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PATH INTEGRALS ON RIEMANNIAN MANIFOLDS WITH SYMMETRY AND STRATIFIED GAUGE STRUCTURE

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Abstract. We study a quantum system in a Riemannian manifold M on which a Lie group G acts isometrically. The path integral on M is decomposed into a family of path integrals on quotient space $Q = M/G$ and the reduced path integrals are completely classified by irreducible unitary representations of G. It is not necessary to assume that the action of G on M is either free or transitive. Hence the quotient space M/G may have orbifold singularities. Stratification geometry, which is a generalization of the concept of principal fiber bundle, is necessarily introduced to describe the path integral on M/G. Using it we show that the reduced path integral is expressed as a product of three factors; the rotational energy amplitude, the vibrational energy amplitude, and the holonomy factor.

1. Basic Observations and the Questions

Let us consider the usual quantum mechanics of a free particle in the one-dimensional space $\mathbb{R}$. A solution for the initial-value problem of the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\phi(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x,t) = \frac{1}{2}\Delta\phi(x,t) \tag{1.1}$$

is given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K(x,y;t)\phi(y,0) \tag{1.2}$$

with the propagator

$$K(x,y;t) = \langle x|\,\mathrm{e}^{-\frac{\mathrm{i}}{2}t\Delta}|y\rangle = \frac{1}{\sqrt{2\pi\mathrm{i}t}}\exp\left[\frac{\mathrm{i}}{2t}(x-y)^2\right]. \tag{1.3}$$

Their physical meanings are clear; the wave function $\phi(x,t)$ represents probability amplitude to find the particle at the location x at the time t. The propagator $K(x,y;t)$ represents transition probability amplitude of the particle to move from y to x in the time interval t.

If the particle is confined in the half line $\mathbb{R}_{\geq 0} = \{x \geq 0\}$, we need to impose a boundary condition on the wave function $\phi(x,t)$ at $x=0$ to make the initial-value problem (1.1) have a unique solution. As one of possibilities we may chose the Neumann boundary condition

$$\frac{\partial\phi}{\partial x}(0,t) = 0\,. \tag{1.4}$$

Then the solution of (1.1) is given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K_N(x,y;t)\phi(y,0) \tag{1.5}$$

with the corresponding propagator

$$K_N(x,y;t) = K(x,y;t) + K(-x,y;t)\,. \tag{1.6}$$

The physical meaning of the propagator $K_N(x,y;t)$ is obvious; the first term $K(x,y;t)$ represents propagation of a wave from y to x while the second term $K(-x,y;t)$ represents propagation of a wave from y to $-x$, which is the mirror image of x. Thus the Neumann propagator $K_N(x,y;t)$ is a superposition of the direct wave with the reflected wave.

As an alternative choice we may impose the Dirichlet boundary condition

$$\phi(0,t) = 0\,. \tag{1.7}$$

Then the solution of (1.1) is given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K_D(x,y;t)\phi(y,0) \tag{1.8}$$

with the corresponding propagator

$$K_D(x,y;t) = K(x,y;t) - K(-x,y;t)\,. \tag{1.9}$$

Thus the Dirichlet propagator $K_D(x,y;t)$ is also a superposition of the direct wave with the reflected wave but reflection changes the sign of the wave.

The half line $\mathbb{R}_{\geq 0}$ can be regarded as an orbifold $\mathbb{R}/\mathbb{Z}_2$. In the above discussion we have assumed the existence of the propagator $K(x, y; t)$ in $\mathbb{R}$ and constructed the propagators in $\mathbb{R}/\mathbb{Z}_2$ from $K(x, y; t)$. There are two inequivalent propagators; the Neumann propagator $K_N(x, y; t)$ obeys the trivial representation of $\mathbb{Z}_2$ whereas the Dirichlet propagator $K_D(x, y; t)$ obeys the defining representation of $\mathbb{Z}_2 = \{+1, -1\}$.

