# AN ATTEMPT TO USE MECHANICAL ENERGY CONSERVATION PRINCIPLE IN CASE OF CHANNEL DEVELOPED TURBULENT FLOW

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**Abstract.** In this work a hypothesis about the fluid flow character is proposed which consists in the splitting of a flow in two parts. The first one is treated dynamically, and the second statistically. The instability generating energy coming from the dynamical flow being distributed by the statistical flow results in an influence on the first one and forms the observed flow. The connection between the two flows is obtained by the requirement of the mechanical energy conservation. A corresponding model is analyzed in some details and applied to the case of developed turbulence in a channel Poiseuille flow. The mean velocity profiles for values of $Re = 13800, 23200, 32800$ are computed and compared with the existing experimental data. Numerically derived velocity profiles with $Re$ from 100 to 143000 are given.

## 1. Introduction

Nowadays the Hamiltonian formalism is well developed and has found many applications in hydrodynamics in case of ideal fluid (for comprehensive reviews see for example [1] and [2]). In the present work we start from different basic considerations aiming to obtain equations of motion which would provide the possibility of description of dissipative behavior of the fluid. As it is usual for the Hamiltonian formalism for fluid, we will work with fluid particles in order to pass to finite degree of freedom. Later on in this paper we will obtain the typical size of these elements for the chosen example. We assume also that fluid elements keep their integrity. In other words we consider flow as non-mixing but interacting trajectories having size of the fluid elements. It is this part of the fluid motion for which we will attempt to write a Lagrangian. Some terms of it are connected with the interaction between neighboring trajectories that accumulate an internal friction energy during the movement. This energy is, of course, non-conservative. We will

split the calculations in two stages. The first one is to obtain the main flow from derived equation of motion and the second one is to distribute the non-conservative energy inside the main flow. We will focus on the non-geophysical flows. Other basic circumstance is that we omit thermodynamical behavior of the fluid movement which leads us to expect that equation of motion of the main fluid will be in the form of the Burgers equation. Recently there are many attempts to use this equation for the description of the turbulence by modelling the external force in different ways including stochastically, cf [3], [4] and [5]. Concerning the dissipative mechanism we are going to create additional model for the eddy dissipation that is energetically connected with non-conservative terms in the Lagrangian. We provide a realization of the above considered project and have tested it on relatively simple example of incompressible channel Poiseuille flow with high $Re$.

## 2. Description of the Model

In order to describe the dynamical part of the flow we will use Lagrangian formalism to obtain the equation of motion. Let us define the kinetic and potential energies that are significant for our task. We construct our model having in mind geometry of the two-dimensional channel Poiseuille flow – a fact that gives possibility for its simplification. For the kinetic energy of the fluid particles in rectangular coordinates $q = q(q_1, q_2)$ (the flow is oriented along $q_1$) we can write

$$T(q) = \frac{\rho}{2} \sum_{i=1}^{2} \int_0^{2a(q)} \dot{q}_i^2 \, da' \tag{1}$$

where $2a$ is the width of the channel and $\rho$ is the fluid density. The integration in respect of $a'$ give us the kinetic energies of the all fluid elements in point $q$. In the chosen test flow the channel width is constant. The potential of the external force $p$ is equal to the pressure difference between the two ends of the channel and can be written as

$$V_p = \int_0^{2a} \left[ \int_0^{q_1} \frac{\partial p}{\partial q_1'} \, dq_1' \right] da' \tag{2}$$

where $p$ is the external force per unit cross length (oriented in the direction of $q_1$). On this stage, as we have mentioned before, we will consider the fluid as continuum of trajectories with finite size. We introduce two types of internal forces with respect to trajectories: the first one is oriented along to trajectories itself and the second one is between the neighbor trajectories at given point. These forces are reactive – they are results of the action of the external force, the geometry of the flow and the physical characteristics of the flow, as viscosity for example. Is it simplest to use the first derivative of the velocity along the trajectory in order to represent the first type of force mentioned above. We can expect to find out this

force before and after obstacles. The potential of this force can we written as

$$V_o = \nu \sum_{i=1}^{2} \int_0^{2a} \frac{\partial^2 \dot{q}_i}{\partial q_i^2} \, \mathrm{d}a' \tag{3}$$

where the kinematic coefficient of viscosity $\nu$ of the fluid is a constant. The flow moves under the following boundary conditions $\dot{q}(q_2 = 0) = \dot{q}(q_2 = 2a) = 0$. From the equation of the continuity follows that the sum (3) is zero. This means that this force, respectively the potential, is conservative.

