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POISSON STRUCTURES IN $\mathbb{R}^3$

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Abstract. The Poisson structures and Hamiltonian formulation of three dimensional systems is considered in general. A class of degenerate structures in higher dimensions is also briefly discussed.

1. Introduction

In a recent work we [1] have considered the Poisson structures in $\mathbb{R}^3$. We showed that, locally all such structures must have the form

$$J^{ij} = \mu \epsilon^{ijk} \partial_k \Psi \tag{1}$$

where μ and Ψ are arbitrary differentiable functions of x^i, $i = 1, 2, 3$ and ϵ^{ijk} is the Levi-Civita symbol. Here we use the summation convention. This has a very natural geometrical explanation. Let $\Psi = c_1$ and $H = c_2$ define two surfaces S_1 and S_2 respectively, in $\mathbb{R}^3$, where c_1 and c_2 are some constants. Then the intersection of these surfaces define a curve C in $\mathbb{R}^3$. The velocity vector $\mathrm{d}x/\mathrm{d}t$ of this curve is parallel to the vector product of the normal vectors $\nabla\Psi$ and ∇H of the surfaces S_1 and S_2, respectively, i.e.,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu \nabla\Psi \times \nabla H \tag{2}$$

where μ is any arbitrary function in $\mathbb{R}^3$. Equation (2) defines a Hamiltonian system in $\mathbb{R}^3$. In [1] we proved that all Hamiltonian systems in $\mathbb{R}^3$ are of the form (2).

In many examples the general form (1) of a Poisson structure is preserved globally, including the irregular points (points where the rank of the structure changes). That is a Poisson structure has the same form on different symplectic leaves [8].

The general form (1) allows to construct the compatible Poisson structures and the corresponding bi-Hamiltonian systems. The bi-Hamiltonian representation of a system is closely related to the notion of integrability. Given a bi-Hamiltonian system one can construct an infinite hierarchy of commuting first integrals, using

Magri's theorem [3]. For a finite dimensional system if the hierarchy has sufficient number of functionally independent first integrals the system is integrable. The conditions on a bi-Hamiltonian structure to give a complete set of functionally independent first integrals were studied in [4], [5]. On the other hand the existence of bi-Hamiltonian structure, that gives a compleat set of independent first integrals, for an integrable Hamiltonian system was considered in [6], [7].

2. Bi-Hamiltonian Equations

Writing the Poisson structure in the form (1) is convenient for construction of bi-Hamiltonian representations of a given Hamiltonian system, following [1] we have

Definition 1. *Two Hamiltonian matrices J and $\widetilde{J}$ are compatible, if the sum $J + \widetilde{J}$ defines also a Poisson structure.*

Lemma 1. *Let Poisson structures J and $\widetilde{J}$ have form* (1)*, so $J^{ij} = \mu\epsilon^{ijk}\partial_k\Psi$ and $\widetilde{J}^{ij} = \widetilde{\mu}\epsilon^{ijk}\partial_k\widetilde{\Psi}$. Then J and $\widetilde{J}$ are compatible if and only if there exist a differentiable function $\Phi(\Psi, \widetilde{\Psi})$ such that*

$$\widetilde{\mu} = \mu\frac{\partial_{\widetilde{\Psi}}\Phi}{\partial_{\Psi}\Phi} \tag{3}$$

provided that $\partial_{\Psi}\Phi \equiv \frac{\partial\Phi}{\partial\Psi} \neq 0$ and $\partial_{\widetilde{\Psi}}\Phi \equiv \frac{\partial\Phi}{\partial\widetilde{\Psi}} \neq 0$.

This suggests that all Poisson structures in $\mathbb{R}^3$ have compatible pairs, because the condition (3) is not so restrictive on the Poisson matrices J and $\widetilde{J}$. Such compatible Poisson structures can be used to construct bi-Hamiltonian systems. A system is called Hamiltonian if it admits a pair (J, H) such that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = J\nabla H. \tag{4}$$

Definition 2. *A Hamiltonian system is said to be bi-Hamiltonian if it admits two Hamiltonian representations H and $\widetilde{H}$, with compatible Poisson structures J and $\widetilde{J}$ respectively, such that*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = J\nabla H = \widetilde{J}\nabla\widetilde{H}. \tag{5}$$

Lemma 2. *Let J be given by* (1) *and $H(x_1, x_2, x_3)$ is any differentiable function then the Hamiltonian equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = J\nabla H = -\mu\nabla\Psi \times \nabla H \tag{6}$$

is bi-Hamiltonian with the second structure $\widetilde{J}$, given by Lemma 1*. Provided that the differentiable functions $\Phi(\Psi, \widetilde{\Psi})$, $h(\Psi, H)$, and $g(\Psi, H)$ satisfy the following* *equation*

$$\frac{\partial g}{\partial \Psi}\frac{\partial h}{\partial H} - \frac{\partial g}{\partial H}\frac{\partial h}{\partial \Psi} = \frac{\Phi_1(\Psi, g)}{\Phi_2(\Psi, g)} \tag{7}$$

where $\Phi_1 = \partial_\Psi \Phi|_{(\Psi,g)}$, $\Phi_2 = \partial_{\widetilde{\Psi}} \Phi|_{(\Psi,g)}$.

