

ANOTHER PERSPECTIVE ON THE RELATION BETWEEN CLASSICAL AND QUANTUM INTEGRABILITY

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Abstract. We describe a framework suggesting how to “deform” in Planck’s constant \hbar the classical concept of integrability. The key point is to use well defined counterparts of Feynman’s measure on the space of paths of the mechanical system. Then we introduce the associated deformation of quantum conservation laws. The method is tested on elementary systems and provide, indeed, more information than expected.

1. Motivation

This is a brief report on some qualitative aspects of the relations between the classical notions of integrability and their quantum counterparts. We wish to advocate an approach to this question which is certainly not mainstream but seems to us conceptually natural, as well as to provide a few arguments showing why we believe that it is indeed promising. The full realization of this research program may, however, take years so we will somehow abuse the hospitality of Professor Ivaïlo Mladenov, hoping to be able to prove in a future Varna Conference on *Geometry, Integrability and Quantization*, that the program sketched in the 2001 edition was, indeed, sound!

In mathematical physics, the motivation for studying together the two above-mentioned notions of integrability can be traced back to the need to define the quantization of classical systems whose behavior is conflicting as much as possible with the one of integrable systems, the ergodic ones. This is one way to approach “quantum chaos”, not optimal however since it is not really an intrinsic quantum mechanical definition [1, 2]. But, at least, this per-

spective forces us to reconsider in a more thorough way the basic concepts of classical and quantum mechanics.

Let us pick an elementary classical system, which phase space is $M = \mathbb{R}^6$ and with Hamiltonian observable

$$h(q, p) = \frac{1}{2}|p|^2 + V(q), \quad (1)$$

for q and p the position and momentum observables. If the scalar potential V is bounded below, the flow $U_t : M \rightarrow M$ generated by the Hamiltonian vector field $v_h = p_i \frac{\partial}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial}{\partial p_i}$ is complete (i. e. defined for all time t).

A quantization is a linear map from the classical Poisson algebra into self-adjoint operators in the Hilbert space, here $\mathcal{H} = L^2(\mathbb{R}^3)$ with the inner product $\langle \cdot | \cdot \rangle$, associating to any classical observable $a : \mathbb{R}^6 \rightarrow \mathbb{R}$ a quantum one denoted by A , such that to q_k corresponds to the multiplication operator Q_k by q_k , to the momentum p_j corresponds $P_j = -i\hbar \frac{\partial}{\partial q_j}$ and to a real constant λ corresponds λI where I is the identity operator. Moreover Dirac's correspondence principle between Poisson bracket and the commutator

$$\{a_1, a_2\} = \frac{1}{i\hbar} [A_1, A_2] \quad (2)$$

should hold. It has been known for a long time that it is incompatible to require simultaneously all these conditions (Grönewald-van Hove Theorem) [15]. However, at the semi-classical limit $\hbar \rightarrow 0$, it is expected that the deformation with respect to the relation (2) should be small. This observation opens the door to various concurrent quantization procedures with this range of validity. When the classical observable a is enough regular, its quantization A is a pseudo-differential operator [3]. The procedure of Weyl dates back to the early days of quantum mechanics and is defined by the following action on the quantum state (or wave function) Ψ in \mathcal{H} :

$$(A\Psi)(Q) = \frac{1}{(2\pi\hbar)^3} \iint e^{\frac{i}{\hbar}(Q-q)p} a\left(\frac{Q+q}{2}, p\right) \Psi(q) dp dq. \quad (3)$$

A defined in this way, is called a \hbar -pseudo-differential operator of Weyl symbol a . For our elementary Hamiltonian given in (1) we obtain, of course, the Schrödinger operator

$$H = -\frac{\hbar^2}{2} \Delta + V \quad (4)$$

which is essentially self-adjoint on the domain $C_0^\infty(\mathbb{R}^3)$, dense in \mathcal{H} .

