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ON PHASE SPACES AND THE VARIATIONAL BICOMPLEX (AFTER G. ZUCKERMAN)

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Abstract. The notion of a phase space in classical mechanics is of course well known. The extension of this concept to field theory however, is a challenging endeavor, and over the years numerous propositions for such a generalization have appeared in the literature. In this contribution we review a Hamiltonian formulation of Lagrangian field theory based on an extension to infinite dimensions of J.-M. Souriau's symplectic approach to mechanics. Following G. Zuckerman, we state our results in terms of the variational bicomplex. We present a basic example, and briefly discuss some possible avenues of research.

1. Introduction

It appears it was H. Bacry [3] who first noted that one can find the equations of motion of (spinning) elementary particles by studying Hamiltonian systems on coadjoint orbits of the Poincaré group. By doing so, he realized that it is natural and important to introduce phase spaces not just as a set of p's and q's equipped with the canonical form $\mathrm{d}q^i \wedge \mathrm{d}p_i$, but as non-trivial symplectic manifolds. His work was put in a general context by J.-M. Souriau in his ground-breaking *Structure des Systemes Dynamiques* [18]. This treatise is the first complete treatment of mechanics which fully utilizes the language and techniques of symplectic geometry.

It is now widely recognized that a fructiferous approach for treating dynamical problems with a finite number of degrees of freedom, is to model them as Hamiltonian systems on (in general non-trivial) symplectic manifolds [2, 7, 12, 13]. It is less clear how to proceed when considering field theory. A completely rigorous point of view based on manifolds modelled on Banach (or Frechet) spaces would perhaps be the approach of choice, but to pursue such an endeavor is very delicate:

results proven along these lines rely heavily on geometry *and* on hard nonlinear analysis, as J. Marsden's lecture notes [12] testify.

A more formal approach to the Hamiltonian structure of (systems of) evolution equations is summarized in P. Olver's treatise [16] (see also [15]): one gives up the manifold description of phase space, and replaces the symplectic form by a "Hamiltonian differential operator." This point of view has been developed and applied with great success in integrable systems, giving rise, for instance, to the important structure of a "bi-hamiltonian system," which appears to encode the intuitive meaning of integrability for partial differential equations of evolutionary type.

If one is interested in the formal properties of evolution equations (*e.g.*, conservation laws, symmetries, recursion operators [15]) it is natural to use this second point of view. On the other hand, if one is interested in the canonical quantization of a dynamical system [21], or in understanding bifurcations and the structure of the space of solutions [12, 13] one most probably needs to possess a detailed understanding of the structure of the phase space of the system at hand.

How do we construct phase spaces in field theory? The most common approach to field theory starts with a Lagrangian. One can then use a method due to Dirac [8, 9] and extensively studied in the 1970's and 1980's [2, 4, 9] to obtain a physical phase space equipped – at least in the classical version of the method – with canonical variables reminiscent of the p's and q's of classical mechanics [5, 8]. By doing so, one loses covariance, a fact usually seen as an imperfection from a physical point of view [5, 14]. From a geometrical point of view, the description of the phase space through canonical coordinates appears incomplete while one would like to present it in an intrinsic, global fashion.

A way to repair these shortcomings, and to move from a Lagrangian point of view to a Hamiltonian picture, is to stay "in between" the formal versions of the Hamiltonian formalism [8, 15, 16], and its completely rigorous symplectic version [12]. One may attempt to obtain a covariant, coordinate-free description of the phase space, as a first step to a description of the dynamics *à la* Marsden, say, and also as a previous step to canonical quantization [21]. This was accomplished by G. Zuckerman – using formal arguments rooted in the rigorous theory of the variational bicomplex – in a beautiful and not so well-known paper [22] written in 1986. He appears to have been the first in explaining, in full generality, how to build phase spaces in a covariant way, using directly the Lagrangian and not going through Dirac's theory of constraints.

Several special cases of Zuckerman'a analysis appeared before [22], notably in Souriau's treatise [18], and in fact in the work of J. Lagrange himself (Mem. Cl. Sci. Math. Phys. Inst. France (1808) p. 1). Also of great interest are the paper by C. Crnković and E. Witten on the covariant description of phase spaces for

Yang-Mills and general relativity [5], the subsequent analysis by C. Crnković of a general first order Lagrangian theory, superstrings and a general first order theory in superspace [5], N. Woodhouse's discussion of first order Lagrangian field theory appearing in [21], and S. Sternberg's analysis of the formal variational calculus of Gel'fand and Dikii [19].

In this contribution we review some aspects of G. Zuckerman's fundamental paper [22]. This is indeed an exciting area of research, and much remains to be done! In fact, the original motivation for the present article was work by Y. Nutku [14], in which he uses ideas from [5] to study the Hamiltonian formulation of the important Monge-Ampère and Korteweg-de Vries equations. This paper is organized as follows: Section 2 is on symplectic and presymplectic manifolds following [17]. The variational bicomplex is studied in Section 3, and Zuckerman's construction is explained in Section 4. We finish in Section 5 with a simple example and a discussion on work to come.

The Einstein summation convention will be used throughout.

2. Hamiltonian Systems and Presymplectic Manifolds

We review the relation between Hamiltonian mechanics and what is called here *Souriau reduction*, that is, the understanding of the equations of motion as a (perhaps local) description of the leaves of a foliation of a presymplectic manifold [18]. The manifolds appearing in this section are all finite-dimensional unless otherwise explicitly stated. All maps, vector fields and tensors are assumed to be of class C^∞.

