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CONSTRUCTION OF MAXIMAL SURFACES IN THE LORENTZ-MINKOWSKI SPACE

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Abstract. The Björling problem for maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$ has been recently studied by the author together with Alías and Chaves. The present paper is a natural extension of that work, and provides several variations of Björling problem. The main scheme is the following. One starts with a spacelike analytic curve in $\mathbb{L}^3$, and asks for the construction of a maximal surface which contains that curve, and satisfies additionally some other geometric condition. The solution of these Björling-type problems are then applied with a twofold purpose: to construct examples of maximal surfaces in $\mathbb{L}^3$ with prescribed properties, and to classify certain families of maximal surfaces.

1. Introduction

In 1844 Björling [2] asked whether, given an analytic strip in $\mathbb{R}^3$, it is possible to construct explicitly a minimal surface in $\mathbb{R}^3$ containing that strip in its interior. The question, known as **Björling problem** for minimal surfaces, was solved in 1890 by Schwarz [9] by means of a complex variable formula which describes minimal surfaces in terms of analytic strips. That formula happened to be quite useful to establish results about minimal surfaces, as well as to construct particular examples of minimal surfaces in $\mathbb{R}^3$ with interesting geometric properties. Modern approaches to the Björling problem in Euclidean space can be found in [3, 8].

This classic geometric setting was extended in [1] to the case of maximal surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$. A surface in $\mathbb{L}^3$ is a **maximal surface** provided it has zero mean curvature and its induced metric is Riemannian. The solution to Björling problem in $\mathbb{L}^3$ states the following.

Let $\beta\colon I \longrightarrow \mathbb{L}^3$ be a regular spacelike analytic curve, and $V\colon I \longrightarrow \mathbb{L}^3$ be a unit timelike analytic vector field along β such that $\langle \beta', V\rangle \equiv 0$. There exists a unique maximal surface in $\mathbb{L}^3$ that contains β and whose Gauss map along β is given by V. This maximal surface can be explicitly constructed by means of a complex variable formula.

It turns out that this result can be used mainly in two directions. On the one hand it yields a procedure for the construction of maximal surfaces in $\mathbb{L}^3$ with specific properties. On the other hand it can be seen as a complex representation for maximal surfaces in $\mathbb{L}^3$, and hence it may be applied to obtain properties of this type of surfaces. For instance, in [1] it is shown by means of this representation formula that every maximal surface is symmetric with respect to any straight line contained in its interior.

In this paper we extend the topics treated in [1], giving some applications of the Björling problem in $\mathbb{L}^3$. Essentially, the results that we present here can be seen as Björling-type problems in $\mathbb{L}^3$ in which, instead of fixing the Gauss map of the maximal surface along β, we ask the maximal surface to satisfy some other geometric condition related to β. More specifically, suppose that we are given a regular spacelike analytic curve β in $\mathbb{L}^3$. Then we study the problem of constructing all the maximal surfaces in $\mathbb{L}^3$ that contain β and such that one of the following properties is fulfilled.

1. The curve $\beta(s)$ is a geodesic, an asymptotic line or a line or curvature of the maximal surface.
2. The curve $\beta(s)$ lies in a semi-Riemannian analytic surface in $\mathbb{L}^3$ that intersects the maximal surface along $\beta(s)$ orthogonally, or more generally, with constant angle.
3. The curve $\beta(s)$ lies in a degenerate plane of $\mathbb{L}^3$ that intersects the maximal surface along $\beta(s)$ with constant angle.

All of this will be treated in Section 3. In Section 4 the case in which $\beta(s)$ is a circle of $\mathbb{L}^3$ is considered. It turns out that the maximal surfaces that contain a circle as a planar line of curvature are precisely the **catenoids of** $\mathbb{L}^3$, that is, the maximal surfaces of revolution in $\mathbb{L}^3$. Finally, in Section 5 we present some examples of maximal surfaces in $\mathbb{L}^3$ constructed via Björling problem, starting from planar spacelike curves with a prescribed geometrical meaning. This examples were first introduced in [1], and can be seen as Lorentzian counterparts to some classical minimal surfaces in $\mathbb{R}^3$.

2. Björling Problem

Let $\mathbb{L}^3$ denote the 3-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle\,,\rangle = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2\,,$$

where (x_1, x_2, x_3) are the canonical coordinates in $\mathbb{R}^3$. In $\mathbb{L}^3$ we can define for any $a, b \in \mathbb{L}^3$ the cross-product $a \times b \in \mathbb{L}^3$, given by

$$a \times b = \Big(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2\Big)\,,$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$. Thus for any $x \in \mathbb{L}^3$ it holds the relation $\langle a \times b, x\rangle = \det(a, b, x)$.

