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CONFORMAL IMMERSIONS OF DELAUNAY SURFACES AND THEIR DUALS

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Abstract. A few explicit formulas providing conformal coordinates of the axially symmetric constant mean curvature surfaces introduced by Delaunay and their duals are derived. These results give also new examples in a long line of research connected with finding isothermic immersions of surfaces and their duals.

1. Introduction

Let us assume that the parametrized surface $\mathcal{S}$ is (locally) an image of the immersion

$$(u, v) \longrightarrow \mathbf{x}[u, v] = (\mathrm{x}(u, v), \mathrm{y}(u, v), \mathrm{z}(u, v)) \tag{1}$$

defined on an open set $\mathcal{D} \subset \mathbb{R}^2$. In these coordinates the pullback of the Riemannian metric on $\mathcal{S}$ can be expressed (using the standard notation) in the form

$$I = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \tag{2}$$

which is known as the first fundamental form of $\mathcal{S}$. The coefficients in I are given by

$$E = \mathbf{x}_u \,.\, \mathbf{x}_u, \qquad F = \mathbf{x}_u \,.\, \mathbf{x}_v, \qquad G = \mathbf{x}_v \,.\, \mathbf{x}_v.$$

One has to notice that these three functions determine completely the Riemannian structure of $\mathcal{S}$, but that they are not determined by it. For we can apply a diffeomorphic change of coordinates $u = u(\tilde{u}, \tilde{v})$, $v = v(\tilde{u}, \tilde{v})$ in order to obtain an isometric structure which is actually the same. Then the new coefficients $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ can be easily found by plugging in the expressions for

$$\mathrm{d}u = u_{\tilde{u}}\,\mathrm{d}\tilde{u} + u_{\tilde{v}}\,\mathrm{d}\tilde{v} \qquad \text{and} \qquad \mathrm{d}v = v_{\tilde{u}}\,\mathrm{d}\tilde{u} + v_{\tilde{v}}\,\mathrm{d}\tilde{v}$$

into the standard notation given above. Using the language of the quadratic forms and their associated matrices, we have

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \text{and} \qquad \tilde{\mathcal{F}}_I = \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}$$

which are not independent but obey to the simple relation

$$\tilde{\mathcal{F}}_I = \mathcal{J}^t \mathcal{F}_I \mathcal{J} \qquad \text{where} \qquad \mathcal{J} = \begin{pmatrix} \dfrac{\partial u}{\partial \tilde{u}} & \dfrac{\partial v}{\partial \tilde{u}} \\ \dfrac{\partial u}{\partial \tilde{v}} & \dfrac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

denotes the Jacobian matrix and $\mathcal{J}^t$ means the transposed matrix.

This setting suggest that one has to try to find the most appropriate coordinates in which the expressions for E, F and G are as simple as possible. This task is greatly facilitated by a knowledge of both the intrinsic and extrinsic geometry of $\mathcal{S}$, e.g., if the set of coordinates is such that $F \equiv 0$, i.e.,

$$I = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2 \tag{3}$$

this system of coordinates is called **orthogonal**. If, in addition, $E \equiv G = \lambda(u, v) > 0$, i.e.

$$I = \lambda(u, v)(\mathrm{d}u^2 + \mathrm{d}v^2) \tag{4}$$

the coordinate system is called **conformal**, because the angle between any two directions on the surface $\mathcal{S}$ is equal to angle of their pre-images in the Euclidean plane (u, v). One has to notice that orthogonal coordinates can always be found and that this is almost a trivial task. It can be proven also that conformal coordinates exist always as well, but it is a much harder problem to find them in explicit form as this often leads to evaluation of (hyper)elliptic integrals (see Kamberov *et al* [8] for more details).

In what follows we will derive such coordinates for the constant mean curvature surfaces of revolution in the Euclidean space $\mathbb{R}^3$ which are known in the literature as Delaunay surfaces.

