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ON THE MULTIPLICITY OF CERTAIN YAMABE METRICS

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Abstract. In this paper we are interested in the multiplicity of the Yamabe metrics for compact Riemannain manifolds. This problem may be approached only when the scalar curvature is positive. This is the case notably of certain locally conformally flat manifolds. Furthermore, there exists a link with the singular Yamabe problem. We give various applications of this study.

1. Introduction

Consider a smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$ where g is a smooth Riemannian metric on M, $\mathrm{d}v$ denotes the volume form of g, and $R(x)$ its scalar curvature. $\mathcal{H}_1^2(M, g)$ is the Sobolev space. For any nonzero function $u \in \mathcal{H}_1^2(M, g)$, we define the functional

$$J(u) = \frac{4\dfrac{n-1}{n-2}\displaystyle\int_M |\nabla u|^2 \,\mathrm{d}v + \int_M R(x)u^2 \,\mathrm{d}v}{\left(\displaystyle\int_M u^{\frac{2n}{n-2}} \,\mathrm{d}v\right)^{\frac{n-2}{n}}} . \tag{1}$$

The critical points of the functional $J(u)$ are solutions of the Euler-Lagrange equation, called Yamabe equation

$$4\frac{n-1}{n-2}\Delta_g u + R(x)u = K|u|^{\frac{4}{n-2}} u \,, \tag{2}$$

where K is a real number.

There is a geometric interest if we assume the solution u to be C^∞ positive. In this case, the conformal metric $g' = u^{\frac{4}{n-2}} g$ has constant scalar curvature K if u is a critical point of the functional $J(u)$ and conversely. So, we know by the resolution of the Yamabe conjecture for the smooth compact manifolds, the infimum of $J(u)$ is always attained. That means every compact n-dimensional Riemannian manifold can be deformed to a Riemannian manifold with constant scalar curvature by a conformal transformation. More precisely, from the analytic point of view, if (S^n, g_0) is the standard sphere we may assert:

Lemma. *There exists a C^∞ nonzero positive function u_0 on (M, g) such that*

$$J(u_0) = \operatorname*{Inf}_{u \in H_1^2(M)} J(u) = \mu(M, g)\,.$$

Moreover, $\mu(M, g) < \mu(S^n, g_0)$, equality holds only if there exits a conformal diffeomorphism between the manifold (M, g) and the standard sphere (S^n, g_0).

Such metric with constant scalar curvature belonging in the conformal class $[g]$ is called Yamabe metric.

It is also known, that the Yamabe metric is unique up to homothety, when (M, g) is such that $\int_M R(x)\,\mathrm{d}V \leq 0$, or when (M, g) is an Einstein manifold. For the positive case of the scalar curvature ($K > 0$), it is possible to get multiple Yamabe solutions for certain conformally flat product Riemannian manifolds. Schoen and Yau have also investigated the global properties of compact locally conformally flat Riemannian manifolds by using the Yamabe equation.

The multiplicity problem of the Yamabe solutions have been intensively studied during the last few years, in the more general context known as the singular Yamabe problem with isolated singularities. The latter consists to find a metric $g = u^{\frac{4}{n-2}} g_0$ on the domain $S^n - \Lambda_k$, where Λ_k is a finite point-set of (S^n, g_0). This metric is complete and has a positive constant scalar curvature $R(g)$.

Concerning the special case $k = 2$, Schoen (see Section 2 of [10]) has partially described the conformal class of the product metric $(S^1 \times S^{n-1},\ \mathrm{d}t^2 + \mathrm{d}\xi^2)$ where S^1 is the circle of length T and (S^n, g_0) is the standard sphere.

He proves the existence of the rotationally invariant Yamabe metrics

$$g_c^j = (u_c^j)^{\frac{4}{n-2}} (\,\mathrm{d}t^2 + \mathrm{d}\xi^2)\,, \qquad j = 1, 2, \ldots, k-1$$

called pseudo-cylindric metrics (sometime also called by different authors Delaunay metrics or Fowler metrics). They naturally depend on T-values. It is now well known that the number of these metrics is finite in a conformal class for the product $[\,\mathrm{d}t^2 + \mathrm{d}\xi^2]$. Moreover, we have been able to determine the lower bound of this number. We may see for that the papers [5, 6, 10]. In particular, we get an unique Yamabe metric if

$$T \leq 2\pi\sqrt{\frac{n-1}{R_0}}\,. \tag{3}$$

This metric is $c^{te}(\,\mathrm{d}t^2 + \mathrm{d}\xi^2)$.

