightarrow{\partial}}{\partial\pi_\alpha}\right)G. \tag{5.6}$$

Now implement the constraints, and set

$$\pi_\alpha \equiv -\frac{\mathrm{i}}{2}\delta_{\alpha\beta}\psi^\alpha. \tag{5.7}$$

The Dirac brackets then become

$$\{F, G\}_D = F\left(\mathrm{i}\frac{\overleftarrow{\partial}}{\partial\psi^\alpha}\frac{\overrightarrow{\partial}}{\partial\psi^\alpha}\right)G. \tag{5.8}$$

The fermionic star product uses the Dirac bracket:

$$F * G = F \exp\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\psi^\alpha}\frac{\overrightarrow{\partial}}{\partial\psi^\alpha}\right) G. \tag{5.9}$$

The canonical anticommutation relations are

$$\{\psi^\alpha, \psi^\beta\}_* = \psi^\alpha * \psi^\beta + \psi^\beta * \psi^\alpha = \hbar\delta^{\alpha\beta}. \tag{5.10}$$

We see that the fermionic star product has effected a *Cliffordization*: the Grassman variables ψ^α have become elements of a Clifford algebra.
The time-evolution function for the fermionic oscillator is

$$\mathrm{Exp}(Ht) = \sum_n \frac{1}{n!}\left(\frac{-\mathrm{i}t}{\hbar}\right)^n H^{n*} = \cos\left(\frac{\omega t}{2}\right) - \frac{2}{\hbar}\psi^1\psi^2 \sin\left(\frac{\omega t}{2}\right) \tag{5.11}$$

where we have used

$$(\psi^1\psi^2) * (\psi^1\psi^2) = \frac{1}{2}\left(\frac{\hbar}{2}\right)^2 (\psi^1\psi^2) 2 \frac{\overleftarrow{\partial}}{\partial\psi^1}\frac{\overrightarrow{\partial}}{\partial\psi^1}\frac{\overleftarrow{\partial}}{\partial\psi^2}\frac{\overrightarrow{\partial}}{\partial\psi^2}(\psi^1\psi^2) = -\left(\frac{\hbar}{2}\right)^2. \tag{5.12}$$

We can rewrite this result for later use as

$$\mathrm{Exp}(Ht) = \cos\left(\frac{\omega t}{2}\right)\left[1 + \frac{2H}{\mathrm{i}\hbar\omega}\tan\left(\frac{\omega t}{2}\right)\right] \tag{5.13}$$

or

$$\mathrm{Exp}(Ht) = \cos\left(\frac{\omega t}{2}\right)\exp\left[\frac{2H}{\mathrm{i}\hbar\omega}\tan\left(\frac{\omega t}{2}\right)\right]. \tag{5.14}$$

The Fourier-Dirichlet expansion is

$$\mathrm{Exp}(Ht) = \pi_{1/2}\mathrm{e}^{-\frac{\mathrm{i}\omega t}{2}} + \pi_{-1/2}\mathrm{e}^{\frac{\mathrm{i}\omega t}{2}} \tag{5.15}$$

with the projectors

$$\pi_{1/2} = \frac{1}{2} - \frac{\mathrm{i}}{\hbar}\psi^1\psi^2, \quad \pi_{-1/2} = \frac{1}{2} + \frac{\mathrm{i}}{\hbar}\psi^1\psi^2. \tag{5.16}$$

The eigenvalue equations are

$$H * \pi_{1/2} = \frac{\hbar\omega}{2}\pi_{1/2}, \quad H * \pi_{-1/2} = -\frac{\hbar\omega}{2}\pi_{-1/2} \tag{5.17}$$

and the energy eigenvalues are

$$E_\pm = \pm\hbar\omega/2. \tag{5.18}$$

5.1. Holomorphic Variables

As in the bosonic case we can go over to the holomorphic variables

$$f = \frac{1}{\sqrt{2}}\left(\psi^2 + \mathrm{i}\psi^1\right), \quad \bar{f} = \frac{1}{\sqrt{2}}\left(\psi^2 - \mathrm{i}\psi^1\right). \tag{5.19}$$

The star product in these variables becomes

$$f * \bar{f} = f\bar{f} + \hbar/2, \quad \bar{f} * f = \bar{f}f - \hbar/2. \tag{5.20}$$

The star anticommutators are

$$\{f, f\} = \{\bar{f}, \bar{f}\} = 0, \quad \{f, \bar{f}\}_* = \hbar, \tag{5.21}$$

so that these variables as well become elements of a Clifford algebra.