The last force that we include in the Lagrangian is expressed through the velocity derivatives between the neighboring trajectories at the given point

$$V_f = \nu \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^{2a} \left[ \int_0^{q_i} \frac{\partial^2 \dot{q}_i}{\partial q_j^2} \, \mathrm{d}q_i' \right] \mathrm{d}a'(q_2). \tag{4}$$

This potential accumulates the energy of interaction between the given trajectory and its neighbors. This needs at least the second derivatives and we choose the simplest case. The potential (4) is non-conservative and this is evident because the corresponding energy increase monotonically. From our point of view this force is responsible for the emergence of the turbulence. We note that in our example the potentials (3) and (4) are constants. Now we can write the Lagrangian:

$$L(q, \dot{q}, t) = \frac{\rho}{2} \sum_{i=1}^{2} \int_0^{2a} \dot{q}_i^2 \, \mathrm{d}a'(q_2) - \int_0^{2a} \left[ \int_0^{q_1} \left( \frac{\partial p}{\partial q_1'} \right) \mathrm{d}q_1' \right] \mathrm{d}a'(q_2)$$

$$+ \nu \sum_{i=1}^{2} \int_0^{2a} \frac{\partial^2 \dot{q}_i}{\partial q_i^2} \, \mathrm{d}a'(q_2) + \nu \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^{2a} \left[ \int_0^{q_i} \frac{\partial^2 \dot{q}_i}{\partial q_j^2} \, \mathrm{d}q_i' \right] \mathrm{d}a'(q_2). \tag{5}$$

The action will be

$$S[q(t)] = \int_{t_0}^{t_1} L(q, \dot{q}, t) \, \mathrm{d}t. \tag{6}$$

The variation of the functional (6) after above made remarks leads to the following equations of motion:

$$\rho \ddot{q}_1 = -\frac{\partial p}{\partial q_1} + \nu \frac{\partial^2 \dot{q}_1}{\partial q_1^2} + \nu \frac{\partial^2 \dot{q}_1}{\partial q_2^2}, \qquad \rho \ddot{q}_2 = \nu \frac{\partial^2 \dot{q}_2}{\partial q_1^2} + \nu \frac{\partial^2 \dot{q}_2}{\partial q_2^2}. \tag{7}$$

Equations (7) are particular cases of the Burgers equation in Lagrangian variables. Now we start to discuss the distribution of the energy described in (4). This is the process that we hope to connect with the turbulence. The initial considerations in this sense is that up to a certain value the stored non-conservative energy dissipates via the molecular diffusion mechanism. Above that value it may dissipate in another way, namely, via the macroscopic dispersion process which reveals the

turbulence process. After decreasing of eddies energy (due to of internal friction between the flow and eddies) to the given value the process turns into molecular one. The density of a non-conservative energy will be denoted by $\varepsilon$. For the latter quantity, in view of the equations (7) and in the case when stationary channel flow $\partial p/\partial q_1$ is constant we can write:

$$
\begin{aligned}
\varepsilon &= \nu \int_{q_1} \frac{\partial^2 \dot{q}_1}{\partial q_2^2}\, \mathrm{d}q_1' + \nu \int_{q_2} \frac{\partial^2 \dot{q}_2}{\partial q_1^2}\, \mathrm{d}q_2' \\
\varepsilon(l_y) &= \nu \int_0^{l_y} \frac{\partial^2 \dot{q}_1}{\partial q_2^2}\, \mathrm{d}q_1' = \int_0^{l_y} \frac{\partial p}{\partial q_1}\, \mathrm{d}q_1' = \nu \int_0^{l_y} A\, \mathrm{d}q_1' = \nu A l_y
\end{aligned}
\tag{8}
$$

where $\varepsilon_{l_y}$ is the non-conservative energy density from a layer with coordinate $q_2$ and length $l_y$ and $A$ is constant.