2. Singular Yamabe Problem and Pseudo-Cylindrical Metrics

Let a finite point-set $\Lambda_k = p_1, p_2, \ldots, p_k$ of the standard sphere S^n. R_0 being its scalar curvature. Consider the following.

Singular Yamabe Problem: *There exists a metric $g = u^{\frac{4}{n-2}} g_0$ in the conformal class $[g_0]$ on the domain $S^n - \Lambda_k$, which is complete and has a constant scalar curvature $R(g)$.* In others words, the following differential equation

$$\begin{cases} 4\dfrac{n-1}{n-2}\Delta_{g_0}u + R_{g_0}u - R_g u^{\frac{n+2}{n-2}} = 0\,, \\ g = u^{\frac{4}{n-2}} g_0 \text{ is complete on } S^n - \Lambda\,, \qquad R_g = \text{const} > 0\,, \end{cases} \tag{4}$$

has a solution.

A such solution g is called pseudo-cylindric metric. The trivial one is the cylindric metric.

Let $\mathcal{L}_g = \Delta_g + \dfrac{n-2}{4(n-1)} R_g$ denotes the conformal Laplacian of any Riemannian manifold (M, g). $\mathcal{L}_g$ posseses a special property:

Lemma. (Invariance property of the conformal Laplacian) *Let (M, g) and (N, h) are two Riemannian manifolds, such that there exists a conformal diffeomorphism $f\colon M \to N$ (this means that if f is a diffeomorphism, there is a function $v\colon M \to \mathbb{R}^+$, such that $f^*h = v^{\frac{4}{n-2}} g$). Thus, for any function $\varphi\colon N \to \mathbb{R}$ of class C^2, we have*

$$\mathcal{L}_h(\varphi) \circ f = v^{\frac{n+2}{n-2}} \mathcal{L}_g(v\varphi \circ f)$$

$\varphi = 1$ and $f = \mathrm{Id}$ gives the Yamabe equation.

Consequently, we may deduce that if v is any positive solution of

$$\mathcal{L}_{g_0} v - \frac{n-2}{4(n-1)} K v^{\frac{n+2}{n-2}} = 0\,, \tag{5}$$

then K is precisely the scalar curvature of the metric $g = v^{\frac{4}{n-2}} g_0$. Moreover, the condition g to be complete requires that $\lim\limits_{x\to\Lambda} v(x) = \infty$.

The interest of the problem was underlined by Schoen; he first proved the existence of weak solutions of Yamabe problem on (S^n, g_0) which have a prescribed singular set Λ_k. In particular, this latter point allows us to prove the existence of metrics g conformally equivalent to g_0, which have positive constant scalar curvature and are complete on $S^n - \Lambda_k$. We refer to the Mazzeo–Pacard [9] for more details about the general problem, and for a simplier version of the Schoen result.

The set Λ_k may be replaced by $F(\Lambda_k) = \Lambda'_k$, for any conformal transformation. In particular, we may suppose that the points p_i sum to zero, considered as vectors in the euclidian space and then Λ_k is contained in an equatorial subsphere $S^k \subset S^n$. By using a reflection argument, one proves there is no complete (positive regular) solution existing for the value $k = 1$. One remarks a similarity between this singular Yamabe problem and the theory of embedded, complete, constant mean curvature surfaces with k ends in $\mathbb{R}^3$. For the case $k = 2$, a family of such surfaces was discovered by Delaunay. For this reason, it was called "Delaunay metrics" the analogous family of solutions for the corresponding singular Yamabe problem. Some authors also called them "Fowler metrics" in relation with radial solutions of Emden–Fowler equations.