The Hamilton function for the fermionic oscillator takes a form similar to that for the bosonic oscillator:

$$H = \omega\bar{f}f. \tag{5.22}$$

The star exponential becomes

$$\mathrm{Exp}(Ht) = \pi_{-1/2}\mathrm{e}^{\mathrm{i}\omega t} * \pi_{1/2}\mathrm{e}^{-\mathrm{i}\omega t/2} \tag{5.23}$$

with the projectors

$$\pi_{-1/2} = \frac{1}{2} - \frac{1}{\hbar}\bar{f}f, \quad \pi_{1/2} = \frac{1}{2} + \frac{1}{\hbar}\bar{f}f. \tag{5.24}$$

The f and $\bar{f}$ act as annihilation and creation operators in the following sense:

$$f * \pi_{-1/2} = \bar{f} * \pi_{1/2} = 0 \tag{5.25}$$

and

$$\bar{f} * \pi_{-1/2} * f = \hbar\pi_{1/2} \quad f * \pi_{1/2} * \bar{f} = \hbar\pi_{-1/2}. \tag{5.26}$$

5.2. The Fermionic Normal Star Product

We define the fermionic normal star product as

$$F * G = F \exp\left(\hbar\overleftarrow{\partial}_f\overrightarrow{\partial}_{\bar{f}}\right) G. \tag{5.27}$$

We then find

$$f * \bar{f} = f\bar{f} + \hbar, \quad \bar{f} * f = f\bar{f}. \tag{5.28}$$

The star exponential becomes

$$\mathrm{Exp}(Ht) = \pi_0 + \pi_1\mathrm{e}^{-\mathrm{i}\omega t} \tag{5.29}$$

with the projectors

$$\pi_0 = 1 - \frac{1}{\hbar}\bar{f}f, \quad \pi_1 = \frac{1}{\hbar}\bar{f}f. \tag{5.30}$$

The star exponential may be rewritten in a form convenient for later use as:

$$\mathrm{Exp}(Ht) = \mathrm{e}^{\frac{H}{\hbar\omega}} \exp\left[\left(\frac{H}{\hbar\omega}\right)\mathrm{e}^{-\mathrm{i}\omega t}\right]. \tag{5.31}$$

The equivalent to the Schrödinger equation is

$$H * \pi_0 = 0, \quad H * \pi_1 = \hbar\omega\pi_1. \tag{5.32}$$

Again, f and $\bar{f}$ act as annihilation and creation operators:

$$f * \pi_0 = \bar{f} * \pi_1 = 0, \tag{5.33}$$

and

$$\bar{f} * \pi_0 * f = \hbar\pi_1, \quad f * \pi_1 * \bar{f} = \hbar\pi_0. \tag{5.34}$$

The energy eigenvalues are

$$E = 0,\ \hbar\omega. \tag{5.35}$$

5.3. The Matrix Formalism

For this simple system the transition from the phase space description to the conventional quantum mechanical description in terms of linear operators in Hilbert space is particularly simple. Since there are only two states the operators act on a two-dimensional representation space and may be represented as 2×2 matrices. We have

$$f = \sqrt{\hbar}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad \bar{f} = \sqrt{\hbar}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{5.36}$$

Another important operator is the involution operator,

$$\tau = \frac{2}{\hbar}\bar{f}f, \tag{5.37}$$

which satisfies $\tau * \tau = 1$, so that its eigenvalues are ± 1. In terms of this operator the projectors onto the eigenspaces are

$$\pi_{\pm 1/2} = \frac{1}{2}(1 \pm \tau). \tag{5.38}$$

In the matrix representation these operators are

$$\tau = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \pi_{-1/2} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \pi_{1/2} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{5.39}$$

These quantities are related to each other in the same way as in the phase-space description, if the star products are replaced by matrix multiplication. In this way, also in the case of more complicated representations, the star product represents in phase space the algebraic relations which hold between quantum mechanical operators in the conventional Hilbert space representation.

