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CURRENT ASPECTS OF DEFORMATION QUANTIZATION

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Abstract. After a short introduction to the program of deformation quantization we indicate why this program is of current interest. One of the reasons is Kontsevich's generalization of the Moyal product of phase-space functions to the case of general Poisson manifolds. We discuss this generalization, including the graphical calculus for presenting the result. We then illustrate the techniques of deformation quantization for quantum mechanical problems by considering the case of the simple harmonic oscillator. We indicate the relations to more conventional approaches, including the formalisms involving operators in Hilbert space and path integrals. Finally, we sketch some new results for relativistic quantum field theories.

1. Introduction

Deformation quantization is an approach to quantum mechanics which uses phase-space techniques from the early days of quantum mechanics [16, 15, 14], and which was formulated as an autonomous theory by Bayen et al [1] in 1978. Using general mathematical techniques it provides a continuous deformation of the commutative algebra of classical observables to the non-commutative algebra of quantum mechanical observables, where the deformation parameter is $\hbar$. The main tool used is the **star product** of functions on phase-space, which corresponds to the product of operators in Hilbert space used in the conventional formulation of quantum mechanics.

For many years the only explicit example of a star product was the **Moyal product** [11] of functions on $\mathbb{R}^n$. It was only comparatively recently that Kontsevich [9] succeeded in constructing a star product on general Poisson manifolds. This development has caused a resurgence of interest in the deformation quantization program, both for quantum mechanics and for relativistic

quantum field theory, inspired by the hope to achieve a more general and powerful formulation which will enable a treatment of the special problems which arise in the more complex systems of interest in the current literature.

The plan of the present article is as follows. In Section 2 we introduce the general framework for dealing with dynamical systems with symmetries, the Poisson manifolds. We then discuss the Moyal and Kontsevich star products for functions on such manifolds, which are used to define the algeras of observables for flat and curved spaces, respectively. In particular we illustrate Kontsevich's graphical calculus for defining the general star product. In Section 3 we look at the relation to more conventional formulations of quantum mechanics, including the formalism of operators in Hilbert space and Feynman's path integrals. Section 4 gives a brief preview of recent applications of deformation quantization in relativistic quantum field theory.

2. Poisson Manifolds and Star Products

The mathematical foundations underlying deformation quantization were laid by Gerstenhaber [5] in his work on the general deformations of algebraic structures. In *classical mechanics* the observables are functions on an even-dimensional phase space W and the commutative algebra of observables is given by the point-wise product of the functions:

$$(fg)(x) := f(x)g(x)\,, \tag{1}$$

where $f, g\colon W \to \mathbb{R}$ are smooth functions and $x \in W$. The $2n$-dimensional phase space W is a symplectic manifold equipped with a closed 2-form $\Omega_{ij}(x)$, such that the Poisson bracket of two functions is

$$\{f(x), g(x)\} := \Omega_{ij}(x)\partial^i f(x)\partial^j g(x)\,. \tag{2}$$

We can use Darboux's theorem to choose canonical coordinates $x = (q_n, p_n)$ in which Ω_{ij} is constant:

$$\Omega_{ij} = (0 \;\; I_n \;\; -I_n \;\; 0)\,, \tag{3}$$

where I_n is the $n \times n$ identity matrix.

The symplectic manifold is a special case of a **Poisson manifold**. This is a smooth manifold equipped with a Poisson structure $\alpha^{ij}(x)$, such that the Poisson bracket of two functions is

$$\{f(x), g(x)\} := \frac{1}{2}\alpha^{ij}(x)\partial_i f(x)\partial_j g(x), \tag{4}$$

where $\alpha^{ij}(x) = -\alpha^{ji}(x)$ is antisymmetric and satifies the Jacobi identity

$$\alpha^{il}\partial_l\alpha^{jk}(x) + \alpha^{jl}\partial_l\alpha^{ki}(x) + \alpha^{kl}\partial_l\alpha^{ij}(x) = 0\,. \tag{5}$$

If $\alpha^{ij}(x)$ is invertible then $(\alpha^{ij}(x))^{-1} = \Omega_{ij}(x)$, and the Poisson manifold reduces to a symplectic manifold. The Jacobi identity for $\alpha^{ij}(x)$ guarantees that $\Omega_{ij}(x)$ is closed as a 2-form.