It is also known that the moduli space $\mathcal{M}_\Lambda$, which is the set of all smooth positive solutions u of the problem

$$4\frac{n-1}{n-2}\Delta_{g_0} u + R_{g_0} u - R_g u^{\frac{n+2}{n-2}} = 0\,, \tag{6}$$

$$g = u^{\frac{4}{n-2}} g_0 \ \text{ is complete on } \ S^n - \Lambda\,, \quad R_g = \text{const} > 0\,.$$

This space is in general a k-dimensional real analytic manifold. In the general case when $\Lambda = S^p$ (the p-subsphere where $1 \le p \le \frac{n-2}{2}$), these solutions belong to an infinite dimensional space. Nevertheless, the moduli space is only explicitly determined if $\dim \Lambda = 0$, i. e. $\Lambda = \Lambda_k$. In the case $k = 2$, we remark that the space $\mathcal{M}_\Lambda$ may be identified with any other, Λ which contains just two elements. This space, denoted by $\mathcal{M}_2$ which contains all the pseudo-cylindric metrics, may be identified with the open set $\Omega \in \mathbb{R}^2$. The boundary of Ω is the homoclinic curve γ_{b_0}. See [5] for details.

Since this problem is conformally invariant, we may choose the set $\Lambda_2 = (p, -p)$; $-p$ is the antipodal point of p.

Let $\Pi\colon S^n\backslash\{pt\} \to \mathbb{R}^n$ is the stereographic projection. We know that the converse Π^{-1} defined by

$$\Pi^{-1}(x) = \left[\frac{2x}{|x|^2+2}, \frac{|x|^2-1}{|x|^2+1}\right]$$

is a conformal diffeomorphism, and we have $(\Pi^{-1})^\star(g_0) = \dfrac{4}{(|x|^2+1)^2}\,\mathrm{d}x^2$, where $\mathrm{d}x^2$ is the euclidean metric of $\mathbb{R}^n$. Thus, the resolution of (4) is equivalent to the resolution of

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \tag{7}$$

on $\mathbb{R}^n - \tilde{\Lambda}_k$ where $\tilde{\Lambda}_k = \Pi(\Lambda_k)$. Moreover, u have to verify $\lim\limits_{x\to\tilde{\Lambda}_k} u(x) = \infty$. On the other hand, Cafarelli, Gidas and Spruck [3] proved that all solutions of (7) on $\mathbb{R}^n - \tilde{\Lambda}$ are radial. We then define the change

$$u(x) = |x|^{\frac{2-n}{2}}\, v \log \frac{1}{|x|}\,.$$

In fact, this change corresponds to the conformal diffeomorphism

$$\beta\colon \mathbb{R}^n\backslash\{0\} \to \mathbb{R}\times S^{n-1}$$

where $\beta(x) = \left(\log \dfrac{1}{|x|}, \dfrac{x}{|x|}\right)$. Consequently, $v\colon \mathbb{R} \to \mathbb{R}^+$ is a solution of the autonomous ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0\,. \tag{8}$$

The analysis of (8) show us, there is only one center say $(\alpha, 0)$ corresponding to the constant solution $\alpha = \left(\dfrac{n-2}{n}\right)^{\frac{n-2}{2}}$. All the periodic orbits denoted by $\gamma_c(t)$, are parametrized by $\left(v_c(t), \dfrac{\mathrm{d}}{\mathrm{d}t} v_c(t)\right)$, c verifying $0 < c < c_0$. They are surrounded by the homoclinic orbit γ_{c_0}. Here $v_{c_0}(t) = (\cosh t)^{\frac{n-2}{2}}$. The period is also depending on c: $T = T(c)$. The metrics

$$g = v_c^{\frac{4}{n-2}}\,(\mathrm{d}t^2 + \mathrm{d}\xi^2)$$

are called pseudo-cylindric type metrics. They are defined on the product manifold $(S^1\times S^{n-1}, \mathrm{d}t^2 + \mathrm{d}\xi^2)$, a circle of length T crossed with the standard $(n-1)$-dimensional sphere. They belong to the conformal class of the cylindric metrics $[\,\mathrm{d}t^2 + \mathrm{d}\xi^2]$.