A smooth immersion $\chi\colon M^2 \longrightarrow \mathbb{L}^3$ of a 2-dimensional connected orientable manifold is said to be a **semi-Riemannian surface** in $\mathbb{L}^3$ if the induced metric on M^2 via χ is non-degenerate. In that case this metric, which as usual is also denoted by $\langle\,,\rangle$, is Riemannian or Lorentzian and the surface is said to be **spacelike** or **timelike**, respectively.

If $\chi\colon M^2 \longrightarrow \mathbb{L}^3$ is a semi-Riemannian surface we can choose a unit normal vector field N globally defined on M^2. This normal field N can be regarded as a map $N\colon M^2 \longrightarrow \mathbb{H}^2$ if the surface is spacelike, and as $N\colon M^2 \longrightarrow \mathbb{S}^2_1$ if the surface is timelike. Here $\mathbb{H}^2$ denotes the 2-dimensional hyperbolic space, that is $\mathbb{H}^2 = \{x \in \mathbb{L}^3;\, \langle x, x\rangle = -1\}$, while $\mathbb{S}^2_1$ stands for the 2-dimensional de Sitter space $\mathbb{S}^2_1 = \{x \in \mathbb{L}^3;\, \langle x, x\rangle = 1\}$. We will refer to N as the **Gauss map** of the semi-Riemannian surface.

Let us restrict our attention to spacelike surfaces. In that case, since M^2 is Riemannian, we can define isothermal coordinates s, t around any point of M^2. Therefore every spacelike surface in $\mathbb{L}^3$ may be seen, at least locally, as a conformal map $\chi(z)\colon \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{L}^3$. Here by writing $\Omega \subseteq \mathbb{C}$ we want to emphasize that the complex parameter $z = s + \mathrm{i}t$ provides isothermal parameters s, t for Ω. The Gauss map of a surface of this kind will be assumed to be $N = \chi_s \times \chi_t / |\chi_s \times \chi_t|$. Here $|\,|$ stands for the norm in $\mathbb{L}^3$, given by $|x| = |\langle x, x\rangle|^{1/2}$ for all $x \in \mathbb{L}^3$.

A spacelike surface in $\mathbb{L}^3$ is said to be a maximal surface if the mean curvature of the immersion vanishes identically. It turns out that if we are given a spacelike surface of the type $\chi(z)\colon \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{L}^3$, then this surface is maximal if and only if the coordinate functions χ_1, χ_2, χ_3 are harmonic functions in the usual Euclidean sense.

The Björling problem for maximal surfaces consists on the following.

Let $\beta\colon I \to \mathbb{L}^3$ be a regular spacelike analytic curve in $\mathbb{L}^3$, and let $V\colon I \to \mathbb{L}^3$ be a unit timelike analytic vector field along β such that $\langle \beta', V\rangle \equiv 0$. Construct a maximal surface in $\mathbb{L}^3$ containing β whose Gauss map along β is given by V.

This Björling problem turns out to have a unique solution. The specific result, which was proved in [1], states what follows.

Theorem 2.1. (Solution to Björling problem) *Let $\beta\colon I \to \mathbb{L}^3$ be a regular analytic spacelike curve in $\mathbb{L}^3$, and let $V\colon I \to \mathbb{L}^3$ be a timelike analytic unit vector field along β such that $\langle \beta', V\rangle \equiv 0$. There exists a unique maximal surface whose image contains $\beta(I)$ and such that its Gauss mapping along β is V. This maximal surface is explicitly given by*

$$\chi(z) = \mathrm{Re}\left(\beta(z) + \mathrm{i}\int_{s_0}^{z} V(w) \times \beta'(w)\,\mathrm{d}w\right). \tag{2.1}$$

Here $\beta(z), V(z)$ are holomorphic extensions of $\beta(s), V(s)$ over a certain simply connected open set $\Omega \subseteq \mathbb{C}$ containing I, and $s_0 \in I$ is fixed but arbitrary.

The uniqueness part means the following: two maximal surfaces that satisfy the conditions of the Theorem must overlap over a non-empty open set that contains $\beta(I)$. Since maximal surfaces are real analytic, this amounts to say that both maximal surfaces determine the same inextendible maximal surface.