2.1. Presymplectic and Symplectic Manifolds

We begin with a two-form ω on a manifold M. We say that ω is a **presymplectic form** on M if it is closed and of constant rank on M. If the presymplectic form ω is non-degenerate, that is, if the rank of ω is equal to $\dim(M)$, we say that (M, ω) is a **symplectic manifold** and that ω is a **symplectic form** on M. From now on, the adjective "presymplectic" will be applied exclusively to closed two-forms of constant rank strictly less than $\dim(M)$.

A standard example of symplectic manifold is the cotangent bundle T^*M of an arbitrary manifold M [2]. In coordinates, if (q^i) is a coordinate chart on M, and $\alpha_q = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ is an element of T^*M, then $\omega_0 = \mathrm{d}q^i \wedge \mathrm{d}p_i$ is a symplectic form on T^*M. The symplectic manifold (T^*M, ω_0) is called the **canonical phase space** of the **configuration space** M.

If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, $\phi : M_1 \to M_2$ is a smooth symplectic mapping if $\phi^*\omega_2 = \omega_1$. If, in addition, ϕ is a diffeomorphism, then (M_1, ω_1) and (M_2, ω_2) are said to be **symplectomorphic**.

The local characterization of (pre)symplectic forms is given by Darboux theorem [2].

Theorem 1. *Suppose that ω is a non-degenerate two-form on a $2n$-dimensional manifold M. Then $\mathrm{d}\omega = 0$ if and only if for any $m \in M$ there exists a chart (U, ϕ) about m such that $\phi(m) = 0$ and*

$$\omega|_U = \mathrm{d}x^i \wedge \mathrm{d}y_i \tag{1}$$

in which $\phi|_U = (x^1, \ldots, x^n, y_1, \ldots, y_n)$. More generally, if (M, ω) is a $(2n+k)$-dimensional presymplectic manifold with $\mathrm{rank}(\omega) = 2n$, for each $m \in M$ there is a chart (U, ψ) about m such that

$$\omega|_U = \mathrm{d}q^i \wedge \mathrm{d}r_i$$

in which $\psi|_U = (q^1, \ldots, q^n, r_1, \ldots, r_n, w^1, \ldots, w^k)$.

2.2. Hamiltonian Systems

Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function on M. The triplet (M, ω, H) is called the **Hamiltonian system** on (M, ω) with **Hamiltonian function** H and phase space (M, ω). The **evolution** of the system is given by the flow of the vector field X_H uniquely determined by the equation

$$i_{X_H}\omega = \mathrm{d}H. \tag{2}$$

That Equation (2) does encode Hamilton's equations is a consequence of Darboux's result:

Proposition 1. *Let (M, ω) be a symplectic manifold and $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ canonical coordinates (i.e. given by Darboux's theorem) on M, and let $H : M \to \mathbb{R}$ be a smooth function on M. Then, the equation $i_{X_H}\omega = \mathrm{d}H$ implies that $X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\right)$. Thus $(q(t), p(t))$ is an integral curve of X_H if and only if*

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \ldots, n. \tag{3}$$

2.3. The Space of Motions

It is not always straightforward to find a symplectic description of a mechanical system [2, 7, 18]. As stated in the Introduction, it is not uncommon to consider systems described by a (singular) Lagrangian [8], and to find canonical formulations for them by means of the Dirac constraint algorithm [2, 4, 5, 8, 9, 14, 21]. The final result of this algorithm is a presymplectic manifold (M, ω). Given such a data, the corresponding phase space is constructed as follows:

For each $v \in M$ set $\ker_v \omega = \{Z_v \in T_v M\,;\, i_{Z_v}\omega = 0\}$, and define the distribution of vector spaces

$$\ker \omega = \bigcup_{v \in M} \ker_v \omega .$$

Since ω is of constant rank, the dimension of $\ker_v \omega$ is independent of v and $\ker \omega$ is a sub-bundle of the tangent bundle TM. Moreover, if Z, Y are vector fields on M such that $Z(v)$ and $Y(v)$ belong to $\ker_v \omega$ for all $v \in M$, then, $i_{[Z,Y]}\omega = L_Z(i_Y\omega) - i_Y(L_Z\omega) = 0 - i_Y(\mathrm{d}(i_Z\omega) + i_Z\mathrm{d}\omega) = 0$, and so $[Z,Y](v) \in \ker_v \omega$ for all $v \in M$. Frobenius' theorem [2, 13] then implies that the distribution $\ker \omega$ is integrable, that is, there exists a foliation $\Phi_\omega = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ of M satisfying $\ker \omega = T(M, \Phi_\omega)$, in which

$$T(M, \Phi_\omega) = \bigcup_{\alpha \in A} \bigcup_{m \in \mathcal{L}_\alpha} T_m \mathcal{L}_\alpha$$

is the tangent bundle of Φ_ω.

Definition 1. *Let (M, ω) be a presymplectic manifold. The* ***space of motions*** *U_M of (M, ω) is the set of leaves of the foliation Φ_ω, that is, $U_M = M/\ker \omega$.*

Henceforth, the presymplectic manifold (M, ω) will be also called the **evolution space** (M, ω). The procedure of constructing the corresponding space of motions U_M will be referred to as **Souriau reduction**, after the fundamental contributions to the subject made by J.-M. Souriau [18].