Notice the appearance of the Liouville measure $dp dq$ in (3). It is always present in such a geometrical approach to the quantization problem [4]. We are, therefore, inspired here by the general concept of classical (mixed) states of the system, as probability measures on the phase space and the fact that the Hamiltonian flow preserves $dp \wedge dq$. But the wave function Ψ does not live on the phase space, it lives on the space of half the dimension of this classical space, here \mathbb{R}^3 , for example the configuration space. This is the origin of conceptual difficulties in the comparison between classical and quantum mechanics. One can try to associate to any wave function Ψ a density ρ_Ψ on the phase space. Such constructions exist (for example the “Husimi functions” ρ_Ψ mentioned in [2]). But, Heisenberg uncertainty principle between the position and the momentum observables precludes a sharp localization of wave functions in these two variables. Consequently, although it is possible to construct reasonable positive densities ρ_Ψ such that $\int_{\mathbb{R}^6} \rho_\Psi(p, q) dp dq = 1$, these do not represent the probability to find our quantum particle at (p, q) , except at the semi-classical limit. Another way to formulate this difficulty is to say that such a probabilistic interpretation makes sense only for very special, “coherent”, quantum states (those for which the uncertainty relation between the standard deviations of position and momentum in the states Ψ turns into an equality).

2. Path Integrals

On the other hand, one may insist, with Born, on a valid probabilistic interpretation of any state Ψ on the Hilbert space, and only there. We avoid, in this way, an immediate conflict with classical mechanics, where a pure state at time t is simply defined by $U_t(p, q) = (e^{tv_h})(p, q) = (p(t), q(t))$ but face the nontrivial problem to justify the mathematical origin of this probabilistic interpretation (the fact that it is physically sound has been confirmed by eighty years of quantum experiments). Feynman’s answer to this challenge is summarized in his famous path integral representation [6] of the wave function which reduces, here, to

$$\Psi(q, t) = \int_{\Omega^{q,t}} \Psi_0(\omega(0)) e^{\frac{i}{\hbar} S_L[\omega(\cdot), t]} \mathcal{D}\omega, \quad (5)$$

where S_L denotes the classical action functional of Lagrangian L for our system of Hamiltonian (1), i. e.

$$S_L[\omega(\cdot), t] = \int_0^t L(\dot{\omega}(\tau), \omega(\tau)) d\tau = \int_0^t \left(\frac{1}{2} |\dot{\omega}(\tau)|^2 - V(\omega(\tau)) \right) d\tau. \quad (6)$$

In (5), $\Omega^{q,t}$ denotes the paths space $\{\omega \in C([0, t], \mathbb{R}^3) \text{ s.t. } \omega(t) = q\}$ and $\mathcal{D}\omega = \prod_{0 \leq \tau \leq t} d\omega(\tau)$, a purely symbolic object regarded as a measure on $\Omega^{q,t}$. The fundamentally new ingredient of Feynman's approach is to claim that if, following Born, it is indeed true that $\int_A |\Psi(q, t)|^2 dq$ represents the probability to find the quantum particle in the subset A of the configuration space \mathbb{R}^3 , this is simply because the quantum trajectories $\tau \mapsto \omega(\tau)$, $0 \leq \tau \leq t$, are sample paths of some \mathbb{R}^3 -valued stochastic process, whose probability density (w.r.t. dq) at time t is given by $|\Psi(q, t)|^2$. All particularities of the quantization procedure should follow from the underlining process, building block of the path integral method.

The underlying process depends, of course, on the starting classical Hamiltonian of the system. For our simple case (1), in the position representation, this should be a pure diffusion process whose informal finite dimensional distributions are alluded to by Feynman [6] in terms of the integral kernel of the equation solved by the wave function (5), namely Schrödinger equation in $L^2(\mathbb{R}^3)$:

$$i\hbar\partial_t\Psi = H\Psi, \quad \text{with} \quad \Psi(q, 0) = \Psi_0(q), \quad (7)$$

and H is given by (4). To make a long story short (cf. [5]) Feynman's above-mentioned diffusion process does not make any mathematical sense, but a closely related one which does is the \mathbb{R}^3 -valued Wiener process W_t (or Brownian motion) whose generator is the Laplacian operator in (4). This constitutes the starting point of the probabilistic (or Euclidean) approaches to quantum theory, where the time parameter of (7) undergoes the "rotation" $t \mapsto \sqrt{-1}t$. Equivalently, we are dealing now with the heat equation

$$\hbar\partial_t\eta = H\eta, \quad \text{with} \quad \eta(q, 0) = \eta_0(q), \quad (8)$$

instead of (7). When the scalar potential $V = 0$, its close relations with the Brownian motion are known since Einstein.