2. Delaunay Surfaces

In this Section we present the result of the classification theory of constant mean curvature surfaces of revolution – the Delaunay surfaces, and some of their properties by which we can recognize them. Before this, let us recall that the **mean curvature** of a surface in $\mathbb{R}^3$ is defined via the coefficients of its first and second fundamental forms. Computing the latter requires choosing a unit normal vector field $\mathbf{n}[u,v]$ to the surface. Assuming that $\mathbf{x}_u \times \mathbf{x}_v$ never vanishes on $\mathcal{D}$, our choice for this normal vector $\mathbf{n}$ to $\mathcal{S}$ from now on will be

$$\mathbf{n} = \mathbf{n}[u,v] = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}. \tag{5}$$

In this setting the coefficients of the second fundamental form

$$II = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2 \tag{6}$$

of the parametrized surface (1) with a fixed unit normal vector $\mathbf{n}[u,v]$ are given by

$$L = L[u,v] = \mathbf{x}_{uu}.\mathbf{n}, \quad M = M[u,v] = \mathbf{x}_{uv}.\mathbf{n}, \quad N = N[u,v] = \mathbf{x}_{vv}.\mathbf{n}.$$

Making use again of the matrix notation, we have

$$\mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \tag{7}$$

and it turns out convenient to introduce also

$$\mathcal{W} := \mathcal{F}_I^{-1}\mathcal{F}_{II} \tag{8}$$

which is known as a **shape operator** or **Weingarten map**. The most important characteristics of $\mathcal{S}$, i.e. its **Gauss** and **mean** (meaning "average") **curvatures**, denoted respectively by K and H, can be easily expressed via the invariants of $\mathcal{W}$. Namely, the formulas are

$$K = \det(\mathcal{W}) = \frac{LN - M^2}{EG - F^2} \text{ and } H = \frac{1}{2}\mathrm{trace}(\mathcal{W}) = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{9}$$

Surfaces of Delaunay were originally defined in [2] as surfaces obtained by revolving profile curves which themselves arise from rolling conics on a line. Such curves are called *roulettes* of conics. In an Appendix to the same paper Sturm characterizes Delaunay's surfaces variationally as those surfaces of revolution having a minimal lateral area at a fixed volume. That in turn revealed why these surfaces make their appearance as soap bubbles and liquid drops [3, 6, 12] or cells under compression [16]. Modern expositions of the classical differential-geometric construction along the variational viewpoint can be found in [12] and [13]. Using entirely different methods the same problem has been treated by Kenmotsu [9, 10] and Konopelchenko and Taimanov [11].

The complete list of Delaunay surfaces is provided by: planes, spheres, catenoids, cylinders, nodoids and unduloids. The generating curves of a nodoid and an unduloid, called respectively the nodary and the undulary are periodic along the symmetry axis and have one local minimum and one local maximum in each period. If X is the surface symmetry axis Delaunay [2] had found (up to integration) the following parametrization in the XOZ plane of these two curves

$$x = a\sin\varphi - a\tan\varphi\sqrt{1+\alpha-\sin^2\varphi} + \int_0^{\varphi} \frac{a\alpha\,\mathrm{d}\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}} + \beta$$

$$z = -a\cos\varphi + a\sqrt{1+\alpha-\sin^2\varphi}, \qquad \varphi\in\mathbb{R}. \tag{10}$$

Here $a \neq 0$, $\alpha > -1$ and β – arbitrary, are three real constants. The mean curvature H of the surfaces obtained by rotating any of the above curves is $1/2a$. The calculation shows also that the coefficients of their first fundamental forms are

$$E = \frac{a^2\left(\sqrt{\alpha+\cos^2\varphi}-\cos\varphi\right)^2}{\sqrt{2}}\left(\alpha+\cos^2\varphi\right), \qquad F=0$$

$$G = a^2\left(\alpha - 2\cos\varphi\left(\sqrt{\alpha+\cos^2\varphi}-\cos\varphi\right)\right)$$

and makes obvious that the coordinate nets are orthogonal but not conformal. Besides, one has to notice that Delaunay had derived the above parametrization by finding first the corresponding evolute and then the generating curve itself. This means that these solutions exist only on restricted intervals on which the evolute can be found.