The space U_M is a manifold if and only if (see [10, 13, 21] and references therein) for every $v \in M$ there exists a local submanifold Σ_v of M such that Σ_v intersects every leaf of the foliation Φ_ω in at most one point (or nowhere), and $T_v\Sigma_v \oplus T_v(M, \Phi_\omega) = T_v M$. The submanifold Σ_v is called a **slice** or **local cross section** for Φ_ω. It follows that if U_M is a manifold, its dimension is equal to $\dim(M) - \dim(\ker \omega)$.

The next theorem, which goes back at least to Souriau [18], see [2, 7, 12, 13, 21], is the main result on this subject:

Theorem 2. *Let (M, ω) be a presymplectic manifold, and assume that the space of motions U_M is a manifold. Then, U_M can be equipped with a symplectic structure $\widetilde{\omega}$ such that $\pi^*\widetilde{\omega} = \omega$, in which $\pi : M \to M/\ker \omega$ is the canonical projection from M onto U_M.*

As stated above, it is not unusual to model a mechanical system on an evolution space (M, ω) [4, 8]. J.-M. Souriau [18] contends that the true phase space for the system is the symplectic manifold $(U_M, \widetilde{\omega})$ constructed in the last theorem. A very interesting application of this point of view is H. Künzle's discovery [10] of a presymplectic description of a spinning particle in a gravitational field: this paper appears to be the first deep physical application of Souriau reduction. Later,

S. Sternberg and his coworkers formulated a program to reduce classical mechanics to the construction of presymplectic manifolds and the corresponding spaces of motion [7].

We now explain the name "space of motions".

2.4. From Hamiltonian Systems to Presymplectic Manifolds and Back

Souriau's original discussion on the connection between the space of motions and Hamiltonian systems is in [18, p.128-132]. Here we follow [17].

Lemma 1. *Let (M, ω, H) be a Hamiltonian system on the symplectic manifold (M, ω). Define $N = M \times \mathbb{R}$, and set $\Omega = p_1^* \omega + (p_1^* \mathrm{d}H) \wedge (p_2^* \mathrm{d}t)$, in which $p_1 : N \to M$ and $p_2 : N \to \mathbb{R}$ are the canonical projection maps. Then, (N, Ω) is a presymplectic manifold.*

Indeed, it is elementary to check that for each $(m, t) \in N$, $\ker_{(m,t)} \Omega = \{\alpha X_H(m) + \alpha \frac{\partial}{\partial t};\ \alpha \in \mathbb{R}\}$, so that the dimension of $\ker_{(m,t)} \Omega$ is equal to 1 for all $(m, t) \in N$. Note that if $m(t)$ is an integral curve of X_H and one sets $n(t) = (m(t), t)$, then clearly $n'(t) \in \ker_{n(t)} \Omega$ for all t. Following Souriau [18], we *identify the motions of the system described by (M, ω, H) with the leaves of the foliation induced by the integrable distribution* $\ker \Omega$. Of course conversely, *after choosing charts*, we recover, up to parameterizations, the integral curves of X_H.

Lemma 2. *Let (M, ω, H) be a Hamiltonian system on (M, ω), and let (N, Ω) be the presymplectic manifold defined in* Lemma 1. *The integral curves of the Hamiltonian system (M, ω, X_H) can be obtained, up to parametrization, by projecting the leaves of the foliation Φ_Ω into M.*

These two lemmas imply the following result:

Proposition 2. *Identify the leaves of the foliation Φ_Ω with the integral curves of X_H. The space of motions $U_N = N/\ker \Omega$, in which N and Ω are defined in* Lemma 1, *is a manifold. Moreover, the original symplectic manifold (M, ω) and $(U_N, \tilde{\Omega})$ are symplectomorphic.*

Proof: The idea of the proof is that the foregoing discussion implies that $\mathcal{L} \in U_N$ can be described as a curve $(m(t), t)$, in which $m(t)$ is an integral curve of X_H. We can then define the map $\lambda : U_N \to M$ by $\lambda(\mathcal{L}) = m(0)$.

More rigorously, we use coordinates: the flow box theorem [2] says that for each $m \in M$ there exists an open set $U_m \subseteq M$ and a smooth map $F : U_m \times I \to M$, in which $I = (-a, a)$, $a > 0$ or $a = \infty$, such that for each $v \in U_m$, the curve $c_v : I \to M$ given by $c_v(s) = F(v, s)$ is the integral curve of X_H passing through v. Now, since the leaves of N through $u \in U_m$ are precisely the integral curves of

X_H, the submanifold $\Sigma = U_m \times \{0\}$ is a slice for the foliation Φ_Ω. Thus, U_N is a manifold, and for $(u, 0) \in \Sigma$, we simply define the function λ as the projection $\lambda(u, 0) = u$. This is of course a bijective smooth symplectic map. □

Thus, we are justified in considering the phase space of a dynamical system as the space of classical solutions of the system at hand. This observation is at the core of the generalization of Souriau's point of view to Lagrangian field theory [5, 21, 22]: in this infinite dimensional context, one *defines* the phase space of the theory precisely as the space of classical solutions to the equations of motion.

We can also start with an evolution space (M, ω) and build symplectic manifolds (P, Ω) containing (M, ω). A short discussion of this fact is in [17]. We now turn to field theory. We first review the construction of the variational bicomplex, following mainly [1].