In *quantum mechanics* the observables form a non-commutative algebra. In the operator formalism this non-commutativity is taken into account by representing the observables as linear *operators* on a Hilbert space. On the contrary, the observables in the deformation quantization approach are still represented by *functions*, but these functions are now multiplied by the use of a non-commutative star product:

$$f * g = \sum_{n=0}^{\infty} (\mathrm{i}\hbar)^n C_n(f,g)\,. \tag{6}$$

This is understood to be a formal power series in the deformation parameter $\hbar$. The C_n are bidifferential operators such that

i) $\sum_{j+k=n} C_j(C_k(f,g),h) = \sum_{j+k=n} C_j(f, C_k(g,h))$,
ii) $C_0(f,g) = fg$,
iii) $C_1(f,g) - C_1(g,f) = \{f,g\}$.

Property (i) guarantees that the star product is associative: $(f*g)*h = f*(g*h)$.

Property (ii) means that in the limit $\hbar \to 0$ the star product $f * g$ agrees with the point-wise product fg.

Property (iii) says that at the lowest order the commutator $[f,g]_* := f * g - g * f$ satisfies $\lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}\,[f,g]_* = \{f,g\}$.

The first concrete star product was introduced by Moyal [11]. It may be expressed as

$$f *_M g = f \exp\left(\frac{\mathrm{i}\hbar}{2}\alpha^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g\,, \tag{7}$$

where $\overleftarrow{\partial}_i$ operates on f and $\overrightarrow{\partial}_j$ on g. In canonical coordinates this formula is

$$(f *_M g)(q,p) = f(q,p)\exp\left(\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)\right) g(q,p)\,. \tag{8}$$

The Moyal product may also be expressed by the following shift formula,

$$(f *_M g)(q,p) = f\left(q + \frac{\mathrm{i}\hbar}{2}\overrightarrow{\partial}_p, p - \frac{\mathrm{i}\hbar}{2}\overrightarrow{\partial}_q\right) g(q,p)\,, \tag{9}$$

or as a Fourier integral:

$$(f *_M g)(q,p) = \frac{1}{\hbar^2\pi^2}\int \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}p_1\,\mathrm{d}p_2 f(q_1,p_1)g(q_2,p_2) \\ \times \exp\left[\frac{2}{\mathrm{i}\hbar}\left(q_2(p_1-p)+p_2(q-q_1)+(pq_1-qp_1)\right)\right]. \tag{10}$$

This last formula may be given an interesting geometrical interpretation [17]. Denote points in phase space by vectors, for example in 2 dimensions

$$\bar{r} = \begin{pmatrix} q \\ p \end{pmatrix}, \qquad \bar{r}_1 = \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \qquad \bar{r}_2 = \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}. \tag{11}$$

Now consider the triangle in phase space shown in Fig. 1. Its area (symplectic volume) is

$$A(\bar{r},\bar{r}_1,\bar{r}_2) = \frac{1}{2}(\bar{r}-\bar{r}_1)\wedge(\bar{r}-\bar{r}_2) \\ = \frac{1}{2}\left[p(q_2-q_1)+p_1(q-q_2)+p_2(q_1-q)\right], \tag{12}$$

which is proportional to the exponent in the previous formula. Hence

$$(f*g)(\bar{r}) = \int \mathrm{d}\bar{r}_1\wedge\mathrm{d}\bar{r}_2 f(\bar{r}_1)g(\bar{r}_2)\exp\left[\frac{4\mathrm{i}}{\hbar}A(\bar{r},\bar{r}_1,\bar{r}_2)\right]. \tag{13}$$

The *nonlocal* nature of the product is manifest in all these forms: in order to calculate the value of the product of two functions at the point (q,p) it is necessary to know all the derivatives of the functions at (q,p), or equivalently their values on the complete phase space.