Now a question arises; how is a propagator in a general orbifold M/G constructed? Here M is a Riemannian manifold and G is a compact Lie group that acts on M by isometries. Such an example is easily found; we may take $M = \mathbb{S}^2$ and $G = \mathbb{U}(1)$. Then the quotient space is $M/G = [-1, 1]$, which has two boundary points.

Let us turn to another aspect of the propagator, namely, the path-integral expression of the propagator. For the general Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\phi(x,t) = H\phi(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x,t) + V(x)\phi(x,t)\,, \quad x \in \mathbb{R}\,, \tag{1.10}$$

its solution is formally given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K(x, y; t)\phi(y, 0)\,. \tag{1.11}$$

The propagator satisfies the composition property

$$K(x'', x; t + t') = \int_{-\infty}^{\infty} \mathrm{d}x'\, K(x'', x'; t')K(x', x; t)\,. \tag{1.12}$$

By dividing the time interval $[0, t]$ into short intervals we get

$$K(x_N, x_0; t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathrm{d}x_{N-1} \cdots \mathrm{d}x_1 K(x_N, x_{N-1}; \epsilon) \cdots K(x_1, x_0; \epsilon) \tag{1.13}$$

with $t = N\epsilon$. For a short distance and a short time-interval the propagator asymptotically behaves as

$$K(x + \Delta x, x; \Delta t) \sim \frac{1}{\sqrt{2\pi \mathrm{i}\Delta t}} \exp\left[\frac{\mathrm{i}}{2}\left(\frac{\Delta x}{\Delta t}\right)^2 \Delta t - \mathrm{i}V(x)\Delta t\right]. \tag{1.14}$$

Then "the limit $N \to \infty$" gives an infinite-multiplied integration, which is called the path integral,

$$K(x', x; t) = \int_x^{x'} \mathcal{D}x\, \mathrm{e}^{\mathrm{i}\int L\mathrm{d}s} = \int_x^{x'} \mathcal{D}x \exp\left[\mathrm{i}\int_0^t \mathrm{d}s\left(\frac{1}{2}\dot{x}(s)^2 - V(x(s))\right)\right]. \tag{1.15}$$

In a rigorous sense, the limit $N \to \infty$ does not exists but physicists use this expression for convenience. The philosophy of the path integral can be symbolically written as

$$\text{propagation of the wave} = \sum_{\text{trajectories}} \text{motion of the particle}. \qquad (1.16)$$

We can construct the path integral on the half line $\mathbb{R}_{\geq 0} = \mathbb{R}/\mathbb{Z}_2$ as well:

$$K_N(x', x; t) = \sum_{n=0}^{\infty} \int_x^{x'} \mathcal{D}x \, \mathrm{e}^{\mathrm{i} \int L \, \mathrm{d}s}, \qquad (1.17)$$

$$K_D(x', x; t) = \sum_{n=0}^{\infty} (-1)^n \int_x^{x'} \mathcal{D}x \, \mathrm{e}^{\mathrm{i} \int L \, \mathrm{d}s}, \qquad (1.18)$$

where the summations are taken with respect to the number of reflections of the trajectory at the boundary $x = 0$.

Now another question arises; what is the definition of path integrals on a general orbifold M/G? Our main concerns are propagators and path integrals in M/G.

2. Reduction of Quantum System

When a quantum system has a symmetry, it is decomposed into a family of quantum systems that are defined in the subspaces of the original. Here we review the reduction method [5] of quantum system.

A quantum system $(\mathcal{H}, H)$ is defined by a pair of a Hilbert space $\mathcal{H}$ and a Hamiltonian H, which is a self-adjoint operator on $\mathcal{H}$. The symmetry of the quantum system is specified by (G, T), where G is a compact Lie group and T is a unitary representation of G over $\mathcal{H}$. The symmetry implies that $T(g)H = HT(g)$ for all $g \in G$. The compact group G is equipped with the normalized invariant measure $\mathrm{d}g$.