Now we will make an important assumption about the space-time structure of the fluid stream. The eddies are borning at a certain time-interval $\Delta t$ of the motion of a given volume in the corresponding layer. Based on this assumption, and taking into account that the trajectories are straight lines and the profile is parabolic with corresponding boundary conditions, we can write for $\varepsilon(l_y)$

$$
\varepsilon(l_y) = An\nu\dot{q}_1(q_2)\Delta t = \nu AnA(a^2 - q_2^2)\Delta t/2. \tag{9}
$$

Here $l_y$ is the trajectory length passed by the fluid from the corresponding layer $(q_2)$ after $n$ eddies germinated. It follows from the above considerations that the stored internal energy is distributed by the corresponding layer in portions equal to

$$
\varepsilon(l_y)|_{n=1} = \nu A^2(a^2 - q_2^2)\Delta t/2 = f. \tag{10}
$$

During the propagation a momentum transfer occurs between trajectories of the dynamical flow. We suppose that this transfer is in all directions with equal probabilities. However, in view of the symmetry, the fluxes with well developed turbulence from the channel center to the walls and contrariwise remain uncompensated. The difference between these two fluxes, added to the dynamically derived flow, determines the resultant motion. We have to note however, that in the zone near the centerline the fluxes from the channel walls to the center are mutually compensated and, therefore, they must be excluded from the momentum transfer balance. Let us denote the half-width of this zone with $Y$. Aiming to determine the effective length $\lambda_y$ of the eddy flow we will make use of the following simple expression for the eddy velocity decrease

$$
\frac{\mathrm{d}w}{\mathrm{d}q_2} = -k_1 A, \qquad w_0 = f(q_2) = \sqrt{\varepsilon} \tag{11}
$$

where $w$ is the eddy velocity at a point $q_2$ of the channel profile, $k_1$ is a fluid constant with length dimension. The dependence of $w$ on $A$ reflects the influence of

the flux turbulization on the process of eddies absorption. In this way the effective length of the eddy originating from given point $q_{2c}$ is obtained as

$$\lambda_{yc} = (f(q_{2c}) - C))/k_1 A \tag{12}$$

where $C$ is the threshold velocity at which the molecular diffusion mechanism is switching on. As can be seen from (9) the values of $\lambda_{yc}$ depend very weekly on $Re$. In this paper we have assumed that the momentum transfer, which is realized due to the turbulence mechanism, is proportional to the velocity gradient $k_2$ (here $k_2$ is the fluid constant). The last assumption expresses the view that momentum exchange is realized between neighboring trajectories while eddies are regarded as a transmitter. Accordingly, at a given point we have the following momentum balance: $\varphi\downarrow$ is the total quantity of momentum carried throw this point by all eddies coming from central line to the walls, $\varphi\uparrow$ is contrariwise quantity of momentum and loss of momentum due to of eddies born at this point. To average the pulsatory character of eddy emergency we use the ratio the eddy eruption time $\Delta t_r$ (accepted as a flow constant) and time period between two consecutive eddy eruptions $\Delta t$. So for the profile $\overset{\star}{q}_1(q_2)$ of the resultant flow we obtain

$$\overset{\star}{q}_1(q_2) = \dot{q}_1(q_2) + [\varphi\downarrow(q_2) + \varphi\uparrow(q_2) - \dot{q}_1(q_2)]\Delta t_r/\Delta t$$

$$\varphi\downarrow(q_2) = k_2 \int_d^0 \left[ \int_{q_{2c}-\lambda_{q_{2c}}}^{q_{2c}} \delta(q_2' - q_2)\frac{\partial \dot{q}_1}{\partial q_2'}\, dq_2' \right]\, dq_2 \tag{13}$$

$$\varphi\uparrow(q_2) = k_2 \int_0^{a-Y} \left[ \int_{q_{2c}}^{q_{2c}+\lambda_{q_{2c}}} \delta(q_2' - q_2)\frac{\partial \dot{q}_1}{\partial q_2'}\, dq_2' \right]\, dq_2$$

where $\delta(\cdot)$ is the Dirac delta function. As to the spatial structure of the flow we note that the numerical experiments have shown good results when the trajectories are treated as having equal width $\Delta x$ for the whole flow.