Another possible star product is the **standard star product**, defined by

$$f *_S g = f\,\mathrm{e}^{\mathrm{i}\hbar\overleftarrow{\partial}_q\overrightarrow{\partial}_p} g\,. \tag{14}$$

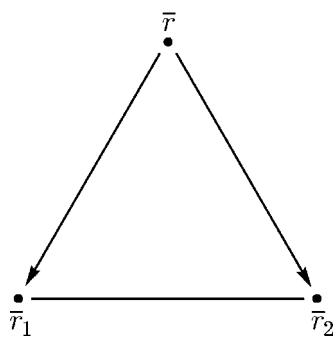

Figure 1. A phase-space triangle

Two star products $*$ and $*'$ are said to be *c-equivalent* if there exists an invertible operator

$$T = \sum_{n=0}^{\infty}\hbar^n T_n\,, \tag{15}$$

where the T_n are differential operators, which satisfies

$$T(f *' g) = Tf * Tg\,. \tag{16}$$

The Moyal and standard star products are c-equivalent, i. e. $T(f *_S g) = Tf *_M Tg$, with the transition operator

$$T = \exp\left[-\frac{\mathrm{i}\hbar}{2}\vec{\partial}_q\vec{\partial}_p\right]. \tag{17}$$

The concept of c-equivalence is a mathematical one (c stands for cohomology [5]), it does not by itself imply any kind of physical equivalence.

Expanding the exponential in Eq. (7) leads to an expression of the Moyal product as a series:

$$f * g = fg + \frac{\mathrm{i}\hbar}{2}\alpha^{ij}(\partial_i f)(\partial_j g) + \frac{1}{2!}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 \alpha^{ij}\alpha^{kl}(\partial_{ik} f)(\partial_{jl} g) + \cdots \tag{18}$$

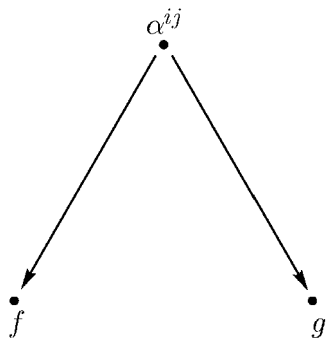

Figure 2. A vertex diagram

Kontsevich uses the following graphical calculus to represent this series. The functions f and g are represented by dots, the differential operators ∂_i by arrows, which terminate on the dots representing the functions on which the operators act. The arrows originate in pairs from vertices which represent the Poisson structure α^{ij}. Thus the term $\alpha^{ij}(\partial_i f)(\partial_j g)$ is represented by the diagram shown in Fig. 2. The complete series is then as shown in Fig. 3. This series exponentiates to yield the Moyal product.

For symplectic spaces the coefficients of the Poisson structure α^{ij} are constants in canonical coordinates. For general Poisson manifolds this is no longer the case. In the general case we get an associative star product by generalizing the series in Fig. 3 to include the action of the differential operators on the Poisson coefficients, as shown in Fig. 4. One is to sum over all admissible graphs, excluding **loop graphs** like those shown in Fig. 5. Kontsevich also gives a prescription for calculating the weights of the different graphs in the sum in terms of certain angular integrals [9]. We shall return to this point in Section 4.

The formula for the first few terms in the series for the Kontsevich product shown in Fig. 4 are

$$\begin{aligned} f * g = fg + \frac{\mathrm{i}\hbar}{2}\alpha^{ij}(\partial_i f)(\partial_j g) + \left(\frac{\mathrm{i}\hbar}{2}\right)^2 \{\frac{1}{2}\alpha^{ij}\alpha^{kl}(\partial_{ik} f)(\partial_{jl} g) \\ + \frac{1}{3}\alpha^{im}[(\partial_m \alpha^{jk})(\partial_{ij} f)(\partial_k g) + (\partial_{ij} g)(\partial_k f)]\} + \cdots \end{aligned}$$

The Kontsevich series does not exponentiate like the Moyal series of Eq. (18). When the Poisson structure is nondegenerate (the symplectic case) then the

Poisson coefficients are constants in the canonical coordinate system and the Kontsevich product reduces to the Moyal product.