In the path integral approach, and the rigorous ones inspired by it, the quantization problem takes quite a new form. Any observable A should be associated with a random variable, a Borel measurable function of the underlying process. If $\Psi_t = U_Q(t, t_0)\Psi_0$, where $U_Q(t, t_0)$ denotes the two-parameter family of unitary propagator associated with the Eq. (7) (later on we will allow V to be time-dependent), the quantum expectation of A in the state Ψ_t , $\langle A \rangle_{\Psi_t} = \langle \Psi_t | A \Psi_t \rangle$ should be associated with the expectation of this random variable with respect to Born's probability density (or its probabilistic counterpart). Maybe the most exciting conceptual aspect of the path integral approach is that it allows, in principle, to contemplate physical observables considerably more general than the self-adjoint operators of von Neumann, for example the one associated with general functionals of the processes.

3. Action Functional and Deformation of Hamilton-Jacobi Equation

For our purposes, a key observation is that the law of the relevant \mathbb{R}^3 -valued process $\omega(t) \equiv Z_t(\omega)$ (the variable ω in the sample space Ω of the trajectories is generally omitted on the r.h.s.) is absolutely continuous with respect to the one of the Wiener W_t (with parameter \hbar , cf. [5]), with **Radon-Nikodym density**

$$\rho[\omega(\cdot)] = \frac{\eta(\omega(t_1), t_1)}{\eta_0(\omega(0))} \exp \left(-\frac{1}{\hbar} \int_0^{t_1} V(\omega(t)) dt \right). \quad (9)$$

This means that this diffusion Z_t , itself, $t \in [0, t_1]$, solves the following Itô's **stochastic differential equation (SDE)**, [14]

$$dZ_t = -\nabla S(Z_t, t) dt + \hbar^{1/2} dW_t, \quad (10)$$

for $S(q, t) = -\hbar \ln \eta(q, t)$ enough regular to admit a gradient $B = -\nabla S$ (a "drift") uniformly Lipschitz on compact time intervals (this can be relaxed). A strong solution of (10) is defined on a probability space with a given Wiener process W_t and a given random initial condition Z_0 and is measurable with respect to the increasing family of sigma-algebras \mathcal{P}_t generated by Z_0 and W_s , $0 \leq s \leq t$. Intuitively, \mathcal{P}_t contains the past information about this random process up to the time t . The time symmetry of our quantized system is manifestly lost by such a probabilistic counterpart. But the quantum invariance under time reversal is due to the fact that the probability density appearing in Born's interpretation of the wave function is of the product form $\bar{\Psi}\Psi(q, t) dq$, where $\bar{\Psi}$ solves the Schrödinger equation conjugated to (7). The probabilistic translation of this is therefore to involve as well a positive solution of the equation adjoint to (8) w.r.t. the time parameter, say

$$-\hbar \partial_t \eta^* = H \eta^* \quad (11)$$

and to show the existence of processes Z_t , itself, $t \in I$, built in terms of (8) and (9), in such a way that the probability

$$P(Z_t \in A) = \int_A \eta^* \eta(q, t) dq. \quad (12)$$

This has been shown to make sense, indeed, in the mid-eighties (cf. references in [7]) and allows us to construct a convincing analogy of the quantization procedure, where many delicate issues take a much clearer form because of the existence of the underlining probability measure (cf. [5]).