## 3. Experimental Material

To evaluate the parameters of the model described above the data from [6] have been used. In Figure 1 the comparison between experimental data for the three cases of developed turbulence ($Re = 13800, 23200, 32300$) and the computed curves of the mean velocity profiles are shown. The statistical results are shown in Table 1. The parameters are: $\Delta t = 0.0003s$, $C = 0.0026$ m/s, $k_1 = 0.00033$ m, $k_2 = 0.0246$ m, $\Delta t_r/\Delta t = 1/10.2$, $\Delta x = 7.10^{-5}$ m. In Figure 2 the eddy's effective lengths for the upward and downward turbulent flow are shown. $Y$ is the semi-width of the central part of the channel where the symmetric turbulent flows directed to the central line are mutually compensated (the value of $Y$ was obtained about $0.16a$). In Figure 3 nine velocity profiles calculated according to

above described model with $Re = 100, 2200, 3000, 10000, 37000, 50000, 73000,$ $97000, 143000$ are shown.
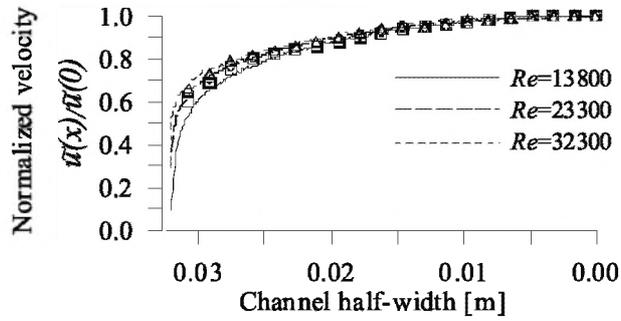


**Figure 1.** Measured and calculated curves of the main velocity profile for the flows with corresponding $Re$
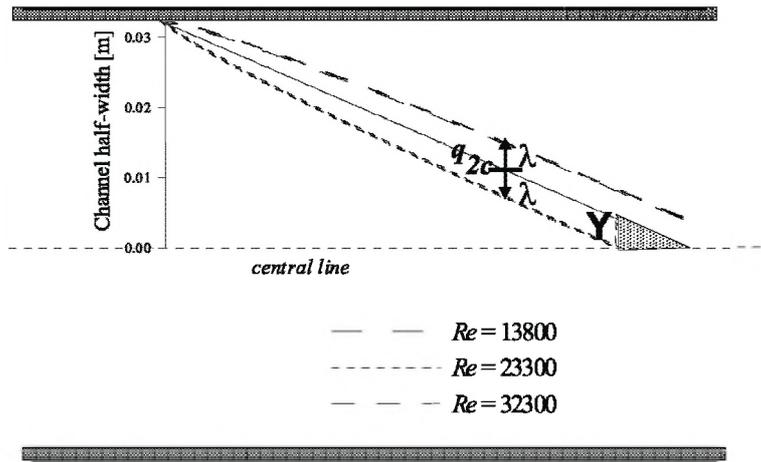


**Figure 2.** The effective eddy's length for the upward and downward flows for each point of the channel profile

Table 1

| Flows($Re$) | Mean velocity MSE(%) | $Re$ calculated | $Re$ MSE(%) |
|---|---|---|---|
| 13800 | 1.3 | 14473 | 4.8 |
| 23200 | 1.5 | 22918 | 1.2 |
| 32300 | 1.7 | 31574 | 2.2 |
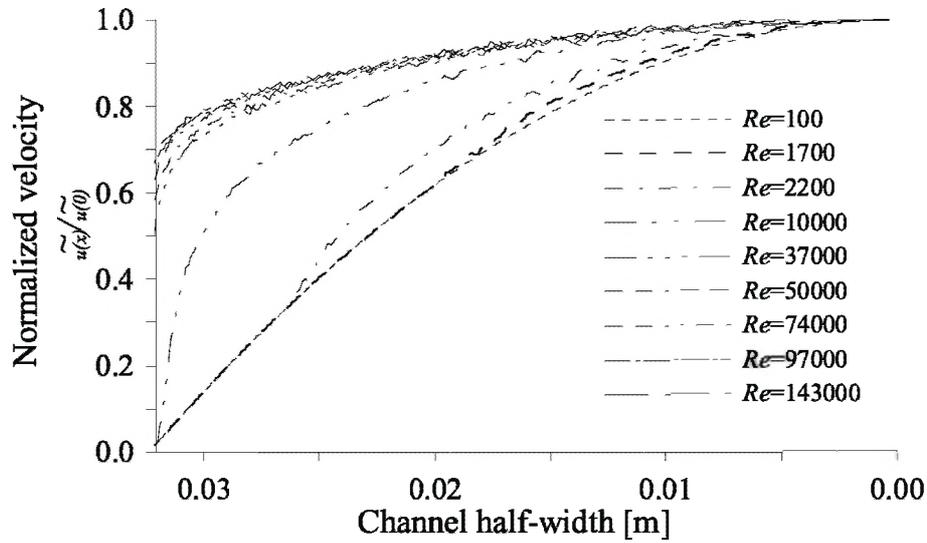
**Figure 3.** Numerically derived profiles for $Re = 100, 2200, 3000,$ 10000, 37000, 50000, 74000, 93000, 143000

