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INEQUALITIES AMONG THE NUMBER OF THE GENERATORS AND RELATIONS OF A KÄHLER GROUP

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Abstract. The present note announces some inequalities on the number of the generators and relations of a Kähler group $\pi_1(X)$, involving the irregularity $q(X)$, the Albanese dimension $a(X)$ and the Albanese genera $g_k(X)$, $1 \leq k \leq a(X)$, of the corresponding compact Kähler manifold X. The principal ideas for their derivation are outlined and the proofs are postponed to be published elsewhere.

Let X be an irregular compact Kähler manifold, i. e., with an irregularity $q = q(X) := \dim_{\mathbb{C}} H^{1,0}(X) > 0$. The **Albanese variety** $\mathrm{Alb}(X) = H^{1,0}(X)^*/H_1(X, \mathbb{Z})_{\text{free}}$ admits a holomorphic Albanese map $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$, given by integration $\mathrm{alb}_X(x)(\omega) := \int_{x_0}^{x} \omega$ of holomorphic $(1,0)$-forms $\omega \in H^{1,0}(X)$ from a base point $x_0 \in X$ to $x \in X$. The complex rank of the Albanese map alb_X is called an **Albanese dimension** $a = a(X)$ of X.
A compact Kähler manifold Y is said to be Albanese general if $\dim_{\mathbb{C}} Y = a(Y) < q(Y)$. The surjective holomorphic maps $f_k\colon X \to Y_k$ of a compact Kähler manifold X onto Albanese general Y_k are referred to as Albanese general k-fibrations of X. The maximum irregularity $q(Y_k)$ of a base Y_k of an Albanese general k-fibration $f_k\colon X \to Y_k$ is called k-th Albanese genus of X and denoted by $g_k = g_k(X)$. The present note states lower bounds on the Betti numbers $b_i(\pi_1(X)) := rk_{\mathbb{Z}} H^i(\pi_1(X), \mathbb{Z})$ of the fundamental group $\pi_1(X)$, in terms of the irregularity $q(X)$, the Albanese dimension $a(X)$ and the Albanese genera $g_k(X)$, $1 \leq k \leq a(X)$.
On the other hand, $b_i(\pi_1(X))$ are estimated above by the number of the generators s and the number of the relations r of $\pi_1(X)$ and, eventually, by the irregularity $q(X)$, exploiting to this end few abstract results on the group cohomologies.

Proposition 1. *Let X be a compact Kähler manifold with Albanese dimension $a \geq 2$, irregularity $q \geq a$ and Albanese genera g_k, $1 \leq k \leq a$. Put*

$$\mu^{2,0} := \max \left\{ \binom{\max\{a, g_k\,;\, g_k > 0, 2 \leq k \leq a\}}{2},\ \delta^0_{g_1}(2q-3) \right\},$$

$$\mu^{1,1} := \max \left\{ \binom{a}{2},\ 2a-1, g_k - 1,\ \delta^0_{g_1}(2q-3);\ g_k > 0,\ 2 \leq k \leq a \right\}$$

where $\delta^0_{g_1}$ stands for Kronecker's delta. Denote by $b_2(\pi_1(X)) := rk_{\mathbb{Z}} H^2(\pi_1(X), \mathbb{Z})$ the second Betti number of the fundamental group of X and suppose that $\pi_1(X)$ admits a finite presentation with s generators and r relations. Then

$$r - s + 2q \geq b_2(\pi_1(X)) \geq 2\mu^{0,2} + \mu^{1,1}\,.$$

If F is a free group and R is a normal subgroup of F then Hopf's Theorem is equivalent to the presence of the exact sequence

$$0 \to H_2(F/R, \mathbb{Z}) \to H_1(R, \mathbb{Z})_F \to H_1(F, \mathbb{Z}) \to H_1(F/R, \mathbb{Z}) \to 0$$

where $H_1(R, \mathbb{Z})_F$ stands for the F-coinvariants of $H_1(R, \mathbb{Z})$ (cf. [4] or [1]). In particular, for a Kähler group $\pi_1(X)$ with s generators and r relations there follows $r - s + 2q \geq b_2(\pi_1(X))$.
The isomorphism $H^1(\pi_1(X), \mathbb{C}) \simeq H^1(X, \mathbb{C})$ of the first cohomologies of $\pi_1(X)$ and X allows to introduce Hodge decomposition $H^1(\pi_1(X), \mathbb{C}) = H^{1,0}(\pi_1(X)) \oplus H^{0,1}(\pi_1(X))$ on the group cohomologies. After constructing an Eilenberg–MacLane space $K(\pi_1(X), 1)$ by glueing to X cells of real dimension ≥ 3, one observes that the complex rank of the cup product of group cohomologies

$$\zeta^{i,j}_{\pi_1(X)} \colon \wedge^i H^{1,0}(\pi_1(X)) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(\pi_1(X)) \to H^{i+j}(\pi_1(X), \mathbb{C})$$

dominates the complex rank of the cup product of de Rham cohomologies

$$\zeta^{i,j}_X \colon \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) \to H^{i+j}(X, \mathbb{C})\,.$$

The quantities $\mu^{i,j}$ are lower bounds on $rk_{\mathbb{C}} \zeta^{i,j}_X$. They are derived by the means of the cohomological descriptions of a and g_k, due to Catanese (cf. [2]). On one hand, $rk_{\mathbb{C}} \zeta^{i,j}_X \geq \binom{a}{i+j}$ for all non-negative integers i, j with $2 \leq i + j \leq a$. On the other hand, $rk_{\mathbb{C}} \zeta^{i,j}_X \geq \binom{g_k - i}{j}$ for $0 \leq i \leq j$ and $2 \leq i + j \leq k \leq a$. Further, $rk_{\mathbb{C}} \zeta^{i,j}_X \geq (i+j)(q-i-j) + 1$ provided $g_1 = \cdots = g_{i+j-1} = 0$. Finally, $rk^{1,1}_X \geq 2a - 1$.

Proposition 2. *Let X be a compact Kähler manifold whose fundamental group $\pi_1(X)$ admits a finite presentation with s generators and r relations. Then the complex rank of the cup products $\zeta^{i,j}_{\pi_1(X)}$ and $\zeta^{i,j}_X$ are bounded below by*

$$\mu^{i,j} := \max\left\{\binom{a}{i+j}, \binom{g_k - i}{j}, \delta^0_{g_1} \cdots \delta^0_{g_{i+j-1}}(i+j)(q-i-j)+1\right\}$$

where a stands for the Albanese dimension of X, g_k, $1 \leq k \leq a$, are the Albanese genera, $q > 0$ is the irregularity, $\delta^0_{g_s}$ denote Kronecker's deltas and the maximum is taken over the positive Albanese genera g_k, labeled by $i+j \leq k \leq a$. The Betti numbers $b_i(\pi_1(X)) := rk_{\mathbb{Z}} H^i(\pi_1(X), \mathbb{Z})$ are subject to the inequalities

$$sr^k \geq b_{2k+1}(\pi_1(X)) \geq 2\sum_{i=0}^{k} \mu^{i,2k+1-i} \quad \text{for } 1 \leq k \leq \frac{a-1}{2}$$

and

$$r^k \geq b_{2k}(\pi_1(X)) \geq 2\sum_{i=0}^{k-1} \mu^{i,2k-i} + \mu^{k,k} \quad \text{for } 2 \leq k \leq \frac{a}{2}\,.$$

The upper bounds on the higher Betti numbers of $\pi_1(X)$ are derived by Gruenberg's free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\pi_1(X)]$-module (cf. [3]).

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