Besides, let us note that the integral in formula (2.1) does not depend neither on s_0 nor on the chosen path between s_0 and z.

3. Maximal Surfaces Generated by Curves

All along this work we will use the following notation: if $f(s)$ is a real analytic function, defined on a real interval I, we will denote its holomorphic extension to an open set of $\mathbb{C}$ by $f(z)$. Of course, $f(z)$ is uniquely determined. In this way, the holomorphic extension of an analytic curve $\beta(s)$ in $\mathbb{L}^3$ will be denoted by $\beta(z)$, and will take its values in $\mathbb{C}^3$.

Let now $\Sigma \subset \mathbb{L}^3$ be a semi-Riemannian surface in $\mathbb{L}^3$ with Gauss map N_Σ, and consider a curve $\beta(s)$ on Σ. We plan to construct maximal surfaces in $\mathbb{L}^3$ that intersect Σ in an interesting way along β. For this, let us begin with a definition. We will say that a spacelike surface intersects Σ *transversally* provided at any point of the intersection (which is assumed to be non-empty) the tangent planes of the surface and Σ are distinct. It is easy to show that in this case the former intersection is a regular spacelike curve. We say that a spacelike surface intersects Σ *orthogonally* if N_Σ is orthogonal to the Gauss map of the surface at any point of the intersection. In this situation, Σ must be a timelike surface.

Theorem 3.1. *Let $\beta\colon I \to \mathbb{L}^3$ be a regular spacelike analytic curve in $\mathbb{L}^3$ contained in a timelike analytic surface $\Sigma \subset \mathbb{L}^3$. There exists a unique maximal surface in $\mathbb{L}^3$ that intersects Σ orthogonally along β. This maximal surface is given explicitly by*

$$\chi(z) = \operatorname{Re}\left(\beta(z) + \mathrm{i}\int^{z} |\beta'(w)| N_\Sigma(w)\,\mathrm{d}w\right). \tag{3.1}$$

Here $N_\Sigma(s)$ stands for the Gauss map of Σ at $\beta(s)$.

Proof: The Gauss map along β of any maximal surface in the above conditions is orthogonal to both β' and N_Σ. Hence, reversing orientation if necessary we get that

$$N(s) = \frac{N_\Sigma(s) \times \beta'(s)}{|N_\Sigma(s) \times \beta'(s)|},$$

and from Theorem 2.1 we obtain the desired existence and uniqueness. For this particular surface it is now easy to check that formula (2.1) simplifies to (3.1). □

A curve $\beta(s)$ on a spacelike surface is said to be a *geodesic* if $\beta''(s)$ is collinear with $N(s)$ for all s, where here $N(s)$ is the Gauss map of the surface along $\beta(s)$. In this way $\beta''(s)$ is timelike whenever it is non zero. If in the preceding Theorem we choose Σ to be a timelike plane Π, it is easy to note that $\beta(s)$ will be a geodesic for the resulting maximal surface whenever it has constant speed, and that the converse trivially holds. We shall call any curve of this kind a **planar geodesic**. In particular, if Π is the timelike x_1, x_3-plane, then $\beta(s) = \big(a(s), 0, b(s)\big)$ and equation (3.1) turns into

$$\chi(z) = \left(\operatorname{Re} a(z), \operatorname{Im}\int^{z} \sqrt{a'(w)^2 - b'(w)^2}\,\mathrm{d}w, \operatorname{Re} b(z)\right). \tag{3.2}$$

Of course, here we are considering the principal branch of the holomorphic square root function, defined in $\mathbb{C} \setminus \{z \in \mathbb{R};\ z \leq 0\}$.

In the case where Σ is the de Sitter space $\mathbb{S}^2_1 = \{x \in \mathbb{L}^3;\ \langle x, x\rangle = 1\}$, if we choose $\beta(s)$ to be a unit spacelike curve the maximal surface generated by

Theorem 3.1 is given by

$$\chi(z) = \mathrm{Re}\left(\beta(z) + \mathrm{i}\int^{z} \beta(w)\,\mathrm{d}w\right).$$

Next, two applications of Theorem 3.1 are presented. The first one can be seen as the solution to a *geodesic* Björling problem, and was first set in [1]. However, the proof we give here is different.