In particular, the function S mentioned in (10) can also be interpreted as a functional of the diffusion, namely

$$S_{L_E}(q, t) = E_{q,t} \int_t^U L_E(\mathcal{D}Z_r, Z_r) dr + E_{q,t} S_U(Z_U). \tag{13}$$

In spite of its complicated appearance, the r.h.s of (13) is a natural counterpart of the classical action S_{L_E} when the integration is computed along the quantum paths, “quite irregular on a very fine scale” as Feynman says ([6] p. 176). In the r.h.s. of Eq. (13) $E_{q,t}$ (or $E[\dots|Z_t = q]$) denotes the conditional expectation, given that $Z_t = q$, whose meaning is clear in the following definition of the mean derivative of any $f \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$,

$$\mathcal{D}f(Z_r, r) = \lim_{\Delta r \rightarrow 0} E \left[\frac{df(Z_r, r)}{dr} | Z(r) \right] \tag{14}$$

where $df(Z_r, r) = f(Z_{r+\Delta r}, r + \Delta r) - f(Z_r, r)$.

By (10) as soon as $\hbar \neq 0$, Z_r inherits the nowhere differentiability of the Brownian paths so the difference quotient in (14) diverges. But the regularization $E[\dots|Z(r)]$ kills its divergent part. It follows from (10) and (14) that the “infinitesimal generator” \mathcal{D} of the diffusion reduces to

$$\mathcal{D}f(Z_r, r) = \left(\partial_r + \hbar \nabla \ln \eta \nabla + \frac{\hbar}{2} \Delta \right) f(Z_r, r). \tag{15}$$

In particular, when $\hbar = 0$, \mathcal{D} reduces to the ordinary derivative along the flow generated by the classical limit of (10).

The index E in the integrand L_E of (13) reminds us that the Euclidean rotation $t \mapsto \sqrt{-1}t$ changes the sign of the force (i. e. of V). Apart from that, the Lagrangian L_E is the same as in the classical action (6), but evaluated along random variables. So (10) leads us to interpret (13) as a quantum deformation of the action function (or functional). Given (8), we know that, as a function, S_{L_E} or simply S for short, solves

$$\partial_t S - \frac{1}{2} |\nabla S|^2 + \frac{\hbar}{2} \Delta S + V = 0. \tag{16}$$

For $\hbar = 0$, i. e. at the classical limit of smooth trajectories, (16) reduces to the backward Hamilton–Jacobi equation, the one associated with the action function of the starting point and with a final condition (cf. the boundary condition in (13)). Generally, it is the Hamilton–Jacobi equation adjoint to this one w.r.t. the time parameter which is considered in textbooks. This is why the velocity is $-\nabla S$ in (10).

Our point is that (16) is an exact quantum deformation of its classical counterpart, associated with an exact deformation (10) of a classical ODE. Let us stress that this does not hold only in the semi-classical regime or, equivalently, for very special states as most of what we said in the first part.

4. Symmetries of the Action Functional and Quantum Consequences

What about symmetries of the functional (13) or, equivalently, of the Lagrangian L_E ? As in the classical case, they are indeed associated with conservation laws:

Theorem 1. ([8]) *Consider a one parameter group of transformations $U_\alpha: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$ such that, for $\alpha \rightarrow 0$, $U_\alpha(q^i, t) = (q^i + \alpha X^i(q, t) + \mathcal{O}(\alpha^2), t + \alpha T(t) + \mathcal{O}(\alpha^2))$, for $i = 1, 2, 3$ and X and T of C^2 class. If our Lagrangian $L_E = L_E(B, q, t)$ (we allow $V = V(q, t)$ here) satisfies the invariance condition*

$$T \partial_t L_E + X^i \partial_{q^i} L_E + (\mathcal{D}X^i - B^i \mathcal{D}T) \partial_{B^i} L_E + L_E \mathcal{D}T = \mathcal{D}\Phi, \quad (17)$$

for some divergence term $\Phi: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ and if the admissible (X^i, T, Φ) solve