3. Conformal Parametrizations

Now, we concentrate on the central subject of this paper and introduce the rotational surface

$$\mathbf{x}[u,v] = \left(\mathrm{e}^{\sigma(u)}\cos v, \mathrm{e}^{\sigma(u)}\sin v, \int_0^u \mathrm{e}^{\sigma(t)}\sin\Omega(t)\,\mathrm{d}t\right). \tag{11}$$

In this coordinate chart the respective coefficients of the I and II fundamental forms are

$$E = \mathrm{e}^{2\sigma(u)}, \quad F=0, \quad G=\mathrm{e}^{2\sigma(u)} \tag{12}$$

and

$$L = -\mathrm{e}^{\sigma(u)}\Omega'(u), \quad M=0, \quad N=-\mathrm{e}^{\sigma(u)}\sin\Omega(u) \tag{13}$$

where $\Omega(u)$ denotes the polar angle of the unit normal vector to the surface. Furthermore, the above coefficients are not independent of each other but must satisfy some compatibility conditions known as Codazzi-Mainardi-Peterson relations and the Gauss equation. The former ones turn out to be satisfied automatically and the latter one is just the celebrated Gauss' *Theorema Egregium* which in our conformal coordinates $E=G=\lambda=\mathrm{e}^{2\sigma(u)}$ states that

$$K = -\frac{1}{2\lambda}\left[\left(\frac{\lambda_u}{\lambda}\right)_u + \left(\frac{\lambda_v}{\lambda}\right)_v\right] = -\frac{1}{2\lambda}\Delta(\log\lambda) = -\sigma''(u)\mathrm{e}^{-2\sigma(u)} \tag{14}$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the standard Laplacian in the parametric plane $\mathbb{R}^2$. On the other hand, the first formula in (9) yields

$$K = \sin\Omega(u)\Omega'(u)\mathrm{e}^{-2\sigma(u)} \tag{15}$$

and the comparison of these results allows us to conclude that the integrability conditions reduces to the equation

$$\sigma'(u) = \cos\Omega(u). \tag{16}$$

The second formula in (9) now gives

$$H = -\frac{1}{2}(\sin\Omega(u) + \Omega'(u))\mathrm{e}^{-\sigma(u)} \tag{17}$$

from which it immediately follows that the class of minimal surfaces of revolution, i.e. those surfaces for which $H \equiv 0$ is singled out by the equation

$$\Omega'(u) = -\sin\Omega(u). \tag{18}$$

This is an equation in which the variables are separated, i.e.,

$$\frac{\mathrm{d}\Omega}{\sin\Omega} = -\mathrm{d}u. \tag{19}$$

and can be easily integrated by making use of the substitution $\cos\Omega(u) = \frac{1-t^2}{1+t^2}$, so one gets

$$\cos\Omega(u) = \frac{\operatorname{sh} u}{\operatorname{ch} u} = \operatorname{th} u \tag{20}$$

and

$$\sin\Omega(u) = (1 - \operatorname{th}^2 u)^{1/2} = \frac{1}{\operatorname{ch} u} = \operatorname{sech} u. \tag{21}$$

Furthermore, one has also

$$\sigma(u) = \int \cos\Omega(u)\,\mathrm{d}u = \int \operatorname{th} u\,\mathrm{d}u = \ln\operatorname{ch} u \tag{22}$$

where the integration constant is omitted because after exponentiation of $\sigma(u)$, i.e.,

$$\mathrm{e}^{\sigma(u)} = \operatorname{ch} u \tag{23}$$

it will contribute just to the scale factor which is not interesting at this moment for our consideration.

Finally,

$$z'(u) = \mathrm{e}^{\sigma(u)}\sin\Omega(u) = \operatorname{ch} u\operatorname{sech} u = 1 \tag{24}$$

and therefore

$$z(u) = u \tag{25}$$

where the new integration constant is omitted again as this time its meaning is nothing other than a translation along the symmetry axis. Combined, all this above gives us the surface

$$\mathbf{x}[u, v] = (\operatorname{ch} u \cos v, \operatorname{ch} u \sin v, u) \tag{26}$$

that can be immediately recognized as a catenoid (the surface on the right-hand side in Fig. 1). A classical theorem in differential geometry (cf. [4, 12, 13, 14])

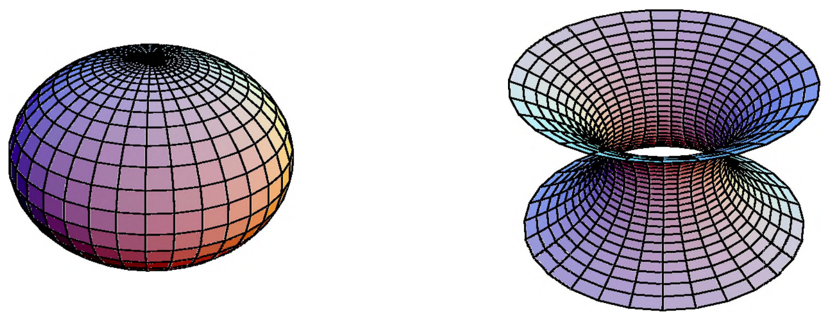

Figure 1. Sphere and its dual surface (the catenoid) on the right side

says that this is the only minimal surface of revolution which is obtained this time by revolving the catenary $x = \operatorname{ch} z$ around the OZ axis.