## 4. Discussion

As a discussion we will make the following notes:

1. This work aims to prove the possibility to consider the fluid motion separately. The splitting concept is well developed in the hydrodynamic calculations and final proof is the quality of solutions. Here the situation is different: we introduce the idea to separate the different physical aspects of fluid flows – advection and diffusivity. This is one of the moments in the speculation of the model – if this kind of the splitting is correct? The tradition is to describe the motion as integrated system – including all knowing facts. Unlike this tradition here we distinguish these two main physical behaviors of the fluid motion as prerequisite.

2. Calculated curves, presented in Fig. 3, posses a stable behavior in large interval of $Re$ – from laminar flow to flows with extremely high developed turbulence, and the shape of curves follow the typical form known from the experiments with increasing $Re$.

3. This model gives some qualitative correct results known from the basic turbulence phenomenology [7]. Relation (4) shows that the energy of instability, accepted in this work as accumulated along the trajectories in the process of friction, increase with velocity. According to above considerations

the emergence of eddies is related with critical level of accumulation of energy determined by (4). This means that after the beginning of the channel in the transition regimes there is laminar part that consequently decrease with increasing of $Re$.

4. It is important to mention also the numerically obtained result connected with time-spatial discretization of the main (dynamically described) flow. Different attempts was used to model it. The better results are obtained in the simplest way of the discretization with respect to time and space. It turned out that the discretization is constant independently of the velocity and the place in the channel. As we noted before here we have evaluated the size of the fluid particles. This means that the trajectories have some constant behaviors in the flow and makes this construction more reliable. Of course in the case of flows with arbitrary geometry at the boundaries the size of the trajectories will be different in the different part of the flow.

5. As for the time discretization the situation is more important. The constancy of the time discretization can be connected with the well known and very complicated phenomenology in the transition regime – alternative changing of the laminar and turbulent behavior of the flow. This fact is easy to explain from the point of view of the above presented model. After the discharging of non-conservative energy some time is necessary to produce the next portion of it and then the new eddies in the central line (and in closest trajectories, usually) can emerge. Let us underline that the constancy of time and space results from the numerical experiments and therefore can not be accepted as a primary requirement of the model.

6. Even these preliminary results show that the predicted interval between 1800 and 1900 $Re$, in which the first eddies emerge is correct.

## 5. Conclusions

This work introduce the concept of the flow splitting in two parts with different physical behavior – dynamical and statistical. Dynamical flow is described by means of a Burgers type equation. The statistical flow distribute accumulated from the dynamical flow dissipative energy via mechanism that simulate some turbulence phenomena in the channel flows. An important flow characteristics are introduced (as time and spatial structure) that can be connected with turbulence behavior. Despite the strong idealization of the turbulence and controversial speculations made in the presented model, we consider the obtained results as interesting and challenging for further study.

# References

[1] Morrison P., *Hamilton Description of the Ideal Fluid*, Rev. Mod. Phys. **70** (1998) 468–521.

[2] Linch P., IMA, University of Minnesota Preprint 1938 (2002).

[3] Ituriaga R. and Khanin K., *Burgers Turbulence and Random Lagrangian Systems*, Com. Math. Phys. **232** (2003) 377–428.

[4] Boldyrev S., *Velocity-differenceprobability Density for Burgers Turbulence*, Phys. Rev. E **55** (1997) 6907–6910.

[5] Chekhlov A. and Yakhot V., *Kolmogorov Turbulence in a Random Force-driven Burgers Equation*, Phys. Rev. E **51** (1995) R2739–R2749.

[6] Hussain A. and Reynolds W., *Measurements in Fully Developed Turbulent Channel Flow*, Transactions of ASME, December (1975) 568–580.

[7] Monin A. and Yaglom A., *Statistical Hydrodinamics*, Nauka, Moscow, 1965.