6. The Supersymmetric Oscillator

Everything we have done up to now can be generalized and unified in the supersymmetric context. The supersymmetric star product is

$$F * G = F \exp\left[\left(\frac{\hbar}{2}\right)\left(\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a + \overleftarrow{\partial}_f\overrightarrow{\partial}_{\bar{f}} + \overleftarrow{\partial}_{\bar{f}}\overrightarrow{\partial}_f\right)\right] G. \tag{6.1}$$

The Hamilton function for the supersymmetric Bose-Fermi oscillator is

$$H_S = \omega\left(\bar{f} * f + \bar{a} * a\right) = \omega\left(\bar{f}f + \bar{a}a\right). \tag{6.2}$$

We define supersymmetric generators relating the bosonic and fermionic sectors by

$$Q_+ = \sqrt{\frac{1}{\hbar}}(a * \bar{f}) = \sqrt{\frac{1}{\hbar}}(a\bar{f}) \quad Q_- = \sqrt{\frac{1}{\hbar}}(\bar{a} * f) = \sqrt{\frac{1}{\hbar}}(\bar{a}a). \tag{6.3}$$

These operators are nilpotent:

$$Q_\pm * Q_\pm = Q_\pm^2 = 0. \tag{6.4}$$

The Hamilton function may be written as

$$H_S = \omega\{Q_+, Q_-\}_*. \tag{6.5}$$

The Hamilton function is supersymmetric:

$$[Q_+, H_S]_* = [Q_-, H_S]_* = 0. \tag{6.6}$$

With the real functions

$$Q_1 = Q_+ + Q_-, \quad Q_2 = -\mathrm{i}(Q_+ - Q_-) \tag{6.7}$$

we obtain

$$H_S = \omega Q_1 * Q_1 = \omega Q_2 * Q_2. \tag{6.8}$$

This is the fundamental structure of all supersymmetry algebras.

6.1. The Product Ansatz

The supersymmetric star product is just the product of its bosonic and fermionic parts. Also the supersymmetric star exponential can be obtained by a factor ansatz:

$$\begin{aligned} \mathrm{Exp}(Ht) &= \frac{1}{\cos\left(\frac{\omega t}{2}\right)} \exp\left[\left(\frac{2H_B}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right] \cos\left(\frac{\omega t}{2}\right) \exp\left[\left(\frac{2H_F}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right] \\ &= \exp\left[\left(\frac{2H_S}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right] \end{aligned} \tag{6.9}$$

where we have used the equations (4.8) and (5.14). The Schrödinger equation is

$$H_S * \pi_{n_F,n_B} = (E_{n_B} + E_{n_F})\pi_{n_F,n_B} \tag{6.10}$$

with the projectors

$$\pi_{n_F,n_B} = \pi_{n_F}\pi_{n_B}. \tag{6.11}$$

The functions $Q_\pm$ act according to

$$Q_+ * \pi_{n_F,n_B} * Q_- = \hbar\pi_{n_F+1,n_B-1}, \quad Q_- * \pi_{n_F,n_B} * Q_+ = \hbar\pi_{n_F-1,n_B+1}. \tag{6.12}$$

For the supersymmetric normal star product

$$F * G = F \exp\left[\hbar(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_f \overrightarrow{\partial}_{\bar{f}})\right] G \tag{6.13}$$

we find the time-evolution function

$$\mathrm{Exp}(Ht) = \mathrm{e}^{-\frac{H_S}{\hbar\omega}} \exp\left[\left(\frac{H_S}{\hbar\omega}\right)\mathrm{e}^{-\mathrm{i}\omega t}\right] \tag{6.14}$$

by the use of equations (4.15) and (5.31).