Corollary 3.1. *If $\beta\colon I \to \mathbb{L}^3$ is a constant speed analytic spacelike curve in $\mathbb{L}^3$ such that $\beta''(s)$ is timelike for all $s \in I$, there exists a unique maximal surface in $\mathbb{L}^3$ that contains β as a geodesic. This maximal surface is given by*

$$\chi(z) = \mathrm{Re}\left(\beta(z) + \mathrm{i}\int^{z} \beta''(w) \times \beta'(w)/|\beta''(w)|\,\mathrm{d}w\right).$$

Proof: Let $\Sigma \subset \mathbb{L}^3$ be a timelike analytic surface that contains $\beta(s)$ and such that $\beta''(s)$ belongs to $T_{\beta(s)}\Sigma$ for all $s \in I$. Then a spacelike surface will contain β as a geodesic if and only if it intersects Σ orthogonally along β. A straightforward application of Theorem 3.1 does the rest. □

The following result explains the situation with respect to asymptotic lines. A curve $\beta(s)$ on a spacelike surface is called *asymptotic line* if $\beta''(s)$ is orthogonal to $N(s)$ for all s. Here $N(s)$ is, as usual, the Gauss map of the surface along $\beta(s)$. Thus $\beta''(s)$ will always be spacelike. Besides, the tangent plane of the surface at $\beta(s)$ will be spanned by $\{\beta'(s), \beta''(s)\}$ whenever those vectors are linearly independent and non zero.

Corollary 3.2. *Let $\beta\colon I \to \mathbb{L}^3$ be a regular analytic spacelike curve such that $\beta''(s)$ is spacelike and non zero for all $s \in I$, and suppose that $\beta'(s)$ and $\beta''(s)$ are always linearly independent. There exists a unique maximal surface in $\mathbb{L}^3$ containing β as an symptotic line. This maximal surface may be constructed as*

$$\chi(z) = \mathrm{Re}\left(\beta(z) + \mathrm{i}\int^{z} (\beta'' \times \beta' \times \beta')(w)/|(\beta'' \times \beta')(w)|\,\mathrm{d}w\right).$$

Proof: This time we choose $\Sigma \subset \mathbb{L}^3$ to be a timelike analytic surface containing $\beta(s)$ and such that $\beta''(s) \times \beta'(s)$ is tangent to Σ at $\beta(s)$ for all $s \in I$. The result follows arguing as in the proof of the previous Corollary. □

If $\beta(s)$ has constant speed, $|\beta'(s)| \equiv c > 0$, this equation simplifies to

$$\chi(z) = \mathrm{Re}\left(\beta(z) + \mathrm{i}c\int^{z} \beta''(w)/|\beta''(w)|\,\mathrm{d}w\right).$$

For what follows now, some definitions are needed. First of all, we recall that if $v, w \in \mathbb{L}^3$ are timelike vectors, there exists a unique $\varphi \geq 0$ such that $|\langle v, w\rangle| = \cosh\varphi|v||w|$. In the same way, if v is spacelike and w is timelike, there exists a unique $\varphi \geq 0$ such that $|\langle v, w\rangle| = \sinh\varphi|v||w|$. This last property is trivial.

Definition 3.1. *Let $\chi\colon M^2 \to \mathbb{L}^3$ be a spacelike surface that intersects a timelike (resp. spacelike) surface $\Sigma \subset \mathbb{L}^3$ along a spacelike curve $\beta(s)$. We say that the surface intersects Σ* with constant angle $\varphi \geq 0$ *provided $|\langle N(s), N_\Sigma(s)\rangle| \equiv \sinh\varphi$ (resp. $|\langle N(s), N_\Sigma(s)\rangle| \equiv \cosh\varphi$). Here $N(s)$ and $N_\Sigma(s)$ stand, respectively, for the Gauss map of the surface at $\beta(s)$ and the Gauss map of Σ at $\beta(s)$.*

Theorem 3.2. *Let $\beta\colon I \to \mathbb{L}^3$ be a regular analytic spacelike curve of a semi-Riemannian analytic surface $\Sigma \subset \mathbb{L}^3$, and choose $\varphi > 0$. There exists exactly two maximal surfaces in $\mathbb{L}^3$ that intersect Σ along β with constant angle φ. If Σ is a non-degenerate plane, those two maximal surfaces are congruent in $\mathbb{L}^3$.*

Proof: If Σ is spacelike with Gauss map N_Σ, we can suppose reversing the orientation of Σ if necessary that $\langle N(s), N_\Sigma(s)\rangle = -\cosh\varphi$. This condition together with $\langle N, N\rangle = -1$ show that

$$N(s) = \cosh\varphi N_\Sigma(s) + \varepsilon\frac{\sinh\varphi}{|\beta'(s)|}\beta'(s) \times N_\Sigma(s)\,, \tag{3.3}$$

where $\varepsilon = \pm 1$. Applying Theorem 2.1 we find exactly two maximal surfaces that satisfy the required conditions, depending on the sign of ε. In the case where Σ is a plane, $N_\Sigma(s)$ is constant and we may assume, composing if necessary with a symmetry with respect to Σ, that $\varepsilon = 1$. Hence the two maximal surfaces are congruent in $\mathbb{L}^3$.