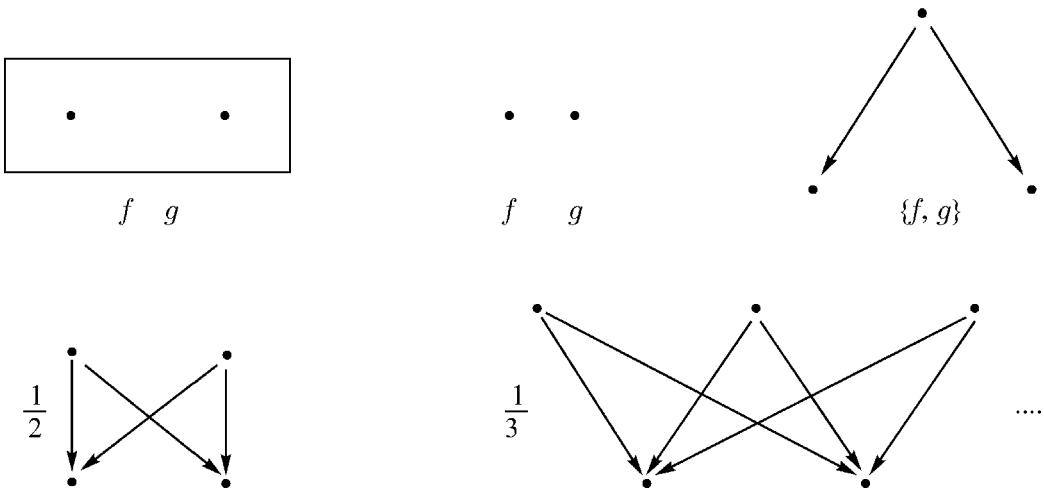

Figure 3. Series expansion for the Moyal product

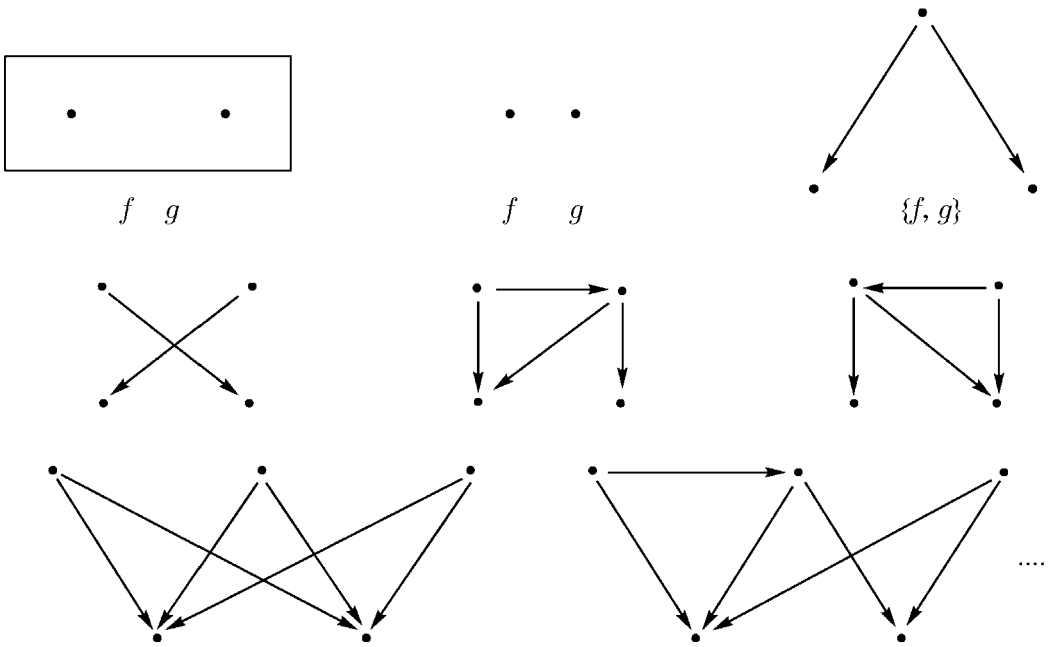

Figure 4. Series expansion for the Kontsevich product

Consider a vector space V with a basis L^A and a Lie algebra structure

$$[L^A, L^B] = c^{AB}_C L^C \,. \tag{19}$$

Then the dual vector space V^* with basis λ_A inherits a Poisson structure

$$\alpha^{AB}(x) = c^{AB}_C x^C \tag{20}$$

for $x = x^C \lambda_C \in V^*$. The Jacobi identity for the structure constants of the Lie algebra ensures the Jacobi identity for the Poisson coefficients α^{AB}. This construction relates the Kontsevich product to the Baker–Campbell–Hausdorff formula

$$\mathrm{e}^A \, \mathrm{e}^B = \mathrm{e}^{C(A,B)} \tag{21}$$

for the product of group elements, where $C(A,B) = A + B + {}^1\!/_2[A,B] + \cdots$. In the same way that the Jacobi identity for the Lie brackets ensures the associativity of the group multiplication, the Jacobi identity for the Poisson structure ensures the associativity of the Kontsevich star product [8].