In the rest of this paper we will concentrate on the more general and interesting case of rotational surfaces for which

$$H = \frac{1}{2a}, \qquad a = \text{constant} \neq 0. \tag{27}$$

From (17) it is clear that the condition $H \equiv$ constant is equivalent to the equation

$$\Omega''(u) = \sin\Omega(u)\cos\Omega(u) \tag{28}$$

which can be easily integrated to the first order second degree equation

$$(\Omega'(u))^2 = \sin^2\Omega(u) + A, \qquad A = \text{constant}\,. \tag{29}$$

The character of the solution of the last equation is strongly influenced by the sign of the constant in (29) so we shall consider separately each of the three possibilities

$$A = -m^2, \quad 0 \quad \text{and} \quad m^2.$$

Let us start with the first case $A \equiv -m^2$, so that we have

$$(\Omega'(u))^2 = \sin^2\Omega(u) - m^2. \tag{30}$$

Introducing as a new variable

$$\xi = \sin\Omega(u), \qquad m^2 < \xi^2 < 1 \tag{31}$$

equation (30) can be rewritten in the form

$$\xi' = -\sqrt{(1-\xi^2)(\xi^2 - m^2)} \tag{32}$$

and integrated once more. This can be done by making use of the Jacobi's elliptic function $\mathrm{dn}(u,k)$ where the elliptic modulus k is related to m via an identification of the latter with the complementary elliptic modulus $\tilde{k}$, i.e. $m \equiv \tilde{k}$

$$\xi = \mathrm{dn}(u,k), \qquad k^2 = 1 - m^2 = 1 - \tilde{k}^2. \tag{33}$$

It is a matter of an easy check to establish that

$$\Omega(u) = \pi - \arcsin(\mathrm{dn}(u,k)) \tag{34}$$

satisfies (28) and in conjunction with (16) one has also

$$\sigma'(u) = \cos\Omega(u) = k\,\mathrm{sn}(u,k). \tag{35}$$

Now we have to integrate the Jacobian function $\mathrm{sn}(u,k)$ which turns out to be an elementary integral. In fact, the composition of substitutions $\mathrm{sn}(u,k) = \xi$ and $\xi^2 = t$ has as a result

$$\begin{aligned} \int \mathrm{sn}(u,k)\,\mathrm{d}u &= \int \frac{\xi\,\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\ &= \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{(1-t)(1-k^2 t)}} \\ &= k^{-1}\ln(\sqrt{1-k^2 t} + k\sqrt{1-t}). \end{aligned} \tag{36}$$

Going back to the original variable $t = \mathrm{sn}^2(u,k)$ gives

$$\sigma(u) = \ln(\mathrm{dn}(u,k) + k\,\mathrm{cn}(u,k)) \tag{37}$$

which after exponentiation produces

$$\mathrm{e}^{\sigma(u)} = \mathrm{dn}(u,k) + k\,\mathrm{cn}(u,k). \tag{38}$$

Finally, the integration of

$$z'(u) = \mathrm{e}^{\sigma(u)}\sin\Omega(u) = (\mathrm{d}n(u,k) + k\,\mathrm{cn}(u,k))\,\mathrm{dn}(u,k) \tag{39}$$

gives us

$$z(u) = E(\mathrm{am}(u,k),k) + k\,\mathrm{sn}(u,k) \tag{40}$$

where $E(\zeta,k)$ denotes the incomplete elliptic integral of the second kind and $\mathrm{am}(u,k)$ is the so called Jacobi amplitude function. For a straightforward exposition and properties of the elliptic functions and integrals see [5] and [7]. Actually, this completes the consideration of the first case as the input data required by (11) for building the surface are provided by (38) and (40). The surface itself is the unduloid depicted on the left side in Fig. 2.