If Σ is timelike the proof keeps the same pattern. □

Even though it has not been stated in the Theorem, both maximal surfaces can be constructed explicitly. For this, we only have to solve Björling problem for the data $\{\beta(s), N(s)\}$, where $N(s)$ is given by (3.3). In particular, if Σ is a spacelike plane in $\mathbb{L}^3$, the only maximal surface (up to congruence) in $\mathbb{L}^3$ that intersects Σ along β with constant angle φ is given by

$$\chi(z) = \mathrm{Re}\left(\beta(z) + \mathrm{i}e \times \beta(z)\cosh\varphi + \mathrm{i}e\sinh\varphi\int^{z} |\beta'(w)|\,\mathrm{d}w\right).$$

A similar formula is obtained when Σ is a timelike plane in $\mathbb{L}^3$.

Next we will extend Definition 3.1 to degenerate planes. First of all, recall that a plane Π of $\mathbb{L}^3$ is said to be *degenerate* if there exists a null vector $v \in \Pi$ such that $\langle v, w\rangle = 0$ for all $w \in \Pi$. Equivalently, Π is degenerate if it does not contain timelike vectors, but it contains a null vector.

If a spacelike surface intersects a degenerate plane Π along a regular curve $\beta(s)$, we say that it intersects Π *with constant angle* if $\langle N(s), v\rangle$ is constant for a null vector $v \in \Pi$. It is immediate that this property is independent of our choice of v. Besides, for any fixed null vector $v \in \Pi$, we say that the surface intersects Π with constant angle $\varphi_v > 0$ if $|\langle N(s), v\rangle| \equiv \varphi_v$.

A regular curve $\beta(s)$ on a spacelike surface is a *line of curvature* if $\beta'(s)$ is a principal direction at $\beta(s)$ for all s. Lines of curvature are characterized by the equation $N'(s) = \lambda(s)\beta'(s)$ for a certain $\lambda(s) \in C^\infty(I)$. The following result, whose proof is straightforward, characterizes the planar lines of curvature of a spacelike surface.

Lemma 3.1. *Let $\beta\colon I \to \mathbb{L}^3$ be a regular curve of a spacelike surface S in $\mathbb{L}^3$. Assume that $\beta(I)$ is contained in a plane Π of $\mathbb{L}^3$. Then β is a line of curvature of S if and only if S intersects Π along β with constant angle.*

Next let us establish the analogue result to Theorem 3.2 for degenerate planes.

Proposition 3.1. *Let $\beta\colon I \to \mathbb{L}^3$ be a regular analytic spacelike curve in $\mathbb{L}^3$ contained in a degenerate plane Π, and choose $v \in \Pi$ null and $\varphi_v > 0$. There exists a unique maximal surface in $\mathbb{L}^3$ that intersects Π along β with constant angle φ_v. This maximal surface can be explicitly constructed.*

Proof: Fix a null vector $v \in \Pi$. Then we can extend v to a basis of $\mathbb{L}^3$, $\{u, v, w\}$, so that Π is generated by v, w, and the metric relations $\langle u, u\rangle = \langle v, v\rangle = \langle u, w\rangle = \langle v, w\rangle = 0$, $\langle u, v\rangle = \langle w, w\rangle = 1$ hold. Such a basis is called a *null frame* of $\mathbb{L}^3$. With respect to this null frame we find that $\beta(s) = \big(0, a(s), b(s)\big)$. In addition, we will assume that $\langle N, v\rangle = \varphi_v$, reversing orientation if necessary. From this and the relation $\langle \beta', N\rangle = 0$, we get that if $N(s) = \big(x(s), y(s), z(s)\big)$, then $x(s) = \varphi_v$ and $z(s) = -\varphi_v(a'(s)/b'(s))$.