7. Supersymmetric Quantum Mechanics

In supersymmetric quantum mechanics we generalize the concept of holomorphic variables by introducing the variables

$$B = \frac{1}{\sqrt{2}}\left(W(q) + \frac{\mathrm{i}p}{\sqrt{m}}\right), \quad \bar{B} = \frac{1}{\sqrt{2}}\left(W(q) - \frac{\mathrm{i}p}{\sqrt{m}}\right), \tag{7.1}$$

instead the variables $a, \bar{a}$ from (4.2). The function $W(q)$ is called a **superpotential**. The appropriate star product is

$$F * G = F \exp\left[\left(\frac{\hbar}{2}\sqrt{m}\right)\frac{\partial W}{\partial q}\left(\partial_B \partial_{\bar{B}} - \partial_{\bar{B}}\partial_B\right)\right] G. \tag{7.2}$$

We easily calculate

$$\{B, \bar{B}\}_* = W^2 + \frac{p^2}{m}, \quad [B, \bar{B}]_* = \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}. \tag{7.3}$$

Instead of

$$H_S = \omega\left[(\bar{a} * a)\pi_{-1/2} + (a * \bar{a})\pi_{1/2}\right] \tag{7.4}$$

we now have

$$\begin{aligned} H_S &= (\bar{B} * B)\pi_{-1/2} + (B * \bar{B})\pi_{1/2} \\ &= \left(\frac{1}{2}\{B, \bar{B}\}_* - \frac{1}{2}[B, \bar{B}]_*\right)\pi_{-1/2} + \left(\frac{1}{2}\{B, \bar{B}\}_* + \frac{1}{2}[B, \bar{B}]_*\right)\pi_{1/2} \\ &= \frac{1}{2}\left(\frac{p^2}{m} + W^2 - \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}\right)\pi_{-1/2} + \frac{1}{2}\left(\frac{p^2}{m} + W^2 + \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}\right)\pi_{-1/2} \\ &= H_1\pi_{-1/2} + H_2\pi_{1/2}. \end{aligned} \tag{7.5}$$

We read off from here the *partner potentials*

$$V_1 = \frac{1}{2}\left(W^2 - \frac{\hbar}{m}\frac{\partial W}{\partial q}\right), \quad V_2 = \frac{1}{2}\left(W^2 + \frac{\hbar}{m}\frac{\partial W}{\partial q}\right). \tag{7.6}$$

We thus have the twin systems

$$\begin{aligned} H_S * \pi_{(-1/2,n_B)} &= H_1 * \pi_{n_B}\pi_{-1/2} = E_1\pi_{n_B}\pi_{-1/2} \\ H_S * \pi_{(1/2,n_B)} &= H_2 * \pi_{n_B}\pi_{1/2} = E_2\pi_{n_B}\pi_{1/2}. \end{aligned} \tag{7.7}$$

From here we read off two bosonic Schrödinger equations:

$$H_1 * \pi_{n_B} = E_1\pi_{n_B}, \quad H_2 * \pi_{n_B} = E_2\pi_{n_B}. \tag{7.8}$$

The twin systems are interrelated by

$$\begin{aligned} H_1 * (\bar{B} * \pi^{(2)}_{n_B} * B) &= E_2(\bar{B} * \pi^{(2)}_{n_B} * B) \\ H_2 * (\bar{B} * \pi^{(1)}_{n_B} * B) &= E_1(\bar{B} * \pi^{(1)}_{n_B} * B) \end{aligned} \tag{7.9}$$

We see that E_2 is also an eigenvalue of the Hamilton function H_1, and E_1 is also an eigenvalue of the Hamilton function H_2. $\bar{B} * \pi^{(2)}_{n_B} * B$ is an eigenfunction of H_1, and $\bar{B} * \pi^{(1)}_{n_B} * B$ is an eigenfunction of H_2.

7.1. An Example

For a simple example of the use of these supersymmetric techniques in non-relativistic quantum mechanics, consider a system described by the superpotential

$$W(q) = A\tanh(\alpha q). \tag{7.10}$$

This leads to the twin potentials

$$\begin{aligned} V_1 &= \frac{1}{2}\left[A^2 - A\left(A + \frac{\hbar\alpha}{\sqrt{m}}\right)\frac{1}{\cosh^2(\alpha q)}\right] \\ V_2 &= \frac{1}{2}\left[A^2 - A\left(A - \frac{\hbar\alpha}{\sqrt{m}}\right)\frac{1}{\cosh^2(\alpha q)}\right]. \end{aligned} \tag{7.11}$$

Now choose $A = \hbar\alpha/\sqrt{m}$. Then

$$V_1 = \frac{\hbar^2\alpha^2}{2m}\left[1 - \frac{2}{\cosh^2(\alpha q)}\right] \tag{7.12}$$

which is known as the Rosen-Morse potential, while

$$V_2 = \frac{\hbar^2\alpha^2}{2m}. \tag{7.13}$$

V_2 is just a constant potential, which describes the motion of a free particle, whereas the Rosen-Morse potential is more complex and describes a system which also supports a bound state. Nevertheless, we can find the eigenvalues and eigenfunctions of the Rosen-Morse system without solving a differential equation, just using as input the known eigenvalues and eigenfunctions of the constant potential, and then transforming these into the corresponding objects in the twin system by using the relations given above.