3. The Variational Bicomplex

3.1. Geometry of Infinite Jets

Let $\pi : E \to M$ be a trivial fiber bundle in which M is the space of independent variables x^i, $1 \leq i \leq n$, and the typical fiber is the space of the dependent variables u^α, $1 \leq \alpha \leq m$. We also let $J^k E$, $k \geq 1$, be the bundle of k-jets of local sections of E.

The **infinite jet bundle** of E, $J^\infty E \to M$, is the inverse limit of the tower of jet bundles $M \leftarrow E \cdots \leftarrow J^k E \leftarrow J^{k+1} E \leftarrow \cdots$ under the standard projections $\pi^k_l : J^k E \to J^l E$, $k > l$. We denote by $\pi^\infty_k : J^\infty E \to J^k E$ the canonical projection from $J^\infty E$ onto $J^k E$. Locally, $J^\infty E$ is described by canonical coordinates $(x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1 i_2 \ldots i_k}, \ldots)$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, obtained from the standard coordinates on the finite-order jet bundles $J^k E$,

$$u^\alpha_{i_1}(j^k(s)(p)) = \frac{\partial s^\alpha}{\partial x^{i_1}}(p), \quad u^\alpha_{i_1 i_2}(j^k(s)(p)) = \frac{\partial^2 s^\alpha}{\partial x^{i_1} \partial x^{i_2}}(p), \quad \ldots \tag{4}$$

in which $p \in M$ and $j^k(s)$ is the k-jet of the local section $s : (x^i) \mapsto (x^i, s^\alpha(x^i))$ of E.

Any local section $s : (x^i) \mapsto (x^i, s^\alpha(x^i))$ of E lifts to a unique local section $j^\infty(s)$ of $J^\infty E$ called the **infinite prolongation** of s. In coordinates, $j^\infty(s)$ is the section

$$\left(x^i, s^\alpha(x^i), \frac{\partial s^\alpha}{\partial x^{i_1}}(x^i), \ldots, \frac{\partial^{i_1 + \cdots + i_k} s^\alpha}{\partial x^{i_1} \ldots \partial x^{i_k}}(x^i), \ldots \right).$$

A function $f : J^\infty E \to \mathbb{R}$ is smooth if it factors through a finite-order jet bundle, that is, if $f = f_k \circ \pi^\infty_k$, for some function $f_k : J^k E \to \mathbb{R}$.

A **vector field** X on $J^\infty E$ is a derivation on the ring of smooth functions on $J^\infty E$. In local coordinates, vector fields are formal series of the form

$$X = A_i \frac{\partial}{\partial x^i} + \sum_{\substack{k \geq 0 \\ 1 \leq i_1 \leq \cdots \leq i_k \leq n}} B^\alpha_{i_1 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_k}} \tag{5}$$

in which A_i, $B^\alpha_{i_1 \ldots i_k}$ are smooth functions on $J^\infty E$. We say that a vector field X given by (5) is π^∞_M-vertical if all the coefficients A_i vanish. Vector fields X on M can be canonically prolonged to vector fields $pr^\infty X$ on $J^\infty E$ by setting

$$pr^\infty X(j^\infty(s)(p)) \cdot f = X(p) \cdot (f \circ j^\infty(s)) \tag{6}$$

for smooth functions f on $J^\infty E$ and $p \in M$. This operation defines the **Cartan connection** $\mathcal{C}$ on $J^\infty E$: the horizontal lift of a vector field X on M, also called the **total derivative** in the X direction, is simply $pr^\infty X$. Locally, horizontal vector fields are linear combinations of the total derivatives D_j, in which $D_j = pr^\infty(\partial/\partial x^j)$, that is,

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u_{i_1 j} \frac{\partial}{\partial u^\alpha_{i_1}} + u_{i_1 i_2 j} \frac{\partial}{\partial u^\alpha_{i_1 i_2}} + \ldots . \tag{7}$$

The prolongation operation (6) satisfies $pr^\infty[X_1, X_2] = [pr^\infty X_1, pr^\infty X_2]$ for all vector fields X_1 and X_2 on M, and therefore the Cartan connection is flat.

Differential forms on $J^\infty E$ are the pull-backs of differential forms on $J^k E$ by the projections π^∞_k. Any differential k-form ω on $J^\infty E$ may be written in canonical coordinates as a finite linear combination of terms

$$A \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \wedge \mathrm{d}u^{\alpha_1}_{j_1 \ldots j_{p_1}} \wedge \cdots \wedge \mathrm{d}u^{\alpha_q}_{l_1 \ldots l_{p_q}} \tag{8}$$

in which $p + q = k$ and A is a smooth function on $J^\infty E$.

A differential form ω on $J^\infty E$ is called a **contact form** if $j^\infty(s)^* \omega = 0$ for all local sections s of E. The set of contact forms determines an ideal $\mathcal{I}$ in the ring of differential forms on $J^\infty E$. Locally, the **contact ideal** $\mathcal{I}$ is generated by the **basic contact one-forms**

$$\theta^\alpha_{i_1 \ldots i_k} = \mathrm{d}u^\alpha_{i_1 \ldots i_k} - \sum_j u^\alpha_{i_1 \ldots i_k j} \, \mathrm{d}x^j \tag{9}$$

for all $k \geq 0$, and it is not hard to check that the exterior derivative of $\theta^\alpha_{i_1 \ldots i_k}$ is given by $\mathrm{d}\theta^\alpha_{i_1 \ldots i_k} = \sum_j \mathrm{d}x^j \wedge \theta^\alpha_{i_1 \ldots i_k j}$. Contact forms are important because they provide a dual description of the Cartan connection, as the set of all vector fields X on $J^\infty E$ satisfying $i_X \omega = 0$ for all one-forms $\omega \in \mathcal{I}$.