$$\begin{aligned} \partial_t X^j &= \nabla^j \Phi, & \partial_t \Phi + \frac{\hbar}{2} \Delta \Phi &= \frac{dT}{dt} V + X^j \nabla_j V + T \partial_t V \\ \frac{dT}{dt} &= 2 \nabla_j X^j \text{ (no summation)}, & & \\ \partial_{q^j} X^i + \partial_{q^i} X^j &= 0, & i \neq j, i = 1, 2, 3, & \end{aligned} \quad (18)$$

then, along the diffusion Z_t , the following stochastic conservation law holds

$$\mathcal{D} (B^i X^i + \epsilon T - \Phi) (Z_t, t) = 0 \quad (19)$$

where, by symmetry with our notation $B^i = \hbar \nabla^i \ln \eta$ for the drift, we have introduced $\epsilon = \hbar \partial_t \ln \eta$. When the l.h.s. of (19) is integrable,

$$E [B^i X^i + \epsilon T - \Phi] = \text{const} . \quad (20)$$

For example, if L_E (i. e. the potential V) does not depend on one of the q^i , say q^3 , then $X = (0, 0, 1)$, $T = 0$ and $\Phi = 0$ are admissible. it follows from (19) that the corresponding component of the drift is conserved: $\mathcal{D}B^3 = 0$.

Now the idea is to construct a \hbar -deformation of the integrability “a la Liouville” (or, better, “a la Hamilton–Jacobi”) starting from eq. (15) and using the stochastic first integral $n(Z_t, t)$ of (19), where

$$n(q, t) = B(q, t)X(q, t) + \epsilon(q, t)T(t) - \Phi(q, t). \tag{21}$$

Notice how different this program is from the traditional way to find a quantum counterpart of Liouville integrability, i. e. of the classical foliation of the phase space into smooth submanifolds [9]. One would simply say, in this traditional context, that our quantized system is integrable if the eigenfunctions of (4) are common to three commuting observables (including this Hamiltonian). Now, if it is clear that the quantum counterparts of the three classical first integrals in involution are three commuting observables, other basic aspects of the classical ideas are much less obvious. For example, what is the quantum counterpart of the action-angle variables?

Although Liouville’s theorem is sufficient for most of the integrable systems, it does not provide a constructive way to find the relevant set of first integrals. The classical method of Hamilton–Jacobi is deeper in this respect, but its quantization is not well defined in regular quantum theory. So the above-mentioned Theorem could be useful, if it had a truly quantum counterpart. It so happens that it does:

Theorem 2. ([10]) *Let us consider any observable*

$$N(t) = X^i(Q(t), t) \circ P^i(t) - T(t)H(t) + i\Phi_S(Q(t), t) \tag{22}$$

where $\Phi_S(q(t), t) = \Phi(q(t), t) + \frac{\hbar}{2}\nabla_j X^j$, \circ denotes Jordan symmetrized product of operators and (X^j, T, Φ) solve the real time version of Eq. (17), i. e.

$$\begin{aligned} \partial_t X^j &= -i\nabla^j \Phi, & i\partial_t \Phi + \frac{\hbar}{2}\Delta\Phi &= \frac{dT}{dt}V + X^j\nabla_j V + T\partial_t V \\ \frac{dT}{dt} &= 2\nabla_j X^j \text{ (no summation)}, & & \\ \partial_{q^k} X^j + \partial_{q^j} X^k &= 0, & j \neq k, i &= 1, 2, 3. \end{aligned} \tag{23}$$

and any operators on the r.h.s. of (22) should be interpreted in Heisenberg picture i. e. $A(t) = U^\top(t, t_0)A_S(t)U(t, t_0)$, for $A_S(t)$ being its Schrödinger form and U^\top the adjoint operator. Notice that here, as in the previous Theorem, the potential V , i. e. the Hamiltonian $H = H_S$ may be explicitly time-dependent. Then $N(t)$ is a constant of motion in the usual quantum mechanical sense:

$$\langle N(t) \rangle_{\Psi_0} = \text{const} \tag{24}$$

for Ψ_0 the initial condition of (7).