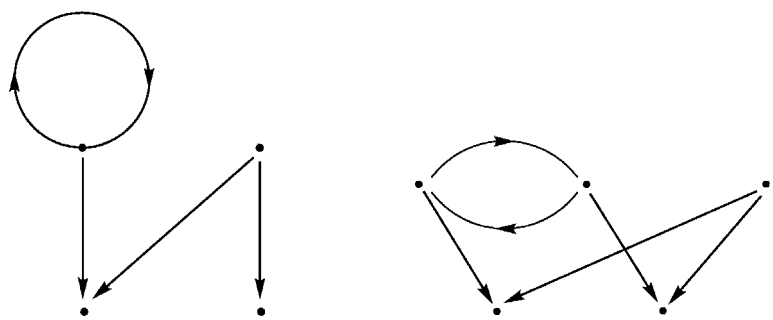

Figure 5. Loop graphs

3. Deformation Quantization

The properties of the star product are well adapted for a description of the algebra of observables. This algebra is associative, as is ensured by Property (i) of the previous section. Property (ii) expresses the **classical limit**: when $\hbar \to 0$ the algebra reduces to the commutative algebra of the classical observables. Property (iii) embodies the **correspondence principle**, which gives the relation between quantum mechanical commutators and Poisson brackets. Instead of demanding equality of these two quantities, higher order corrections in $\hbar$ are allowed.

By choosing a star product one chooses a quantization scheme, which fully specifies the passage from the classical system to its quantum counterpart. We shall discuss in the following to what extent the quantization scheme in some particular context is unique.

We shall show that, having chosen a quantization scheme, we may calculate the quantities of interest for our quantum system. In our simple examples the classical system is completely specified by its Hamilton function. In more general cases one may have to decide what constitutes a *sufficiently large* set of good observables for a complete specification of the system [1].

In this approach the central object of interest in the quantum theory is the *time-evolution function*:

$$\mathrm{Exp}(Ht) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{\mathrm{i}\hbar}\right)^n (H*)^n \,, \tag{22}$$

where $(H*)^n = \underbrace{H * H * \cdots * H}_{n \text{ times}}$. The Hamiltonian is the generator of the time evolution:

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Exp}(Ht) = H * \mathrm{Exp}(Ht) \,. \tag{23}$$

The solution of this equation may be written as

$$\mathrm{Exp}(Ht) = \sum_{\lambda \in I} \pi_\lambda \, \mathrm{e}^{\frac{\lambda t}{\mathrm{i}\hbar}} \,. \tag{24}$$

This is called the **Fourier–Dirichlet expansion** of the time-evolution function. Here I is a sequence in $\mathbb{C}$ and π_λ is a function on W for $\lambda \in I$. I is the *spectrum* of H, $\lambda \in I$ is an *eigenvalue*, and π_λ the *projector* associated with λ. The projection property is

$$\pi_\lambda * \pi_{\lambda'} = \delta_{\lambda\lambda'} \pi_\lambda \,. \tag{25}$$

The projectors and the eigenvalues satisfy the *star-genvalue equation*:

$$H * \pi_\lambda = \lambda \pi_\lambda \,. \tag{26}$$

The Hamiltonian has the **spectral decomposition**:

$$H = \sum_{\lambda \in I} \lambda \pi_\lambda \,. \tag{27}$$

We illustrate how these techniques work for the case of the simple harmonic oscillator. The classical Hamilton function is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \,. \tag{28}$$

In terms of the **holomorphic coordinates**

$$a = \sqrt{\frac{m\omega}{2}} \left(q + \mathrm{i}\frac{p}{m\omega}\right) \,, \qquad \bar{a} = \sqrt{\frac{m\omega}{2}} \left(q - \mathrm{i}\frac{p}{m\omega}\right) \,, \tag{29}$$

this becomes

$$H = \omega a \bar{a} \,. \tag{30}$$

We first choose a quantization scheme characterized by the **normal star product**

$$f *_N g = f\, \mathrm{e}^{\hbar \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}} g\,. \tag{31}$$