3.2. The Variational Bicomplex

To define the variational bicomplex we bigrade the differential forms on $J^\infty E$: A p-form on $J^\infty E$ is *of type* (r,s), in which $r+s=p$, if $\omega(X_1,\dots X_p)=0$ whenever either a) more than s of the vector fields $X_1,\dots X_p$ are π^∞_M-vertical, or b) more than r of the vector fields $X_1,\dots X_p$ are horizontal. In coordinates, a p-form ω is of type (r,s) if it can be written as a finite sum of terms of the form

$$A\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_r}\wedge\theta^{\alpha_1}_{j_1\dots j_{p_1}}\wedge\dots\wedge\theta^{\alpha_s}_{l_1\dots l_{p_s}}. \tag{10}$$

Let $\Omega^p(J^\infty E)$ denote the set of p-forms on $J^\infty E$, and $\Omega^{r,s}(J^\infty E)$ denote the set of p-forms of type (r,s). Then,

$$\Omega^p(J^\infty E)=\bigoplus_{r+s=p}\Omega^{r,s}(J^\infty E).$$

The exterior derivative splits, $\mathrm{d}:\Omega^{r,s}(J^\infty E)\to\Omega^{r+1,s}(J^\infty E)\oplus\Omega^{r,s+1}(J^\infty E)$, and we can write $\mathrm{d}=\mathrm{d}_H+\mathrm{d}_V$, in which $\mathrm{d}_H:\Omega^{r,s}(J^\infty E)\to\Omega^{r+1,s}(J^\infty E)$, and $\mathrm{d}_V:\Omega^{r,s}(J^\infty E)\to\Omega^{r,s+1}(J^\infty E)$ are the *horizontal* and *vertical* exterior derivatives, respectively. The equation $\mathrm{d}^2=0$ implies that $\mathrm{d}_H^2=\mathrm{d}_V^2=0$ and $\mathrm{d}_H\mathrm{d}_V+\mathrm{d}_V\mathrm{d}_H=0$. In local coordinates, d_H and d_V are computed as follows:

$$\begin{aligned}
\mathrm{d}_H f &= \sum_i (D_{x^i}f)\mathrm{d}x^i && (11)\\
\mathrm{d}_V f &= \frac{\partial f}{\partial u^\alpha}\theta^\alpha+\frac{\partial f}{\partial u^\alpha_i}\theta^\alpha_i+\frac{\partial f}{\partial u^\alpha_{ij}}\theta^\alpha_{ij}+\dots && (12)\\
\mathrm{d}_H(\mathrm{d}x^i) &= 0,\quad \mathrm{d}_H\theta^\alpha_{i_1\dots i_k}=\sum_j \mathrm{d}x^j\wedge\theta^\alpha_{i_1\dots i_k j} && (13)\\
\mathrm{d}_V(\mathrm{d}x^i) &= 0,\quad \mathrm{d}_V\theta^\alpha_{i_1\dots i_k}=0. && (14)
\end{aligned}$$

Thus, for example, $\mathrm{d}_V x^i=0$ and $\mathrm{d}_V u^\alpha_{i_1\dots i_k}=\theta^\alpha_{i_1\dots i_k}$.

The **variational bicomplex** of E is the double complex $(\Omega^{*,*}(J^\infty E),\mathrm{d}_H,\mathrm{d}_V)$ of differential forms on the infinite jet bundle $J^\infty E$. In detail, writing $\Omega^{*,*}$ for $\Omega^{*,*}(J^\infty E)$, this important bicomplex looks like follows:

$$\begin{array}{ccccccccccccc}
 & & & & \vdots & & & & & & & & \vdots\\
 & & & & \uparrow \mathrm{d}_V & & \uparrow \mathrm{d}_V & & & & \uparrow \mathrm{d}_V & & \uparrow \mathrm{d}_V\\
 & & 0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{\mathrm{d}_H} & \Omega^{1,2} & \xrightarrow{\mathrm{d}_H} & \cdots & \xrightarrow{\mathrm{d}_H} & \Omega^{n-1,2} & \xrightarrow{\mathrm{d}_H} & \Omega^{n,2}\\
 & & & & \uparrow \mathrm{d}_V & & \uparrow \mathrm{d}_V & & & & \uparrow \mathrm{d}_V & & \uparrow \mathrm{d}_V\\
 & & 0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{\mathrm{d}_H} & \Omega^{1,1} & \xrightarrow{\mathrm{d}_H} & \cdots & \xrightarrow{\mathrm{d}_H} & \Omega^{n-1,1} & \xrightarrow{\mathrm{d}_H} & \Omega^{n,1}\\
 & & & & \uparrow \mathrm{d}_V & & \uparrow \mathrm{d}_V & & & & \uparrow \mathrm{d}_V & & \uparrow \mathrm{d}_V\\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{\mathrm{d}_H} & \Omega^{1,0} & \xrightarrow{\mathrm{d}_H} & \cdots & \xrightarrow{\mathrm{d}_H} & \Omega^{n-1,0} & \xrightarrow{\mathrm{d}_H} & \Omega^{n,0}
\end{array} \tag{15}$$

If the fiber bundle $E \to M$ is simply $\mathbb{R}^{m+n} \to \mathbb{R}^n$, all the sequences appearing in (15), both horizontal and vertical, are exact. This important result has been proven by several researchers, notably I. Anderson, L. Dickey, F. Takens, W. Tulczyjew, T. Tsujishita, and A. Vinogradov. Original references appear in [1, 22].