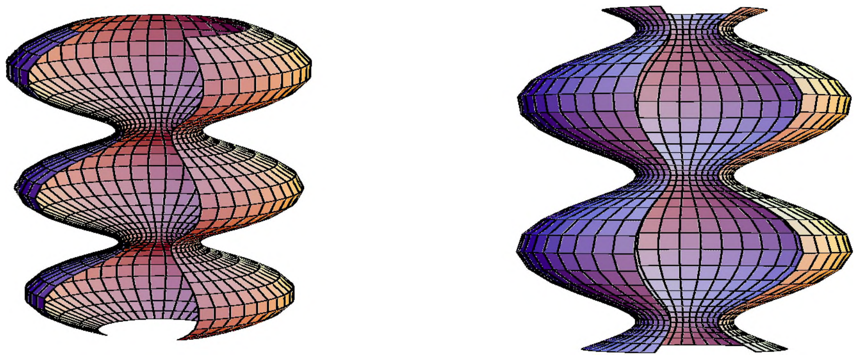

Figure 2. Some open parts of the unduloid and its dual surface on the right side for $k = 0.555556$

As the treatment of the other two cases is quite similar we will fix only the main points. So, let us continue and consider the next possibility

$$A \equiv 0$$

which means that we have to solve

$$\Omega'(u) = \sin \Omega(u).$$

As the last equation differs from (19) by a sign, we can write immediately

$$\cos \Omega(u) = -\operatorname{th} u, \quad \sin \Omega(u) = \operatorname{sech} u, \quad \Omega(u) = \pi - \arcsin(\operatorname{sech} u) \tag{41}$$

and furthermore

$$\sigma(u) = -\ln \operatorname{ch} u, \quad e^{\sigma(u)} = \operatorname{sech} u \quad \text{and} \quad z(u) = \operatorname{th} u. \tag{42}$$

So, this settles the second case and the surface which we have obtained is the ordinary sphere shown on the left-hand side in Fig. 1.

Finally, let us consider the last case when $A \equiv m^2 > 0$. Now,

$$(\Omega'(u))^2 = \sin^2 \Omega(u) + m^2 \tag{43}$$

and the substitution $\xi = \sin \Omega(u)$ converts the above equation into

$$\xi' = \sqrt{(1 - \xi^2)(\xi^2 + m^2)}. \tag{44}$$

It can be integrated again via the Jacobian elliptic functions, this time using

$$\xi = \operatorname{cn}(u/k, k) \quad \text{where} \quad k^2 = \frac{1}{1 + m^2} \quad \text{and} \quad \tilde{k}^2 = \frac{m^2}{1 + m^2}. \tag{45}$$

From all of the above we have also

$$\Omega(u) = \pi - \arcsin(\operatorname{cn}(u/k,k)) \tag{46}$$

and consequently

$$\begin{aligned}
\sigma'(u) &= \cos\Omega(u) = \operatorname{sn}(u/k,k)\\
\sigma(u) &= \ln(\operatorname{dn}(u/k,k) + k\operatorname{cn}(u/k,k))\\
\mathrm{e}^{\sigma(u)} &= \operatorname{dn}(u/k,k) + k\operatorname{cn}(u/k,k) \qquad (47)\\
z'(u) &= \mathrm{e}^{\sigma(u)}\sin\Omega(u) = (\operatorname{dn}(u/k,k) + k\operatorname{cn}(u/k,k))\operatorname{cn}(u/k,k)\\
z(u) &= k\operatorname{sn}(u/k,k) + E(\operatorname{am}(u/k,k),k) - \frac{1-k^2}{k}u.
\end{aligned}$$

The surface generated by these data is just the nodoid mention above and drawn on the left side in Fig. 3.

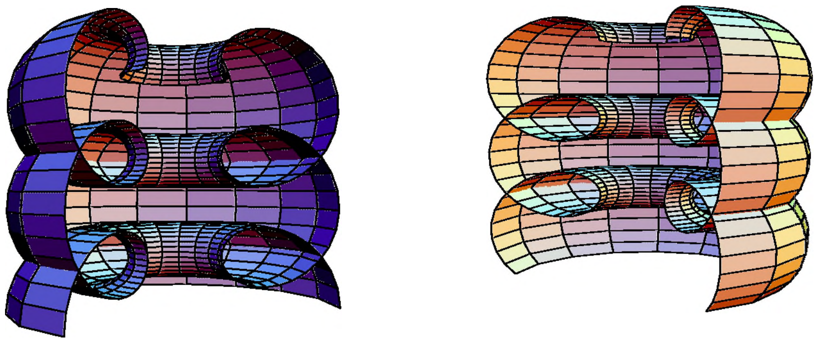

Figure 3. Slices of the immersions of the nodoid and its dual surface on the right side for $k = 0.555556$