8. Non-relativistic Spin and the Pauli Equation

To describe a non-relativistic particle with spin we shall use besides the usual position and momentum variables three Grassman variables labelled θ_1, θ_2, θ_3. We introduce the *Pauli star product* given by

$$F * G = F \exp\left(\frac{\hbar}{2}\sum_i \overleftarrow{\partial}_{\theta_i}\overrightarrow{\partial}_{\theta_i}\right) G. \tag{8.1}$$

We find that the variables θ_i satisfy the relations of a Clifford algebra:

$$\{\theta_i, \theta_j\}_* = \hbar\delta_{ij}. \tag{8.2}$$

Consider the Pauli elements

$$\sigma^i = \frac{1}{\mathrm{i}\hbar}\epsilon^{ijk}\theta_j\theta_k. \tag{8.3}$$

They are easily seen to fulfill the relations

$$\{\sigma^i, \sigma^j\}_* = 2\delta^{ij}, \quad [\sigma^i, \sigma^j]_* = 2\mathrm{i}\epsilon^{ijk}\sigma^k. \tag{8.4}$$

We now want to describe a charged spin one-half particle moving under the influence of a constant magnetic field. We then have to take the position and momentum variables explicitly into account, so we work with the *Pauli-Moyal star product*

$$F * G = F \exp\left[\left(\frac{\mathrm{i}\hbar}{2}\right)\sum_i \left(\overleftarrow{\partial}_{q^i}\overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p^i}\overrightarrow{\partial}_{q_i} - \mathrm{i}\overleftarrow{\partial}_{\theta_i}\overrightarrow{\partial}_{\theta_i}\right)\right] G. \tag{8.5}$$

Define the supercharges

$$\begin{aligned} Q_1 &= \frac{1}{2\sqrt{m}}\left[-\left(p_2 - \frac{e}{c}A_2\right)\sigma^1 + \left(p_1 - \frac{e}{c}A_1\right)\sigma^2\right] \\ Q_2 &= \frac{1}{2\sqrt{m}}\left[\ \left(p_1 - \frac{e}{c}A_1\right)\sigma^1 + \left(p_2 - \frac{e}{c}A_2\right)\sigma^2\right] \end{aligned} \tag{8.6}$$

where $\bar{A}(q^1, q^2)$ is the vector potential of the magnetic field. Then we have the supersymmetry algebra

$$\begin{aligned}
&\{Q_1, Q_2\}_* = 0, \\
&Q_1 * Q_1 = Q_2 * Q_2 = \frac{1}{2m}\left[\left(p_1 - \frac{e}{c}A_1\right)^2 + \left(p_2 - \frac{e}{c}A_2\right)^2\right] \\
&\qquad + \frac{1}{2m}\left[\left(p_1 - \frac{e}{c}A_1\right), \left(p_2 - \frac{e}{c}A_2\right)\right]_* (\sigma^1 * \sigma^2)
\end{aligned} \tag{8.7}$$

together with the Hamilton function

$$H_P = Q_1 * Q_1 = \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2 - \frac{e\hbar}{2mc}(\bar{\sigma} \cdot \bar{B}) \tag{8.8}$$

which is Pauli's Hamilton function with gyromagnetic ratio $g = 2$. Note that the quantities $\{Q_1, Q_2, H_P\}$ form a supersymmetry algebra only for this value of g.

9. Relativistic Quantum Mechanics and the Dirac Equation

For relativistic systems we use the fermionic variables f, $\bar{f}$ to couple the particle and anti-particle sectors.