To decompose $(\mathcal{H}, H)$ into a family of reduced quantum systems, we introduce $(\mathcal{H}^\chi, \rho^\chi)$, where $\mathcal{H}^\chi$ is a finite dimensional Hilbert space of the dimensions $d^\chi = \dim \mathcal{H}^\chi$. Besides, ρ^χ is an irreducible unitary representation of G over $\mathcal{H}^\chi$. The set $\{\chi\}$ labels all the inequivalent representations. For each $g \in G$, $\rho^\chi(g) \otimes T(g)$ acts on $\mathcal{H}^\chi \otimes \mathcal{H}$ and defines the tensor product representation. The **reduced Hilbert space** is defined as the subspace of the invariant vectors of $\mathcal{H}^\chi \otimes \mathcal{H}$,

$$(\mathcal{H}^\chi \otimes \mathcal{H})^G := \{\psi \in \mathcal{H}^\chi \otimes \mathcal{H} \,;\, \forall h \in G,\, (\rho^\chi(h) \otimes T(h))\psi = \psi\}. \qquad (2.1)$$

Let the set $\{e_1^\chi, \ldots, e_d^\chi\}$ be an orthonormal basis of $\mathcal{H}^\chi$. Then the *reduction operator* $S_i^\chi \colon \mathcal{H} \to (\mathcal{H}^\chi \otimes \mathcal{H})^G$ is defined by

$$f \in \mathcal{H} \mapsto S_i^\chi f := \sqrt{d^\chi} \int_G \mathrm{d}g\, (\rho^\chi(g) e_i^\chi) \otimes (T(g) f) \,. \tag{2.2}$$

Theorem 2.1. *S_i^χ is a partial isometry. Namely, $(S_i^\chi)^* S_i^\chi$ is an orthogonal projection operator acting on $\mathcal{H}$ while $S_i^\chi (S_i^\chi)^*$ is the identity operator on $(\mathcal{H}^\chi \otimes \mathcal{H})^G$.*

Theorem 2.2. *The family of the projections $\{(S_i^\chi)^* S_i^\chi\}$ forms a resolution of the identity as*

$$\sum_{\chi, i} (S_i^\chi)^* S_i^\chi = I_{\mathcal{H}} \,. \tag{2.3}$$

Hence, the Hilbert space is decomposed as

$$\mathcal{H} = \bigoplus_{\chi, i} \mathrm{Im} (S_i^\chi)^* S_i^\chi \cong \bigoplus_{\chi, i} (\mathcal{H}^\chi \otimes \mathcal{H})^G \tag{2.4}$$

and this decomposition is compatible with the Hamiltonian action. Namely, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{S_i^\chi} & (\mathcal{H}^\chi \otimes \mathcal{H})^G \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes H} \\ \mathcal{H} & \xrightarrow{S_i^\chi} & (\mathcal{H}^\chi \otimes \mathcal{H})^G \end{array} \tag{2.5}$$

Then $((\mathcal{H}^\chi \otimes \mathcal{H})^G, \mathrm{Id} \otimes H)$ defines a ***reduced quantum system****.*

The projection $P^\chi \colon \mathcal{H}^\chi \otimes \mathcal{H} \to (\mathcal{H}^\chi \otimes \mathcal{H})^G$ onto the reduced space is defined by

$$P^\chi := \int_G \mathrm{d}g\, \rho^\chi(g) \otimes T(g) \,. \tag{2.6}$$

The **reduced time-evolution operator** of the reduced system is

$$U^\chi := P^\chi (\mathrm{Id} \otimes \mathrm{e}^{-\mathrm{i}Ht}) \,. \tag{2.7}$$

Theorems 2.1 and 2.2 are easily proved by an application of the Peter–Weyl theorem, which states that the set of the matrix elements of irreducible unitary representations $\{\sqrt{d^\chi}\, \rho_{ij}^\chi(g)\}_{\chi, i, j}$ forms a complete orthonormal set of $L_2(G)$. Our main purpose is to give a path-integral expression to the time-evolution operator U^χ. To describe it we need to introduce some related notions.