4. Hamiltonian Formalism for Lagrangian Field Theories

We fix a fiber bundle $\pi : E \to M$ and let $J^\infty E$ be the infinite jet bundle of E. The space $\Omega^{n,1}(J^\infty E)$ possesses a distinguished subspace $\mathcal{E}^{n+1}(E)$ of all source forms on $J^\infty E$: we say that a differential form $\omega \in \Omega^{n,1}(J^\infty E)$ is a **source form** if in any local system of coordinates (x^i, u^α) on E,

$$\omega = P_\beta(x^i, u^\alpha, u^\alpha_i, \dots, u^\alpha_{i_1\dots i_k})\, \mathrm{d}u^\beta \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

or, equivalently, if $\omega = P_\beta(x^i, u^\alpha, u^\alpha_i, \dots, u^\alpha_{i_1\dots i_k})\theta^\beta \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. An intrinsic characterization of the space of source forms is in the first two references of [1]. Their importance is due to the following

Lemma 3 ([1, 22]). *Assume that $\omega \in \Omega^{n,1}(J^\infty E)$. Then, ω can be written uniquely as*

$$\omega = \omega_1 + \mathrm{d}_H \eta \tag{16}$$

in which $\omega_1 \in \mathcal{E}^{n+1}(E)$ is a source form and $\eta \in \Omega^{n-1,1}(J^\infty E)$.

Suppose now that we fix a Lagrangian density $\lambda \in \Omega^{n,0}(J^\infty E)$, so that in local coordinates (x^i, u^α), $\lambda = L(x^i, u^\alpha, u^\alpha_i, \dots)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, see [1, 16]. The vertical exterior derivative $\mathrm{d}_V\lambda$ belongs to $\Omega^{n,1}(J^\infty E)$, and the last lemma implies that one can write

$$\mathrm{d}_V \lambda = E(\lambda) + \mathrm{d}_H \eta \tag{17}$$

uniquely, in which $E(\lambda)$ is a source form (essentially the Euler-Lagrange operator evaluated at L, that is, $E(\lambda) = E_\alpha(L)\mathrm{d}u^\alpha \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, in which

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u^\alpha_i}\right) + D_i D_j \left(\frac{\partial L}{\partial u^\alpha_{ij}}\right) - \cdots$$

see [1, 16]) and $\eta \in \Omega^{n-1,1}(J^\infty E)$. We define $U(\lambda) \in \Omega^{n-1,2}(J^\infty E)$ by

$$U(\lambda) = \mathrm{d}_V \eta. \tag{18}$$

The differential form $U(\lambda)$ is Zuckerman's universal current [1, 22]. We observe that $\mathrm{d}_V U(\lambda) = 0$, and that, less trivially, $\mathrm{d}_H U(\lambda)$ vanishes on solutions $u^\alpha(x^i)$ to the Euler-Lagrange equations. Indeed, on solutions to $E_\alpha(L) = 0$, equation (17) becomes $\mathrm{d}_V\lambda = \mathrm{d}_H\eta$, and therefore

$$0 = \mathrm{d}_V \mathrm{d}_V \lambda = \mathrm{d}_V \mathrm{d}_H \eta = -\mathrm{d}_H \mathrm{d}_V \eta = -\mathrm{d}_H U(\lambda). \tag{19}$$

After G. Zuckerman [22], we say that the differential $(n-1,2)$-form $U(\lambda)$ is a conserved current for the Euler-Lagrange equations $E_\alpha(L)=0$:

Definition 2. *Fix a form $\lambda \in \Omega^{n,0}(J^\infty E)$ as above. A differential form $K \in \Omega^{n-1,q}(J^\infty E)$, $q \geq 0$, is a conserved current for the Euler-Lagrange equations $E_\alpha(L)=0$ if*

$$\mathrm{d}_H K = 0$$

whenever $u^\alpha(x^i)$ is a solution to the equations $E_\alpha(L)=0$.

The conserved currents of Definition 2 are a generalization of the standard conservation laws of field theory. Usually conservation laws are defined as differential forms $K \in \Omega^{n-1,0}(J^\infty E)$ which are closed on solutions, see [1, 2, 16] and references therein. The importance of these new conserved currents, also called *higher-degree or form-valued conservation laws,* has been recognized only recently [1].

Of course, Definition 2 extends to arbitrary systems of partial differential equations. Thus, for example, it is a straightforward exercise to check that the canonical symplectic form $\omega_0 = \mathrm{d}q^i \wedge \mathrm{d}p_i$ is a form-valued conservation law for Hamilton's equations.