We then have

$$\bar{a} *_N a = a\bar{a}\,, \qquad a *_N \bar{a} = a\bar{a} + \hbar\,, \tag{32}$$

so that

$$[a, \bar{a}]_{*_N} = \hbar\,. \tag{33}$$

Eq. (23) for this case is

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Exp}_N(Ht) = (H + \hbar\omega\bar{a}\partial_{\bar{a}}) \operatorname{Exp}_N(Ht), \tag{34}$$

with the solution

$$\operatorname{Exp}_N(Ht) = \mathrm{e}^{-a\bar{a}/\hbar} \exp\left(\mathrm{e}^{-\mathrm{i}\omega t} a\bar{a}/\hbar\right)\,. \tag{35}$$

Developing the last exponential leads to the Fourier–Dirichlet expansion:

$$\operatorname{Exp}_N(Ht) = \mathrm{e}^{-a\bar{a}/\hbar} \sum_{n=0}^{\infty} \frac{1}{\hbar^n n!} \bar{a}^n a^n\, \mathrm{e}^{-\mathrm{i}n\omega t}\,. \tag{36}$$

Comparing coefficients in (24) and (36) leads to

$$\begin{aligned} \pi_0^{(N)} &= \mathrm{e}^{-a\bar{a}/\hbar}\,, \\ \pi_n^{(N)} &= \frac{1}{\hbar^n n!} \pi_0 \bar{a}^n a^n = \frac{1}{\hbar^n n!} \bar{a}^n *_N \pi_0^{(N)} *_N a^n\,, \\ \lambda_n &= n\hbar\omega\,. \end{aligned} \tag{37}$$

Note that the spectrum obtained in Eq. (37) does not include the zero-point energy.

We now consider the Moyal quantization scheme. Writing Eq. (8) in terms of holomorphic coordinates leads to

$$f *_M g = f\, \mathrm{e}^{\frac{\hbar}{2}(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)} g\,. \tag{38}$$

Here one has

$$a *_M \bar{a} = a\bar{a} + \frac{\hbar}{2}\,, \qquad \bar{a} *_M a = a\bar{a} - \frac{\hbar}{2}\,, \tag{39}$$

and again

$$[a, \bar{a}]_{*_M} = \hbar\,. \tag{40}$$

The value of the commutator of two phase space variables is fixed by property (iii) of the star product, and cannot change when one goes to an equivalent star product.

The Moyal star product is c-equivalent to the normal star product with the transition operator

$$T = \mathrm{e}^{-\frac{\hbar}{2}\vec{\partial}_a\vec{\partial}_{\bar{a}}}\,, \tag{41}$$

hence we can obtain the $\pi_n^{(M)}$ projectors from the $\pi_n^{(N)}$ projectors by

$$\begin{aligned}\pi_0^{(M)} &= T\pi_0^{(N)} = 2\,\mathrm{e}^{-2a\bar{a}/\hbar}\,,\\ \pi_n^{(M)} &= T\pi_n^{(N)} = \frac{1}{\hbar^n n!}\bar{a}^n *_M \pi_0^{(M)} *_M a^n = 2(-1)^n\,\mathrm{e}^{-\frac{2H}{\hbar\omega}}\,L_n\left(\frac{4H}{\hbar\omega}\right).\end{aligned} \tag{42}$$

The star-genvalue equation is

$$H *_M \pi_n^{(M)} = \omega\left(\bar{a} *_M a + \frac{\hbar}{2}\right) *_M \pi_n^{(M)} = \hbar\omega\left(n + \frac{1}{2}\right)\pi_n^{(M)}\,. \tag{43}$$

We now have for the spectrum

$$\lambda_n = \left(n + \frac{1}{2}\right)\hbar\omega\,, \tag{44}$$

which is the textbook result. We conclude that for this problem the Moyal quantization scheme is the correct one. This is an example of the fact that c-equivalent star products can lead to different spectra. In fact, if two star products induce the same spectra, they are actually identical.

Inserting the expressions for the projectors from Eq. (42) into the Fourier–Dirichlet expansion yields

$$\mathrm{Exp}_M(Ht) = \frac{1}{\cos\frac{\omega t}{2}}\exp\left[\left(\frac{2H}{\mathrm{i}\hbar\omega}\right)\tan\frac{\omega t}{2}\right]. \tag{45}$$

Here we have used the generating function for the Laguerre polynomials:

$$\frac{1}{1+s}\exp\left[\frac{zs}{1+s}\right] = \sum_{n=0}^{\infty} s^n(-1)^n L_n(z), \tag{46}$$

with $s = \mathrm{e}^{-\mathrm{i}\omega t}$.