Assume that the base space M is equipped with the measure $\mathrm{d}x$. Then the space of the square-integrable functions $L_2(M)$ becomes a Hilbert space $\mathcal{H}$. Moreover, assume that the compact Lie group G acts on M preserving the measure $\mathrm{d}x$. Then $g \in G$ is represented by the unitary operator $T(g)$ on $f \in L_2(M)$ by

$$(T(g)f)(x) := f(g^{-1}x)\,. \tag{2.8}$$

Let $p\colon M \to Q = M/G$ be the canonical projection map. Then a measure $\mathrm{d}q$ of $Q = M/G$ is induced by the following way. Let $\phi(q)$ be a function on Q such that $\phi(p(x))$ is a measurable function on M. The induced measure $\mathrm{d}q$ of Q is then defined by

$$\int_Q \mathrm{d}q\,\phi(q) := \int_M \mathrm{d}x\,\phi(p(x))\,. \tag{2.9}$$

On the other hand, suppose that the time-evolution operator $\mathbb{U}(t) := \mathrm{e}^{-\mathrm{i}Ht}$ is expressed in terms of an integral kernel $K\colon M \times M \times \mathbb{R}_{>0} \to \mathbb{C}$ as

$$(\mathbb{U}(t)f)(x) = \int_M \mathrm{d}y\,K(x,y;t)f(y) \tag{2.10}$$

for any $f(x) \in L_2(M)$.

Let us turn to the reduced Hilbert space (2.1) and characterize it for the case $\mathcal{H} = L_2(M)$. A vector $\psi \in \mathcal{H}^\chi \otimes L_2(M)$ can be identified with a measurable map $\psi\colon M \to \mathcal{H}^\chi$. The tensor product $\rho^\chi(g) \otimes T(g)$ acts on ψ as

$$((\rho^\chi(g) \otimes T(g))\psi)(x) = \rho^\chi(g)\psi(g^{-1}x)\,, \qquad g \in G \tag{2.11}$$

via the definition (2.8). The definition (2.1) of the invariant vector $\psi \in (\mathcal{H}^\chi \otimes L_2(M))^G$ implies

$$((\rho^\chi(g) \otimes T(g))\psi)(x) = \rho^\chi(g)\psi(g^{-1}x) = \psi(x)\,, \tag{2.12}$$

which is equivalent to

$$\psi(gx) = \rho^\chi(g)\psi(x)\,. \tag{2.13}$$

A function $\psi\colon M \to \mathcal{H}^\chi$ satisfying the above property is called an **equivariant function**. Hence the reduced Hilbert space is identified with the space of the equivariant functions $L_2(M, \mathcal{H}^\chi)^G$.

The projection operator $P^\chi\colon L_2(M; \mathcal{H}^\chi) \to L_2(M, \mathcal{H}^\chi)^G$, is now given by

$$(P^\chi\psi)(x) = \int_G \mathrm{d}g\,\rho^\chi(g)\psi(g^{-1}x)\,. \tag{2.14}$$

From (2.7–2.10) and (2.14) the reduced time-evolution operator is given by

$$(U^\chi(t)\psi)(x) = \int_G \mathrm{d}g \int_M \mathrm{d}y\, \rho^\chi(g) K(g^{-1}x, y; t)\psi(y) \tag{2.15}$$

and thus the corresponding **reduced propagator** is $K^\chi \colon M \times M \times \mathbb{R}_{>0} \to \operatorname{End}\mathcal{H}^\chi$ is defined by

$$K^\chi(x, y; t) := \int_G \mathrm{d}g\, \rho^\chi(g) K(g^{-1}x, y; t)\,. \tag{2.16}$$

Our aim is to express the reduced propagator in terms of path integrals.

3. Stratification Geometry

To write down a concrete form of the path integral we need to equip the base space M with a Riemannian structure. Namely, now we assume that M is a differential manifold equipped with a Riemannian metric g_M and that the Lie group G acts on M preserving the metric g_M. Then the volume form induced from the metric defines an invariant measure $\mathrm{d}x$ of M. We do *not* assume that the action of G on M is free. Therefore $p \colon M \to M/G$ is not necessarily a principal bundle.