Of course, any such $N(t)$ (N for “Noetherian” cf. [10]) is an unbounded operator, whose dense domain has to be constructed in the Hilbert space, so that the “details” behind the claim of this theorem are not so trivial to provide (cf. [10]). But if we agree to interpret the quantum observables $Q(t), P(t), H(t)$ in (22) as the real time version of the diffusion process Z_t , its drift vector field $B(Z_t, t)$ and the scalar $\epsilon(Z_t, t)$ used in the probabilistic theorem, as well as to see (24) as a counterpart of (20), then the analogy is striking. The only aspect of the quantum version not given (and not needed) by the probabilistic one is Jordan’s symmetrization, turning $N(t)$ into a respectable candidate for the title of quantum observable. It is not our point here to explain why this is not an accidental analogy (this has been done in [5]) but we will use this opportunity to discuss some of the implications for the general concept of integrability we are after. The existence of explicitly time-dependent first integrals in classical mechanics has been known from the very beginning of this field. In fact, it is not always realized that the three commuting (or “in involution”) observables $n(q, p, t)$ needed for the classical Liouville integrability are allowed to be time-dependent, i. e. to satisfy

$$\partial_t n + \{n, h\} = 0 \quad (25)$$

for $\{f, g\} = \sum_{j=1}^3 (\partial_{q^j} f \partial_{p_j} g - \partial_{p_j} f \partial_{q^j} g)$ the canonical Poisson bracket. Each such observable n generates a canonical transformation (which does not preserve the Hamiltonian h , in general, since n is time-dependent).

Although this is familiar, few general results seems to be known showing how to use such time-dependent first integrals in the actual process of integration of a given system. Nevertheless, those integrals are anything but exceptional, even for manifestly integrable systems. Some of them are, in fact, trivial. Consider the simplest possible example, the one dimensional ($M = \mathbb{R}$) classical free case, i. e. $h(p, q) = \frac{1}{2}p^2$, or $L(\dot{q}, q) = L_E(\dot{q}, q) = \frac{1}{2}\dot{q}^2$ (since the potential $V = 0$, the Euclidean version needed for theorem 1 does not make any difference here). We can take informally the limit $\hbar \rightarrow 0$ of the quantum Theorem 2 and see what it means classically. A particular solution of the one dimensional version of (23) is

$$X(q, t) = -t, \quad T(t) = 0, \quad \Phi(q, t) = -iq \quad (26)$$

so, by (22), $n(t) = -tp(t) + q(t)$ should be a first integral. Now the solution of the underlying Hamiltonian system, $\dot{p} = 0$, $\dot{q} = p$ with $p(0) = p_0$, $q(0) = q_0$ is given by

$$p(t) = p_0, \quad q(t) = p(t)t + q_0. \quad (27)$$

This means that, in this case, we have rediscovered the fact that $q(t) - tp(t) = n_1(q, p, t)$ is a constant of motion, indeed the initial position q_0 ! Since the equations of motion are linear, the same conclusion holds true after substitution of q, p by their quantum (Heisenberg's) counterparts Q, P . It has been said that this elementary example (27) contains all the same "the complete geometrical essence of the idea of integrability via a complete set of conservation laws" [11]. However, for general systems, the commuting first integrals needed for Liouville integrability cannot be substituted for trivial local constants like the initial conditions mentioned before. Now, there are other solutions of (23) more interesting in this respect. Consider, for example, $X(q, t) = -qt$, $T(t) = -t^2$, $\Phi(q, t) = -\frac{1}{2}q^2$. Then the classical limit of (22) says that

$$n_2(q, p, t) = -qtp + \frac{1}{2}t^2p^2 + \frac{1}{2}q^2 \quad (28)$$

is a first integral, a priori not a trivial one. In this case, there is a slight quantum deformation of the conservation law, due to the presence of \hbar in the quantum equation for Φ (cf. (23)). But this deformation is compensated by the extra term due to Jordan's symmetrization in (22) so