In the conventional approach the time-development of the system is characterised by the appropriate matrix element of the **time-development operator**, namely the **Feynman kernel**:

$$K(q_2, t; q_1, 0) = \langle q_2|\,\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}|q_1\rangle\,, \tag{47}$$

where $\hat{H}$ is the Hamilton operator. The Feynman kernel is the Fourier transform of the time-evolution function:

$$K(q_2, t; q_1, 0) = \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{e}^{\mathrm{i}p(q_1-q_2)/\hbar}\,\mathrm{Exp}_M\left(H\left(\frac{q_1+q_2}{2}, p\right)t\right). \tag{48}$$

For the harmonic oscillator we find

$$K(q_2,t;q_1,0)=\sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega t}}\exp\left[\tfrac{\mathrm{i}m\omega}{2\hbar\sin\omega t}\left((q_1^2+q_2^2)\cos\omega t-2q_1q_2\right)\right]. \quad (49)$$

The path integral expression for the Feynman kernel is

$$K(q_2,t;q_1,0)=\int Dq(t)\,\mathrm{e}^{\mathrm{i}S[q]/\hbar}\,, \quad (50)$$

where $S[q]$ is the classical action functional and the notation $Dq(t)$ indicates an integration over all paths with the fixed endpoints q_1 and q_2. For the harmonic oscillator the semi-classical approximation is exact, hence the path integral can be evaluated by inserting the classical solution into the action functional. The result of this procedure agrees with the RHS of Eq. (49) [6]. Hence there is also a direct relationship between the path integral and the star exponential. This has been directly calculated by Sharan [12] for the coordinate representation, and by Dito [3] for the holomorphic representation.

4. Quantum Field Theory

A real scalar field is given in terms of the coefficients $a(\mathbf{k}),\bar{a}(\mathbf{k})$ by

$$\phi(x)=\int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx}+\bar{a}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}\right], \quad (51)$$

where $\hbar\omega_{\mathbf{k}}=\sqrt{\hbar^2\mathbf{k}^2+m^2}$ is the energy of a single quantum of the field. The corresponding quantum field operator is

$$\Phi(x)=\int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\left[\hat{a}(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx}+\hat{a}^\dagger(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}\right], \quad (52)$$

where $\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k})$ are the annihilation and creation operators for a quantum of the field with momentum $\hbar\mathbf{k}$. The Hamiltonian is

$$H=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\hbar\omega_{\mathbf{k}}\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\,. \quad (53)$$

$N(\mathbf{k})=\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})$ is interpreted as the number operator, and the above expression is then just the generalization of Eq. (37), the expression for the energy of the harmonic oscillator in the normal ordering scheme, to an infinite number of degrees of freedom. Had we chosen the Weyl ordering scheme we would have been lead, by the generalization of Eq. (44), to an infinite vacuum energy. Hence the requirement of vanishing vacuum energy implies the choice of the normal ordering scheme in free field theory. In the framework of deformation quantization this leads to the choice of the normal star product for treating free

scalar fields, as pointed out by Dito [3]: only for this choice is the star product well-defined.

In realistic physical field theories involving interacting relativistic fields we are limited up to now to perturbative calculations. The objects of interest are products of the fields. The analogue of the *Moyal product* of Eq. (7) for systems with an infinite number of degrees of freedom is

$$\phi(x_1) * \phi(x_2) * \cdots * \phi(x_n) = \exp\left[\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\delta}{\delta\phi(x)}\Delta(x-y)\frac{\delta}{\delta\phi(y)}\right] \times \phi(x_1),\ldots\phi(x_n)\,, \tag{54}$$

where the expressions $\delta/\delta\phi(x)$ indicate functional derivatives. Here we have used the antisymmetric **Schwinger function**

$$\Delta(x-y) = \langle 0|\,[\Phi(x),\Phi(y)]\,|0\rangle\,. \tag{55}$$

The Schwinger function is uniquely determined by relativistic invariance and causality from the **equal-time commutator**

$$\left[\Phi(x),\dot{\Phi}(y)\right]\Big|_{x^0=y^0} = \mathrm{i}\hbar\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{56}$$