Using at this point that $\langle N, N\rangle = -1$ we obtain finally

$$N(s) = \frac{-1}{2\varphi_v}\Big(-2\varphi_v^2, 1 + \varphi_v^2\left(a'(s)^2/b'(s)^2\right), 2\varphi_v^2\left(a'(s)/b'(s)\right)\Big).$$

Therefore $N(s)$ is uniquely determined, and the solution to Björling problem provides the desired result. □

4. Catenoids and Circles in $\mathbb{L}^3$

A spacelike surface in $\mathbb{L}^3$ is said to be a surface of revolution if it can be obtained by the action over a plane curve $\Gamma \subset \Pi$ of the one-parameter group of rigid motions of $\mathbb{L}^3$ which fix a certain axis $\ell \subset \Pi$ such that $\Gamma \cap \ell = \emptyset$. It is well known that in this case the plane Π is always timelike. Then every surface of revolution in $\mathbb{L}^3$ is congruent to a surface of revolution for which Π is the x_1, x_3-plane and ℓ is the x_1-axis, the x_3-axis or the $x_1 = x_3$-line, depending on the causal character of ℓ (see [4] for details). The classification of the maximal surfaces of revolution in $\mathbb{L}^3$, also called *catenoids*, asserts the following (see [5, 7] and [1]).

- Any maximal surface of revolution with spacelike axis is congruent to a **hyperbolic catenoid**, which is parametrized as

$$\chi(s + \mathrm{i}t) = A\Big(s, \cos s \sinh t, \cos s \cosh t\Big). \tag{4.1}$$

  Here $A > 0$ and $\chi(s + \mathrm{i}t)$ is defined on $\Omega = (-\pi/2, \pi/2) \times \mathbb{R} \subset \mathbb{C}$.
- Any maximal surface of revolution with timelike axis is congruent to an **elliptic catenoid**, parametrized as

$$\chi(s + \mathrm{i}t) = A\Big(\sinh s \cos t, \sinh s \sin t, s\Big), \tag{4.2}$$

  where $A > 0$ and $\chi(s + \mathrm{i}t)$ is defined whenever $s > 0$.
- Any maximal surface of revolution with null axis is congruent to a **parabolic catenoid**. A conformal parametrization of this surface with respect to the canonical null frame $\mathcal{F}$ of $\mathbb{L}^3$, given by

$$\mathcal{F} = \Big\{\Big(\sqrt{2}/2, 0, -\sqrt{2}/2\Big), \Big(\sqrt{2}/2, 0, \sqrt{2}/2\Big), (0, 1, 0)\Big\}, \tag{4.3}$$

  is

$$\chi(s + \mathrm{i}t) = A\Big(s + B, -\frac{t^2}{2}(s + B) + \frac{1}{6}s^3 + \frac{B}{2}s^2 + \frac{B^2}{2}s, t(s + B)\Big). \tag{4.4}$$

  Here $A > 0$, $B \in \mathbb{R}$ and $\chi(s + \mathrm{i}t)$ is defined whenever $s > -B$.

Besides, we define a **circle in** $\mathbb{L}^3$ as a planar spacelike curve with non zero constant curvature (see [6] for details). Thus, depending on the causal character of the plane $\Pi \subset \mathbb{L}^3$ in where the circle lies, we have three type of circles in $\mathbb{L}^3$: circles in spacelike planes, which draw Euclidean circles; circles in timelike planes, that describe Euclidean hyperbolas, and finally circles in degenerate planes, which trace Euclidean parabolas of a certain type.

An alternative definition of circle is the following: a spacelike curve α in $\mathbb{L}^3$ is a circle if there exists a straight line ℓ in $\mathbb{L}^3$ such that α describes the non-linear orbit of a point $p \in \mathbb{L}^3 \setminus \ell$ under the action of the 1-parameter group of rigid motions of $\mathbb{L}^3$ that fix ℓ pointwise. In this case, α is a regular analytic curve contained in the plane Π of $\mathbb{L}^3$ that passes through p and that is perpendicular to ℓ.

With this second definition, it becomes clear that any curve of the form $\alpha(t) = \chi(s_0 + \mathrm{i}t)$ on a hyperbolic catenoid (resp. elliptic catenoid, parabolic catenoid) is a circle of a timelike (resp. spacelike, degenerate) plane.

In addition to this relation between circles and catenoids, we note that the following characterization of the hyperbolic catenoid was obtained in [1].