Using Lemma 2 we can construct infinitely many compatible Hamiltonian representations by choosing functions Φ, g, h satisfying (7). For instance, if we fix functions Φ, and g then equation (7) became linear first order partial differential equations for h.

3. Examples

Example 1. *Consider Kermac-Mackendric system*

$$\begin{aligned} \dot{x}_1 &= -r x_1 x_2 \\ \dot{x}_2 &= r x_1 x_2 - a x_2 \\ \dot{x}_3 &= a x_2 \end{aligned} \tag{8}$$

where $r, a \in \mathbb{R}$ are constants. This system admits Hamiltonian representation. The Poisson structure J is given by (1) with $\Psi = x_1 + x_2 + x_3$, $\mu = x_1 x_2$ and the Hamiltonian is $H = r x_3 + a \ln x_1$. This structure has irregular planes $x_1 = 0$ and $x_2 = 0$ (coordinate planes) where its form (1) is preserved [1]. *Using* Lemma 1, *we can easily construct a second compatible representation taking $\widetilde{\Psi} = a x_1\, \mathrm{e}^{r x_3}$, $\widetilde{\mu} = x_1 x_2$ and $\widetilde{H} = (a x_1)^{-1}\, \mathrm{e}^{-r x_3}\,(x_1 + x_2 + x_3)$.*

Example 2. *Consider the Euler equations*

$$\begin{aligned} \dot{x}_1 &= \frac{I_2 - I_3}{I_2 I_3} x_2 x_3 \\ \dot{x}_2 &= \frac{I_3 - I_1}{I_3 I_1} x_3 x_1 \\ \dot{x}_3 &= \frac{I_1 - I_2}{I_1 I_2} x_1 x_2 \end{aligned} \tag{9}$$

where $I_1, I_2, I_3 \in \mathbb{R}$ are some (non-vanishing) real constants. This system admits Hamiltonian representation. The Poisson structure J is given by (1) with $\Psi = -\frac{1}{2}\,(x_1^2 + x_2^2 + x_3^2)$, $\mu = 1$, and the Hamiltonian is $H = \dfrac{x_1^2}{2I_1} + \dfrac{x_2^2}{2I_2} + \dfrac{x_3^2}{2I_3}$. This structure has an irregular point $(0,0,0)$ (the origin) where its form (1) is preserved [1].

Example 3. *Consider Lotka-Voltera system*

$$\begin{aligned} \dot{x}_1 &= -abcx_1x_3 - bc\mu_0 x_1 + cx_1x_2 + c\nu x_1 \\ \dot{x}_2 &= -a^2bcx_2x_3 - abc\mu_0 x_2 + x_1x_2 \\ \dot{x}_3 &= -abcx_2x_3 - abc\nu_0 x_3 + bx_1x_3 \end{aligned} \tag{10}$$

where $a, b, c, \mu_0, \nu_0 \in \mathbb{R}$ are constants. The Poisson structure J is given by (1) *with $\Psi = -\ln x_1 - b\ln x_2 + c\ln x_3$, $\mu = x_1x_2x_3$ and the Hamiltonian is $H = abx_1 + x_2 - ax_3 + \nu_0 \ln x_2 - \mu_0 \ln x_3$. This structure has irregular lines given by $x_i = 0$ and $x_j = 0$, $i, j = 1, 2, 3$, $j \neq i$ (coordinate lines). Both Ψ and H are not defined at these points. So, the system does not have a Hamiltonian formulation at these points.*

Many other examples of three dimensional systems are considered in [1].

4. Higher Dimensions

Solutions of Jacobi equations similar to (1) can be given in higher dimensions. Following [2] we have the next two theorems:

Theorem 1. *Let μ and $f_1, \ldots, f_{n-2}$ be smooth functions on an open and dense subset of an n-dimensional manifold M then*

$$J^{ij} = \mu\epsilon^{ijk_1\ldots k_{n-2}}\partial_{k_1} f_1 \ldots \partial_{k_{n-2}} f_{n-2} \tag{11}$$

defines a Poisson structure of rank 2. The functions $f_1, \ldots, f_{n-2}$ are Casimir functions of the Poisson structure (11). *Conversely, if J is a Poisson structure of rank 2 then there exist functions μ and $f_1, \ldots, f_{n-2}$ such that J has the form* (11).

This theorem describes all degenerate Poisson structures in $\mathbb{R}^4$ since they are necessarily of rank 2 or 0.

Theorem 2. *Any degenerate Poisson structure in $\mathbb{R}^4$ has the form*

$$J^{ij} = \mu\epsilon^{ijkl}\partial_k f_1 \partial_l f_2 \tag{12}$$

where μ, f_1, and f_2 are smooth functions. The functions f_1 and f_2 are Casimir functions of the Poisson structure (12).

It is clear from these theorems that three dimensions is of special importance. In three dimensions all Poisson structures have the form (1).

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