$$N_2(t) = (-Qt) \circ P + \frac{1}{2}t^2P^2 + \frac{1}{2}Q^2 \quad (29)$$

is, as well, the conserved quantum observable. Also, notice that if it is indeed true that $\{n_1, n_2\} = 0$, those two first integrals are not functionally independent, in fact $n_2 = \frac{1}{2}(n_1)^2$, and so n_2 does not provide any new information. Clearly our algebra of first integrals is not commutative; if $n_3 = \frac{1}{2}p^2$, corresponding to $X = 0$, $T = 1$ and $\Phi = 0$, then $\{n_1, n_3\} = p$. It is known that Liouville's original theorem can be generalized to include this situation (cf. [12]). What is lying behind the above mentioned theorem is a single idea which we have summarized in the (three dimensional) probabilistic context of Theorem 1. As indicated before, the construction is founded on some positive solutions of (a pair of) heat equations. It provides both the properties of the underlying diffusion (10) and the associated deformation (15) of the classical Hamilton-Jacobi equation. Any symmetry of this framework corresponds to the action of a group of transformations G acting on a subset of the space of independent and dependent variables (respectively (q, t) and η) for Eq. (8) and such that if $\eta(q, t)$ is a solution, then $g \cdot \eta$ is another one for $g \in G$. So, completing the transformations of (q, t) given on Theorem 1 by $\eta + \frac{\alpha}{\hbar}\Phi\eta(q, t) + \mathcal{O}(\alpha^2)$ for the dependent variable, such a requirement imposes conditions on (X^j, T, Φ) ; these are precisely (17). If we denote by

$$\hat{N} = \hbar X^j \partial_{q^j} + \hbar T \partial_t - \Phi \quad (30)$$

the infinitesimal generator associated with these transformations, one shows easily that this amount to impose the commutation relations

$$[\hat{N}, \hbar\partial_t - H] = -\frac{dT}{dt} (\hbar\partial_t - H). \quad (31)$$

Now pick a (regular) positive solution η of (8) in the domain of \hat{N} to which is associated a diffusion Z_t . Then

$$\left(\frac{\hat{N}\eta}{\eta} \right) (q, t) \quad (32)$$

coincides with the stochastic first integral n given by (21). Up to a multiplicative factor $i = \sqrt{-1}$ (and up to a symmetrization of its first term), the operator (30) is a quantum object, namely (22).

5. Open Problems and Prospects

When $V = 0$, the symmetry Lie algebra of Theorems 1 and 2 has dimension thirteen (and when $M = \mathbb{R}$, has dimension six). In fact, all Hamiltonians with quadratic potentials V and any additional linear factor $\nabla\eta$ have symmetry Lie algebras isomorphic to the first one, explicitly computable (cf. [10]). So, Theorems 1 and 2 provide more that what is needed for Liouville integrability (the term “super-integrable” has been used by some authors, in these situations) and a way to compute the first integrals. The method advocated here is much more general than what our elementary class suggests, but many aspects of it deserves further considerations. For example, what is its relation with the inverse scattering method (cf. [13])? The analogy of (31) with Lax’s representation of the KdV equation as a commutation condition is striking. Since the inverse scattering method allows to construct action-angle variables for integrable systems, can we transpose this in our present quantum context?

Another interesting aspect is the role of the scalar (or divergence) field Φ in Theorems 1 and 2. Its presence is related with the cohomology of our symmetry Lie algebra (or equivalently, to the fact that the action (13) is not exactly invariant under the group of transformations but only invariant up to a divergence term $\mathcal{D}\Phi(Z_t, t)$).

In any case, it seems that a concept of “complete integrability” for the Hamilton-Jacobi-Bellman equation (16) is accessible via the stochastic first integrals provided by Theorem 1. This \hbar -deformation of the classical theory should be of intrinsic interest, given the importance of (16) in stochastic control theory [16]. Moreover, and in the perspective of Prof. Mladenov’s meeting, Theorem 2 strongly suggests that such a result should, as well, be of direct relevance for a very natural definition of quantum integrability.

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