Proposition 4.1. *Every maximal surface in* $\mathbb{L}^3$ *containing a circle as a geodesic is a piece of a hyperbolic catenoid.*

To end up this Section we will improve this last result. For this, we do not require the circle to be a geodesic of the maximal surface, but we ask it to be a line of curvature. As follows from Lemma 3.1, this condition amounts to ask the maximal surface to intersect the plane that contains the circle with constant angle.

Theorem 4.1. *Every maximal surface in* $\mathbb{L}^3$ *that contains a circle of a timelike (resp. spacelike, degenerate) plane* Π *as a line of curvature is a piece of a hyperbolic catenoid (resp. elliptic catenoid, parabolic catenoid) whose axis is orthogonal to* Π.

Proof: Let $\alpha(s)$ be a circle in $\mathbb{L}^3$. Recalling the second definition of circle, we can suppose composing with a translation in $\mathbb{L}^3$ that both Π and ℓ pass through the origin. In this situation, note that $\langle \alpha(s), \alpha(s)\rangle$ is constantly equal to $\langle p, p\rangle$. Then we define the *radius* of $\alpha(s)$ as $R = |\langle p, p\rangle|^{1/2}$. It follows immediately that R is well defined, and that $R > 0$, since $R = 0$ would imply that $\alpha(s)$ is contained in a null line. All of this shows that a circle in $\mathbb{L}^3$ is determined, up to a rigid motion of $\mathbb{L}^3$, by its radius and by the causal character of the plane in which it is contained.

Moreover, suppose that $\alpha(s)$ is a circle of radius R lying in a non-degenerate plane Π, and choose $\varphi > 0$. Then Theorem 3.2 and the previous comments show that the data R, φ together with the causal character of Π determine up to a rigid motion the only maximal surface in $\mathbb{L}^3$ that intersects Π along $\alpha(s)$ with constant angle φ.

Once here, suppose that Π is timelike and choose $R > 0$, $\varphi \geq 0$. If we define $s_0 = \arctan(\sinh \varphi) \in [0, \pi/2)$, we can consider the hyperbolic catenoid given in (4.1) for the specific choice of $A = R/\cos s_0$. On this particular catenoid, the curve $\alpha(t) = \chi(s_0, t)$ is a circle of radius R. Moreover, this hyperbolic catenoid intersects the plane in which $\alpha(t)$ is contained with constant angle φ. Thus the previous discussion shows that every maximal surface that contains a circle of a timelike plane as a planar line of curvature is congruent to a hyperbolic catenoid.

In the case where the plane Π is spacelike, a similar discussion shows that every maximal surface containing a circle of Π as a line of curvature is congruent to an elliptic catenoid.

Finally, consider a circle $\alpha(s)$ of radius R lying in a degenerate plane Π. Recalling the canonical null frame of $\mathbb{L}^3$, $\mathcal{F} = \{F_1, F_2, F_3\}$, the plane Π will be assumed to be the degenerate plane spanned by F_2, F_3. Thus $v = F_2$ is a null vector on Π. It follows then from Proposition 3.1 that the only maximal surface that intersects Π along α with constant angle $\varphi_v \equiv \varphi > 0$ is uniquely determined up to rigid motions by the data R, φ. Once here it is easy to show, as we did for the hyperbolic catenoid, that any curve of the form $\chi(s_0, t)$ on a parabolic catenoid parametrized by (4.4) is a circle which is also a line of curvature of the surface. Besides, fixing R, φ, there exist constants A, B, s_0 depending on R and φ so that the curve $\alpha(t) = \chi(s_0, t)$ of the parabolic catenoid (4.4) determined by our choice of A, B is a circle of radius R lying in a degenerate plane. Moreover, the parabolic catenoid intersects this degenerate plane with constant angle φ along $\alpha(t)$. Consequently, every maximal surface that contains a circle of a degenerate plane as a planar line of curvature is congruent to a parabolic catenoid. This finishes the proof. □

5. Examples

In Section 3 we noted as a consequence of Theorem 3.1 that, given a regular analytic spacelike curve $\beta(s)$ contained in a timelike plane Π, there exists a unique maximal surface that intersects Π orthogonally along $\beta(s)$. That is, there is exactly one maximal surface that contains $\beta(s)$ as a planar geodesic. Furthermore, this maximal surface can be explicitly constructed by means of (3.2).