The following definition replaces the space of motions of Section 2:

Definition 3. a) *The solution variety S_L associated with a Lagrangian L is the set of all local smooth sections*

$$\psi : (x^i) \mapsto (x^i, u^\alpha(x^i))$$

of the bundle E such that $u^\alpha(x^i)$ is a solution to the Euler-Lagrange equations $E_\alpha(L)=0$.

b) *For each $\psi \in S_L$, the tangent space $T_\psi S_L$ at ψ is the set of all vector fields*

$$\delta\psi = G^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1} G^\alpha \frac{\partial}{\partial u^\alpha_{i_1}} + D_{i_1} D_{i_2} G^\alpha \frac{\partial}{\partial u^\alpha_{i_1 i2}} + \dots \qquad (20)$$

on the infinite jet bundle $J^\infty E$ such that $G^\alpha(\psi)$ (the pull-back of G^α by the section $\psi \in S_L$) satisfy the Jacobi equations, that is, the linearization of the Euler-Lagrange equations at ψ.

Zuckerman's main result [22] is the following:

Theorem 3. *For any Lagrangian density $\lambda = L\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \in \Omega^{n,0}(J^\infty E)$, consider the associated differential forms $\eta \in \Omega^{n-1,1}(J^\infty E)$ and $U(\lambda) \in \Omega^{n-1,2}(J^\infty E)$ defined in* (17) *and* (18)*, respectively. Then, $U(\lambda)$ is a conserved current for the Euler-Lagrange equations $E_\alpha(L)=0$. Moreover,*

a) *Suppose that C is a compact, oriented $(n-1)$-dimensional submanifold of M. Define differential forms θ_C and ω_C on S_L as follows: For any solution $\psi \in S_L$*

and any two vectors $\delta_1\psi, \delta_2\psi \in T_\psi S_L$,

$$\theta_C(\psi)\cdot\delta_1\psi = \int_C \psi^*(i_{\delta_1\psi}\eta) \quad \textit{and} \quad \omega_C(\psi)\cdot(\delta_1\psi,\delta_2\psi) = \int_C \psi^*(i_{\delta_1\psi}i_{\delta_2\psi}U(\lambda)). \tag{21}$$

Then, the one-form θ_C *and the two-form* ω_C *satisfy* $\omega_C = \mathrm{d}\,\theta_C$ *and* $\mathrm{d}\,\omega_C = 0$.
b) *The two-form* ω_C *does not depend on the submanifold* C.

The two-form ω_C is the (pre)symplectic form *on the space of solutions* S_L we were trying to obtain. The important question of when ω_C is in fact symplectic will not be treated here. It is briefly considered in the last work of [5], in [19] and in [22]. Some general remarks on this issue can also be found in [20].

5. An Example

We consider the simple example of a Lagrangian density of the form [20, 21]

$$\lambda = L(x_i, u^\alpha, u_i^\alpha)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n.$$

Set $\nu = \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$, and $\nu_i = (-1)^i\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{i-1}\wedge\mathrm{d}x^{i+1}\wedge\cdots\wedge\mathrm{d}x^n$. One has

$$\mathrm{d}_V\lambda = \left(\frac{\partial L}{\partial u^\alpha}\theta^\alpha + \frac{\partial L}{\partial u_i^\alpha}\theta_i^\alpha\right)\wedge\nu = E_\alpha(L)\theta^\alpha\wedge\nu + \left[\frac{\partial L}{\partial u_i^\alpha}\theta_i^\alpha + D_i\left(\frac{\partial L}{\partial u_i^\alpha}\right)\theta^\alpha\right]\wedge\nu$$

and, on the other hand, one easily computes

$$\mathrm{d}_H\left(\frac{\partial L}{\partial u_i^\alpha}\theta^\alpha\wedge\nu_i\right) = \left[\frac{\partial L}{\partial u_i^\alpha}\theta_i^\alpha + D_i\left(\frac{\partial L}{\partial u_i^\alpha}\right)\theta^\alpha\right]\wedge\nu.$$

Thus, $\mathrm{d}_V\lambda$ can be written as $\mathrm{d}_V\lambda = E_\alpha(L)\theta^\alpha\wedge\nu + \mathrm{d}_H\eta$ in which

$$\eta = \frac{\partial L}{\partial u_i^\alpha}\theta^\alpha\wedge\nu_i. \tag{22}$$

The "presymplectic potential" θ_C defined in equation (21) now reads

$$\theta_C(\psi)\cdot\delta\psi = \int_C \psi^*(i_{\delta\psi}\eta) = \int_C \frac{\partial L}{\partial u_i^\alpha}G^\alpha\nu_i$$

in which $\delta\psi$ is given by (20). This formula, found here from general principles, coincides with the ones Woodhouse [21, p. 132] and Crnković [5] found by formal manipulations. The corresponding (pre)symplectic form ω_C becomes

$$\begin{aligned}\omega_C(\psi)\cdot(\delta_1\psi,\delta_2\psi) = \int_C \psi^*(i_{\delta_1\psi}i_{\delta_2\psi}U(\lambda)) = \int_C & \frac{\partial^2 L}{\partial u^\beta\partial u_i^\alpha}[G_1^\beta G_2^\alpha - G_2^\beta G_1^\alpha] \\ & + \frac{\partial^2 L}{\partial u_j^\beta\partial u_i^\alpha}[(D_jG_1^\beta)G_2^\alpha - (D_jG_2^\beta)G_1^\alpha]\end{aligned}$$

in which $\delta\psi_1$ and $\delta\psi_2$ are vectors in $T_\psi S_L$ as in (20).