For each point $x \in M$, $G_x := \{g \in G\,;\, gx = x\}$ is called the **isotropy group** of x and $\mathcal{O}_x := \{gx \,|\, g \in G\}$ is the **orbit** through x. It is easy to see that $\mathcal{O}_x \cong G/G_x$. Note that the dimensions of the orbit $\mathcal{O}_x$ can change suddenly when the point $x \in M$ is moved. The subspace of the tangent space T_xM, $V_x := T_x\mathcal{O}_x$, is called the **vertical subspace** and its orthogonal complement $H_x := (V_x)^\perp$ is called the **horizontal subspace**. $P_V \colon T_xM \to V_x$ is the **vertical projection** while $P_H \colon T_xM \to H_x$ is the **horizontal projection**. A curve in M whose tangent vector always lies in the horizontal subspace is called a **horizontal curve**. Although these terms have been introduced in the theory of principal fiber bundle, we use them for a more general manifold that admits group action.

Let $\mathfrak{g}$ denote the Lie algebra of the group G. For each $x \in M$, $\mathfrak{g}_x$ is the Lie subalgebra of the isotropy group G_x. The group action $G \times M \to M$ induces infinitesimal transformations $\mathfrak{g} \times M \to TM$ by differentiation. The induced linear map $\theta_x \colon \mathfrak{g} \to T_xM$ has $\ker\theta_x = \mathfrak{g}_x$ and $\Im\theta_x = V_x$. Then it defines an isomorphism $\tilde{\theta}_x \colon \mathfrak{g}/\mathfrak{g}_x \to V_x$. Now we define the **stratified connection form** ω by

$$\omega_x := (\tilde{\theta}_x)^{-1} \circ P_V \colon T_xM \to \mathfrak{g}/\mathfrak{g}_x\,. \tag{3.1}$$

Actually ω is not smooth over the whole M but it is smooth on each stratum of M.

4. Reduction of Path Integral

The Riemannian structure (M, g_M) defines the Laplacian Δ_M. Suppose that $V\colon M \to \mathbb{R}$ is a potential function such that $V(gx) = V(x)$ for all $x \in M$, $g \in G$. Then the Hamiltonian $H = \frac{1}{2}\Delta_M + V(x)$, which acts on $L_2(M)$, commutes with the action of G, which is defined in (2.8). Let us assume that the path integral in M is formally given by

$$K(x', x; t) = \int_x^{x'} \mathcal{D}x \exp\left[\mathrm{i}\int_0^t \mathrm{d}s \left(\frac{1}{2}\|\dot{x}(s)\|^2 - V(x(s))\right)\right]. \tag{4.1}$$

Now we repeat our question; what is the path-integral expression for the reduced propagator (2.16) on $Q = M/G$? The answer is our main result which is given below.

Theorem 4.1. *The reduced path integral on $Q = M/G$ is*

$$\begin{aligned} K^\chi(x', x; t) = \int_q^{q'} \mathcal{D}q\, \rho^\chi(\gamma)\rho^\chi_*\left(\mathcal{P}\exp\left[-\frac{\mathrm{i}}{2}\int_0^t \mathrm{d}s\, \Lambda(\tilde{q}(s))\right]\right) \\ \times \exp\left[\mathrm{i}\int_0^t \mathrm{d}s \left(\frac{1}{2}\|\dot{q}(s)\|^2 - V(q(s))\right)\right]. \end{aligned} \tag{4.2}$$

To read the above equation we need explanation of the symbols. The canonical projection map $p\colon M \to Q = M/G$ induces the metric g_Q of Q by asserting that the map p is a stratified Riemannian submersion. For $x, x' \in M$ we put $q = p(x)$ and $q' = p(x')$. The map $q\colon [0, t] \to Q$ is a curve connecting $q = q(0)$ and $q' = q(t)$. The map $\tilde{q}\colon [0, t] \to M$ is a horizontal curve such that $\tilde{q}(0) = x$ and $p(\tilde{q}(s)) = q(s)$ for $s \in [0, t]$. The element $\gamma \in G$ is a holonomy defined by $x' = \gamma \cdot \tilde{q}(t)$.