The last part of the present work is devoted to the construction of particular examples applying the above result. Specifically, we will present the natural analogues in $\mathbb{L}^3$ to the classic minimal surfaces of Enneper, Henneberg and Catalan in Euclidean space $\mathbb{R}^3$ (see [3, 8]). This was first done in [1], where some computer pictures of the resulting maximal surfaces are also shown.

All along this last section Π will stand for the timelike x_1, x_3-plane in $\mathbb{L}^3$.

Example 5.1. (Henneberg's maximal surface) *Consider on* Π *the* Neil's parabola

$$2(x_3+1)^3 = 9x_1^2\,.$$

This equation implicitly determines a spacelike curve that can be parametrized as

$$\beta(s) = \left(2\cosh s + \frac{2}{3}\cosh 3s, 0, 2\cosh 2s\right), \tag{5.1}$$

with $|\beta'(s)| = 8\cosh s \sinh^2 s$. *Applying now Theorem* 3.1 *we can construct a maximal surface in* $\mathbb{L}^3$, *which by* (3.2) *is written in coordinates as*

$$\begin{aligned}
\mathcal{H}^1(s,t) &= 2\cosh s\cos t + \frac{2}{3}\cosh 3s\cos 3t \\
\mathcal{H}^2(s,t) &= \frac{2}{3}\cosh 3s\sin 3t - 2\cosh s\sin t \\
\mathcal{H}^3(s,t) &= 2\cosh 2s\cos 2t
\end{aligned}$$

Inspired by the Euclidean situation we will name this surface as ***Henneberg's maximal surface****. Note that thanks to Theorem* 3.1 *Henneberg's maximal surface is the only maximal surface in* $\mathbb{L}^3$ *that contains Neil's parabola as a planar geodesic.*

Example 5.2. (Enneper's maximal surface) *Consider on* Π *the parabola*

$$x_1^2 = \frac{8}{3}x_3 - \frac{8}{9}\,.$$

This curve can be parametrized as

$$\gamma(s) = \frac{1}{3}\left(4s, 0, 2s^2+1\right)$$

and it is plain that γ *is spacelike if* $s \in (-1,1)$. *For each* s *we can construct a unique line in* Π *about which the points* $\gamma(s)$ *and* $(0,0,-1/3)$ *of* Π *are symmetric. The envelope of this family of straight lines is the curve*

$$\beta(s) = \left(\frac{1}{3}s^3 + s, 0, s^2\right)\,.$$

This is a spacelike curve on Π *with* $|\beta'(s)| = 1 - s^2$. *Applying Theorem* 3.1 *we get a maximal surface* $\mathcal{E}\colon \mathbb{D} \subseteq \mathbb{C} \to \mathbb{L}^3$ *intersecting* Π *orthogonally along* α, *given by*

$$\mathcal{E}(s,t) = \left(\frac{1}{3}s^3 - st^2 + s, \frac{1}{3}t^3 - s^2t + t, s^2 - t^2\right) .$$

Regarding the construction we have just done, we call this surface ***Enneper's maximal surface****, in analogy with the Euclidean situation (see [8], p. 80).*

For the next example we recall that the cycloid in $\mathbb{R}^2$ may be described as the orbit of the origin when we roll the unit circle centered at $(-1,0)$ upwards over the y-axis. This curve can be seen as an application which assigns to each $t \in \mathbb{R}$ the image of the point $(1,0)$ under the rotation in $\mathbb{R}^2$ of angle $-t$ composed with the translation $(-1,t)$.

We also note that in a Lorentz plane $\mathbb{L}^2$ a rotation of angle t is given by the matrix

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} .$$

Example 5.3. (Catalan's maximal surface) *Fix the point* $(1,0,0) \in \Pi$. *Then the equivalent process in* Π *to the one above that describes the cycloid in* $\mathbb{R}^2$ *produces a curve that can be parametrized as*

$$\beta(s) = (\cosh s - 1, 0, s - \sinh s) . \tag{5.2}$$

This curve is spacelike whenever $s > 0$. *A standard application of Theorem* 3.1 *for* β *produces a maximal surface given by*

$$\mathcal{C}(z) = \left(\cosh s \cos t - 1, 4 \sinh \frac{s}{2} \sin \frac{t}{2}, s - \sinh s \cos t\right) . \tag{5.3}$$

Here $z = s + \mathrm{i}t$ *and* $s > 0$. *We call this surface* ***Catalan's maximal surface****, since classical Catalan's surface is the only minimal surface in* $\mathbb{R}^3$ *which contains the cycloid as a planar geodesic (see [8, 3]).*

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