which is the characterization of the canonical structure in the field theoretic framework. The Moyal product is however not the suitable star product to use in this context. In relativistic quantum field theory it is necessary to incorporate *causality* in the form advocated by Feynman: positive frequencies propagate *forward* in time, whereas negative frequencies propagate *backwards* in time. This is achieved by using the **Feynman propagator**:

$$\Delta_F(x-y) = \begin{cases} \Delta^+(x) & \text{for } x^0 > 0 \\ -\Delta^-(x) & \text{for } x^0 < 0\,, \end{cases} \tag{57}$$

where $\Delta^+(x), \Delta^-(x)$ are the propagators for the positive and negative frequency components of the field, respectively. In the operator language

$$\Delta_F(x-y) = \mathcal{T}(\Phi(x)\Phi(y)) - \mathcal{N}(\Phi(x)\Phi(y))\,, \tag{58}$$

where $\mathcal{T}$ indicates the time-ordered product of the fields and $\mathcal{N}$ the normal-ordered product. Since the second term in this equation is a normal ordered product with vanishing vacuum expectation value, the Feynman propagator may be simply characterized as the vacuum expectation value of the time-ordered product of the fields. The antisymmetric part of the Feynman propagator is the Schwinger function:

$$\Delta_F(x) - \Delta_F(-x) = \Delta^+(x) + \Delta^-(x) = \Delta(x)\,. \tag{59}$$

Since going over to an c-equivalent product leaves the antisymmetric part of the differential operator in the exponent of Eq. (54) invariant, this suggests that the use of the Feynman propagator instead of the Schwinger function merely involves the passage to a c-equivalent star product. This is indeed easy to verify. The new star product is called a **Wick product**. It corresponds to a time- ordered product of the operators. This is just the **Wick theorem**, which is the basic tool of relativistic perturbation theory; in operator language [10]

$$\mathcal{T}(\Phi(x_1),\ldots\Phi(x_n)) = \exp\left[\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\delta}{\delta\Phi(x)}\Delta_F(x-y)\frac{\delta}{\delta\Phi(y)}\right] \times \mathcal{N}(\Phi(x_1),\ldots\Phi(x_n))\,. \tag{60}$$

The relation between relativistic perturbation theory and deformation quantization has recently been discussed by Dütsch and Fredenhagen [4].

An even more direct relation between deformation quantization and quantum field theory has been uncovered by studies of the Poisson-Sigma model [2, 7, 13]. This model involves a set of scalar fields X^i which map a 2-dimensional manifold Σ_2 onto a Poisson space M, as well as generalized gauge fields A_i, which are one-forms on Σ_2 mapping to one-forms on M. The action is given by

$$S_{PS} = \int_{\Sigma_2} (A_i\,\mathrm{d}X^i + \alpha^{ij}A_iA_j)\,, \tag{61}$$

where α^{ij} is the Poisson structure of M. The remarkable formula found by Cattaneo and Felder [2] is

$$(f*g)(x) = \int DXDAf(X(1))g(X(2))\,\mathrm{e}^{\mathrm{i}S_{PS}/\hbar}\,, \tag{62}$$

where f, g are functions on M, $*$ is Kontsevich's star product [9], and the functional integration is over all fields X which satisfy the boundary condition $X(\infty) = x$. Here Σ_2 is taken to be a disc in $\mathbb{R}^2$; 1, 2 and ∞ are three points on its circumference. By expanding the functional integral in the above equation according to the usual rules of perturbation theory one finds that the coefficients of the powers of $\mathrm{i}\hbar$ reproduce the graphs and weights that characterize Kontsevich's star product. The rule forbidding loop graphs corresponds nicely to the absence of tadpole graphs in the renormalized quantum field theory.

For the case in which the Poisson structure is invertible we can perform the Gaussian integration in Eq. (62) involving the fields A_i. The result is

$$(f*g)(x) = \int DXf(X(1))g(X(2))\exp\left[\frac{\mathrm{i}}{\hbar}\int \Omega_{ij}\,\mathrm{d}X^i\,\mathrm{d}X^j\right]. \tag{63}$$

This is just Eq. (10) for the Moyal product, to which the Kontsevich product reduces in the symplectic case. Here $\Omega_{ij} = (\alpha^{ij})^{-1}$ as before, and $\int \Omega_{ij}\, \mathrm{d}X^i\, \mathrm{d}X^j$ is the symplectic volume of the manifold M.

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