To describe the symbol Λ, which is called the **rotational energy operator**, we need more explanation. The metric $g_M\colon TM \otimes TM \to \mathbb{R}$ defines an isomorphism $\hat{g}_M\colon TM \to T^*M$. Then its inverse map $\hat{g}_M^{-1}\colon T^*M \to TM$ defines a symmetric tensor field $g_M^{-1}\colon M \to TM \otimes TM$. Thus combining it with the stratified connection $\omega_x\colon T_xM \to \mathfrak{g}/\mathfrak{g}_x$ we define the rotational energy operator by

$$\Lambda(x) := -(\omega_x \otimes \omega_x) \circ g_M^{-1}(x) \in (\mathfrak{g}/\mathfrak{g}_x) \otimes (\mathfrak{g}/\mathfrak{g}_x). \tag{4.3}$$

The unitary representation ρ^χ of the group G in $\mathcal{H}^\chi$ induces the representation ρ^χ_* of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then we have $\rho^\chi_*(\Lambda(x)) \in \operatorname{End}\mathcal{H}^\chi$. Moreover,

$$\lambda(\tau) = \rho_*^\chi \left(\mathcal{P} \exp \left[-\frac{\mathrm{i}}{2} \int_0^\tau \mathrm{d}s \, \Lambda(\tilde{q}(s)) \right] \right) \in \operatorname{End} \mathcal{H}^\chi \tag{4.4}$$

is defined as a solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \lambda(\tau) = -\frac{\mathrm{i}}{2} \rho_*^\chi \left(\Lambda(\tilde{q}(\tau)) \right) \lambda(\tau) \,, \qquad \lambda(0) = I \in \operatorname{End} \mathcal{H}^\chi \,. \tag{4.5}$$

Now we can read off the physical meaning of the reduced path integral (4.2). The path integral is expressed as a product of three factors:

i) the rotational energy amplitude $\exp[-\frac{\mathrm{i}}{2} \int_0^t \mathrm{d}s \, \Lambda(\tilde{q}(s))]$, which represents motion of the particle along the vertical directions of $p \colon M \to M/G$;
ii) the vibrational energy amplitude $\exp[\mathrm{i} \int_0^t \mathrm{d}s \, (\frac{1}{2} \|\dot{q}(s)\|^2 - V(q(s)))]$, which represents motion of the particle along the horizontal directions;
iii) the holonomy factor γ, which is caused by non-integrability of the horizontal distributions.

Here we give the outline of the proof of the main Theorem 4.1. For the detail see the reference [6]. Essentially, it is only a matter of calculation; from the path integral on M (4.1)

$$K(x', x; t) = \int_x^{x'} \mathcal{D}x \, \mathrm{e}^{\mathrm{i}I[x]} \,, \qquad I[x] = \int_0^t \mathrm{d}s \, \left(\frac{1}{2} \|\dot{x}(s)\|^2 - V(x(s)) \right) \tag{4.6}$$