4. Duals of the Delaunay's Surfaces

Another important property of the classical conformal immersions is the phenomenon of dual surfaces. Two immersions $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are considered as dual to each other if they share the same tangent plane at corresponding points. This definition follows from the simple observation that if the first fundamental form induced by $\mathbf{x}$ is given by

$$I = \mathrm{e}^{2\tau(u,v)}(\mathrm{d}u^2 + \mathrm{d}v^2) \tag{48}$$

then a direct computation shows that the form

$$\boldsymbol{\omega} = \mathrm{e}^{-2\tau(u,v)}(\mathbf{x}_u\,\mathrm{d}u - \mathbf{x}_v\,\mathrm{d}v) \tag{49}$$

is closed and therefore can be considered (locally) as a differential of the immersion $\tilde{\mathbf{x}}$ given by

$$\mathrm{d}\tilde{\mathbf{x}} = \mathrm{e}^{-2\tau(u,v)}(\mathbf{x}_u\,\mathrm{d}u - \mathbf{x}_v\,\mathrm{d}v). \tag{50}$$

The situation is a little bit more simple for the surfaces of revolution as in this case there exists an explicit expression for their dual immersions (for more details see [8]). For, if such a surface is parameterized by the isothermal coordinates (u, v) in the form

$$\mathbf{x}[u, v] = (r(u)\cos v, r(u)\sin v, z(u)) \tag{51}$$

then its dual is given by

$$\tilde{\mathbf{x}}[u, v] = \left(-\frac{\cos v}{r(u)}, -\frac{\sin v}{r(u)}, \tilde{z}(u)\right) \tag{52}$$

where the function $\tilde{z}(u)$ satisfies the separable ordinary differential equation

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}u} = \frac{1}{r^2}\frac{\mathrm{d}z}{\mathrm{d}u}. \tag{53}$$

Rewriting (52) in the conformal coordinates (11) we will have

$$\tilde{\mathbf{x}}[u, v] = \left(-\mathrm{e}^{-\sigma(u)}\cos v, -\mathrm{e}^{-\sigma(u)}\sin v, \int_0^u \mathrm{e}^{-\sigma(t)}\sin\Omega(t)\,\mathrm{d}t\right) \tag{54}$$

and using this formula one can easily determine the respective dual surface in explicit form. As the x and y components of $\tilde{\mathbf{x}}$ are clear in each of the three cases under consideration we will concentrate on their z components.

Taking into account (34), (38) and (40) we have for the unduloid

$$\tilde{z} = \int_0^u \frac{\mathrm{dn}(\tilde{u}, k)\,\mathrm{d}\tilde{u}}{\mathrm{dn}(\tilde{u}, k) + k\,\mathrm{cn}(\tilde{u}, k)} = (E(\mathrm{am}(u, k), k) - k\,\mathrm{sn}(u, k))/\tilde{k}^2 \tag{55}$$

and its dual surface is shown on the right-hand side in Fig. 2.

The sphere case is straightforward as well and one gets

$$\tilde{\mathbf{x}}[u, v] = (-\operatorname{ch} u\cos v, -\operatorname{ch} u\sin v, u) \tag{56}$$

which coincides with the catenoid (26) up to a reflection with respect to the symmetry axis in the meridional planes.

Finally, for the nodoid case we have to evaluate the integral

$$\int_0^u \frac{\mathrm{cn}(\tilde{u}/k, k)\,\mathrm{d}\tilde{u}}{\mathrm{dn}(\tilde{u}/k, k) + k\,\mathrm{cn}(\tilde{u}/k, k)} \tag{57}$$

which after some work with the Jacobian elliptic functions gives us

$$\tilde{z} = (k\,\mathrm{sn}(u/k, k) - E(\mathrm{am}(u/k, k), k))\,/\tilde{k}^2 + u/k. \tag{58}$$

The respective dual surface is shown on the right side in Fig. 3.

This completes also our study on the subject given in the title of the present work and makes obvious the need of more detailed investigation of the new surfaces obtained as duals to nodoids and unduloids.

We hope to report on this topic soon elsewhere.

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