Dirac's Hamilton function for a massless particle is

$$H_D = Q = D\bar{f} + \bar{D}f \tag{9.1}$$

where D, $\bar{D}$ have even Grassmann parity. For spin one-half particles take

$$D = \bar{D} = \frac{c}{\sqrt{\hbar}}(\bar{\sigma} \cdot \bar{p}). \tag{9.2}$$

In matrix notation we have

$$\hat{H}_D = \begin{pmatrix} 0 & c\bar{\sigma} \cdot \bar{p} \\ c\bar{\sigma} \cdot \bar{p} & 0 \end{pmatrix} =: \bar{\alpha} \cdot \bar{p}. \tag{9.3}$$

The Dirac star product is

$$F * G = F \exp\left[\left(\frac{\hbar}{2}\right)\left(\overleftarrow{\partial}_f \overrightarrow{\partial}_{\bar{f}} + \overleftarrow{\partial}_{\bar{f}} \overrightarrow{\partial}_f + \sum_i \left(\overleftarrow{\partial}_{q^i} \overrightarrow{\partial}_{p_i} - \mathrm{i}\overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q^i} + \overleftarrow{\partial}_{\theta_i} \overrightarrow{\partial}_{\theta_i}\right)\right)\right] G. \tag{9.4}$$

We then find

$$H_D * H_D = c^2(\bar{\sigma} \cdot \bar{p}) * (\bar{\sigma} \cdot \bar{p}) = c^2\bar{p}^2 \tag{9.5}$$

which corresponds to the relation $E = |\bar{p}|c$ for massless particles.

For massive particles and anti-particles we extend the Hamilton function by a term involving the involution operator:

$$H_D = Q + M * \tau \tag{9.6}$$

where

$$M = M_+\pi_{1/2} + M_-\pi_{-1/2} \tag{9.7}$$

and $M_\pm$ are bosonic variables. We have

$$M * \tau = M_+\pi_{1/2} - M_-\pi_{-1/2} \tag{9.8}$$

from which we see that $M_\pm$ are the masses of the particle and anti-particle, respectively. We now calculate

$$H_D * H_D = (c^2\bar{p}^2 + M_+ * M_+)\pi_{1/2} + (c^2\bar{p}^2 + M_- * M_-)\pi_{-1/2}. \tag{9.9}$$

For $M_\pm = mc^2$ this corresponds to the relativistic energy-momentum relation

$$E^2 = |\bar{p}|^2c^2 + (mc^2)^2. \tag{9.10}$$

In the non-relativistic limit the Hamilton function (9.1) reduces to the Pauli Hamilton function of Equation (8.8). In Reference [8] we demonstrated this fact by using the resolvent method. We have also performed a calculation based on the Foldy-Wouthusen method which yields in addition relativistic corrections to the Pauli equation [9].

References

[1] Abrikosov A., Ginzburg V. and Legget A., e-archiv www.almz.com/nobel/.

[2] Alexandrov M., Kontsevich M., Schwarz A. and Zabronsky O., *Geometry of the Master Equation*, Int. J. Mod. Phys. **A 12** (1997) 1405–1430.

[3] Cooper F., Khare A. and Sukhatme U., *Supersymmetry and Quantum Mechanics*, Phys. Reports **251** (1995) 267–385.

[4] Cornell E., Ketterle W. and Wieman C., e-archiv www.almaz.com/nobel/.

[5] Dirac P.A.M., *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[6] Henneaux M. and Teitelboim C., *The Quantization of Gauge Systems*, Princeton University Press, Cambridge MA, 1992.

[7] Hirshfeld A., *Deformation Quantization in Quantum Mechanics and Field Theory*. In: Geometry, Integrability and Quantization IV, I. Mladenov and G. Naber (Eds), Coral Press, Sofia, 2003, pp 11–41.

[8] Hirshfeld A. and Henselder P., *Deformation Quantization for Systems with Fermions*, Annals of Physics (NY) **302** (2002) 59–77.

[9] Hirshfeld A., Henselder P. and Spernat T., Dortmund Preprint 2003.

[10] Inoue A., *What is Superanalysis and How to Use It*,
e-archiv http://www.math.titech.ac.jp/ inoue/ibaraki-conf03.pdf

[11] Jolie J., *Uncovering Supersymmetry*, Scientific American **287** (2002) 54–61.

[12] Kostelecky V. and Campbell D. (Eds), *Supersymmetry in Physics*, North Holland, 1985.