with the reduction procedure (2.16) we get

$$\begin{aligned}
K^\chi(x', x; t) &:= \int_G \mathrm{d}h \, \rho^\chi(h) K(h^{-1}x', x; t) = \int_G \mathrm{d}h \, \rho^\chi(h) \int_x^{h^{-1}x'} \mathcal{D}x \, \mathrm{e}^{\mathrm{i}I[x]} \\
&= \int_G \mathrm{d}h \, \rho^\chi(h) \int_q^{q'} \mathcal{D}q \int_e^{h^{-1}\gamma} \mathcal{D}g \, \mathrm{e}^{\mathrm{i}I[g\tilde{q}]} = \int_q^{q'} \mathcal{D}q \int_G \mathrm{d}h \, \rho^\chi(h) \int_e^{h^{-1}\gamma} \mathcal{D}g \, \mathrm{e}^{\mathrm{i}I[g\tilde{q}]} \\
&= \int_q^{q'} \mathcal{D}q \int_G \mathrm{d}h \, \rho^\chi(\gamma h) \int_e^{h^{-1}} \mathcal{D}g \, \mathrm{e}^{\mathrm{i}I[g\tilde{q}]} \\
&= \int_q^{q'} \mathcal{D}q \rho^\chi(\gamma) \int_G \mathrm{d}h \, \rho^\chi(h) \int_e^{h^{-1}} \mathcal{D}g \, \mathrm{e}^{\mathrm{i} \int \mathrm{d}s \, \frac{1}{2} \|\dot{g}\|^2} \, \mathrm{e}^{\mathrm{i} \int \mathrm{d}s \, \{ \frac{1}{2} \|\dot{q}\|^2 - V(q) \}} \\
&= \int_q^{q'} \mathcal{D}q \rho^\chi(\gamma) \rho_*^\chi \left(\mathcal{P} \exp \left[-\frac{\mathrm{i}}{2} \int_0^t \mathrm{d}s \, \Lambda(\tilde{q}(s)) \right] \right) \mathrm{e}^{\mathrm{i} \int \mathrm{d}s \, \{ \frac{1}{2} \|\dot{q}\|^2 - V(q) \}} \,.
\end{aligned} \tag{4.7}$$

5. Example

Finally, we show an example of application of our formulation. Let us begin with the plane $M = \mathbb{R}^2$, which has the standard metric $g_M = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. It admits the symmetry action of $G = \mathbb{SO}(2)$. The quotient space is a half line $Q = \mathbb{R}^2/\mathbb{SO}(2) = \mathbb{R}_{\geq 0}$. The invariant potential is a function $V(r)$ only of r.

The group action

$$\mathbb{SO}(2) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \quad \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.1}$$

induces the action of the Lie algebra

$$\mathfrak{so}(2) \times \mathbb{R}^2 \longrightarrow T\mathbb{R}^2; \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.2}$$

which defines the vertical distribution

$$\theta \colon \mathfrak{so}(2) \times \mathbb{R}^2 \longrightarrow T\mathbb{R}^2; \quad \left(\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \phi \frac{\partial}{\partial \theta}. \tag{5.3}$$

Then the stratified connection becomes

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\theta. \tag{5.4}$$

In the cotangent space the metric is given as

$$(g_M)^{-1} = \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}. \tag{5.5}$$

The rotational energy operator is

$$\Lambda = -(\omega \otimes \omega) \circ (g_M)^{-1} = -\frac{1}{r^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.6}$$

The irreducible unitary representations of $\mathbb{SO}(2)$ are labeled by the integers $n \in \mathbb{Z}$ and defined by

$$\rho_n \colon \mathbb{SO}(2) \longrightarrow \mathbb{U}(1); \quad \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mapsto \mathrm{e}^{\mathrm{i}n\phi}. \tag{5.7}$$

The differential representation of the Lie algebra of $\mathbb{SO}(2)$ is

$$(\rho_n)_* \colon \mathfrak{so}(2) \longrightarrow \mathfrak{u}(1); \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mapsto \mathrm{i}n\phi. \tag{5.8}$$

The rotational energy operator is then represented as

$$(\rho_n)_*(\Lambda) = -\frac{(\mathrm{i}n)^2}{r^2} = \frac{n^2}{r^2}\,. \tag{5.9}$$

Finally the reduced path integral is given by

$$\begin{aligned} K_n(r',\theta',r,\theta;t) = & \int_r^{r'} \mathcal{D}r\, \mathrm{e}^{\mathrm{i}n(\theta'-\theta)} \\ & \times \exp\left[\mathrm{i}\int_0^t \mathrm{d}s \left\{ -\frac{n^2}{2r^2} + \frac{1}{2}\dot{r}^2 - V(r) \right\}\right]. \end{aligned} \tag{5.10}$$

So the effective potential for the radius coordinate r is given by

$$V_{\mathrm{eff}}(r) = V(r) + \frac{n^2}{2r^2}\,, \tag{5.11}$$

where the second term represents the centrifugal force.

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