Finally, we would like to remark that even though some work in the area has appeared in the literature (see for example [14, 20]) no complete, rigorous exposition of Zuckerman's ideas seems to be available (except for a review of [22] by P. Deligne and D. Freed [6]). Also, as pointed out by Nutku [14], the Lagrangian methods do not much appear in integrable systems, the exception being the intriguing paper [19] by S. Sternberg. We wonder if the analysis of [19] can be generalized to other integrable hierarchies, and also if Zuckerman's approach can be related to the Hamiltonian operators of [15, 16].

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References

[1] Anderson I., *Introduction to the Variational Bicomplex*, Contemporary Mathematics, **132** (1992) 51–73; *The Variational Bicomplex*, Utah State University Technical Report, 1989. (`http://www.math.usu.edu/~fg_mp/Pages/Publications/Publications.html`);
Anderson I. and Kamran N., *Conservation Laws and the Variational Bicomplex for Second-Order Scalar Hyperbolic Equations in the Plane*, Acta Applicandae Mathematicae **41** (1995) 135–144.

[2] Abraham R. and Marsden J., *Foundations of Mechanics*, Second Edition, Addison-Wesley, Reading, MA, 1978.

[3] Bacry H., *Space-Time and Degrees of Freedom of the Elementary Particle*, Commun. Math. Phys. **5** (1967) 97–105.

[4] Cariñena J. and López C., *Symplectic Structure on the Set of Geodesics of a Riemannian Manifold*, Int. J. Mod. Phys. **A 6** (1991) 431–444.

[5] Crnković C., *Symplectic Geometry and (Super-)Poincare Algebra in Geometrical Theories*, Nuclear Physics **B 288** (1987) 419–430;
Symplectic Geometry of the Covariant Phase Space, Class. Quantum Grav. **5** (1988) 1557–1575;
Crnković C. and Witten E., *Covariant Description of Canonical Formalism in Geometrical Theories*. In: Three hundred years of gravitation, S. Hawking and W. Israel (Eds.), Cambridge, 1987, pp 676–684.

[6] Deligne P. and Freed D., *Classical Field Theory*. In: Quantum Fields and Strings: A Course for Mathematicians, P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison and E. Witten (Eds.), American Mathematical Society and the Institute for Advanced Studies, Providence and Princeton, 1999, pp 137–225.

[7] Guillemin V. and Sternberg S., *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[8] Hanson A., Regge T. and Teitelboim C., *Constrained Hamiltonian Systems*, Accad. Naz. dei Lincei, Rome, 1976.

[9] Henneaux M and Teitelboim C., *Quantization of Gauge Systems*, Princeton University Press, Princeton, New Jersey, 1992.

[10] Künzle H., *Canonical Dynamics of Spinning Particles in Gravitational and Electromagnetic Fields*, J. Math. Phys. **13** (1972) 739–744.

[11] Marle C., *Sous-variétés des rang constant et sous-variétés symplectiquement régulières d'une variété symplectique*, C. R. Acad. Sc. Paris, Série I **295** (1982) 119–122.

[12] Marsden J., *Lectures on Geometric Methods in Mathematical Physics*, SIAM, Philadelphia, 1981.

[13] Marsden J. and Ratiu T., *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems.* Second Edition, Springer, New York, 1999.

[14] Nutku Y., *Hamiltonian Structure of Real Monge–Ampère Equations*, J. Phys. A: Math. & Gen. **29** (1996) 3257–3280; *Lagrangian Approach to Integrable Systems Yields New Symplectic Structures for KdV.* In: Integrable Hierarchies and Modern Physical Theories, H. Aratyn and A. Sorin (Eds.), Kluwer, Netherlands, 2001, pp 203–213.

[15] Olver P., *Recursion Operators and Hamiltonian Systems.* In: Symmetries and Nonlinear Phenomena. Paipa, Colombia 1988. D. Levi and P. Winternitz, (Eds), World Scientific, Singapore, 1989, pp 222–249.

[16] Olver P., *Applications of Lie Groups to Differential Equations*, Second Edition, Springer, New York, 1993.

[17] Reyes E., *On the Motion of Particles and Strings, Presymplectic Mechanics and the Variational Bicomplex*, Preprint, 2003.

[18] Souriau J.-M., *Structure des Systemes Dynamiques*, Dunod, Paris, 1970. (English Translation: *Structure of Dynamical Systems: A Symplectic View of Physics*, Translated by C. Cushman-de Vries and with a preface by R. Cushman and G. Tuynman, Translation Eds., Birkhäuser, Boston, MA, 1997).

[19] Sternberg S., *Some Preliminary Remarks on the Formal Variational Calculus of Gel'fand and Dikii.* In: Differential Geometrical Methods in Mathematical Physics (Bonn 1977), K. Bleuler, H. Petry and A. Reetz (Eds.), LNM vol. **676**, Springer, Berlin, 1978, pp 399–407.

[20] Torre C., *Covariant Phase Space Formulation of Parametrized Field Theories*, J. Math. Phys. **33** (1992) 3802–3812.

[21] Woodhouse N., *Geometric Quantization*, Second Edition, Oxford Mathematical Monographs, Oxford, 1992.

[22] Zuckerman G., *Action Principles and Global Geometry.* In: Mathematical Aspects of String Theory (San Diego, CA, 1986), S. Yau (Ed.), World Scientific, Singapore, 1987, pp 259–284.