3. Multiplicity Cases of Yamabe Metrics

3.1. Multiplicity of the Pseudo-cylindric Metrics

Many authors have tried to describe the conformal class $[g_0]$ of the product metric $g_0 = \mathrm{d}t^2 + \mathrm{d}\xi^2$ on $S^1(T) \times S^{n-1}$ — the product Riemannian manifold of a circle of length T and the euclidian $(n-1)$-sphere of radius 1 in order to complete the Schoen analysis concerning this problem. In particular, we refer to [4] and [5], and for a more geometrical interpretation to Mazzeo-Pacard [9]. The following result resume the known properties of the pseudo-cylindric metrics (see [5, 6, 9 and 10]).

Proposition 1. *Let $(S^1(T) \times S^{n-1},\ g_0 = \mathrm{d}t^2 + \mathrm{d}\xi^2)$ be a product Riemannian manifold of a circle of length T and the euclidian $(n-1)$-sphere of radius 1. If the value of T satisfies the condition*

$$T_{k-1} = \frac{2\pi(k-1)}{\sqrt{n-2}} < T \leq T_k = \frac{2\pi k}{\sqrt{n-2}}\,, \tag{9}$$

for each integer $k \geq 1$, then there exist at least, in the conformal class $[g_0]$ k rotationally invariant pseudo-cylindric metrics

$$g_c^j = (u_c^j)^{\frac{4}{n-2}}\,(\mathrm{d}t^2 + \mathrm{d}\xi^2)\,, \qquad j = 0, 1, 2, \ldots, k-1 \ \ \textit{with} \ \ g_c^0 = g_0\,.$$

The case $k = 1$ only corresponds to the trivial product metric $C^{te}(\mathrm{d}t^2 + \mathrm{d}\xi^2)$. In particular, when $T \leq T_1$ we get an unique Yamabe metric, namely the product metric g_0.

4. Remarks on the Multiplicity of the Yamabe Solutions

4.1. Yamabe Metrics on Manifolds

Recall that according to Obata, all Yamabe solutions are minimal for the functional $J(u)$ on the standard sphere (S^n, g_0) and their sectional curvatures turn out to be constant. There actually exists a conformal diffeomorphism between the manifolds $(S^n, u^{\frac{4}{n-2}} g_0)$ and (S^n, g_0). All the metrics are obtained by pulling back the canonical one, by a natural family of conformal transformations. Among these metrics, we get an explicit one parameter family of metrics (of constant scalar curvature R)

$$g_t = \left(\sqrt{1+t^2} + t\cos\alpha x\right)^{-2} g_0 \quad \text{with} \quad \alpha = \sqrt{\frac{R}{n(n-1)}}\,. \tag{10}$$

We can verify that all the functions $u_t(x) = (\sqrt{1+t^2} + t\cos\alpha x)^{-\frac{n-2}{2}}$ are solutions of the Yamabe equation (2) for $R \equiv K$. These functions are minimizing for the functional, i. e. $J(u_t) = \mu(S^n, g_0)$.
More generally, consider a Riemannian manifold (M, g) with constant scalar curvature R which in addition admits a conformal isometric vector field $\mathcal{X}$. Then, $\dfrac{R}{n-1}$ belongs to the spectrum of the Laplace operator Δ_g. By pulling back g by the flow of $\mathcal{X}$, we get a one parameter family of metrics which are conformal to g and have the same scalar curvature R.

4.2. The Degenerate Case

$T = T_k$ corresponds to the degenerate case which has previously been considered. The product metrics in general are not Morse critical points. Notice that the Morse index of the solution $g = u^{\frac{4}{n-2}} g_0$ is equal to the number of eigenvalues of the Laplace operator Δ_g in $[0, n]$. The assumption that the solution is not degenerate implies that the strict inequality $\lambda_1 > n$ holds. According to Lichnerowicz, we get $\lambda_1 \geq n$, whenever the metric g is Yamabe solution [11]. More generally, Schoen proves that, if all Yamabe solutions are non degenerate in a conformal class $[g_0]$ of a Riemannian compact manifold (M, g_0), then the functional has only a finite number of Morse critical points. Thus, for a generic conformal class of metrics this hypothesis on g_0 is true, and $[g_0]$ contains at most a finite number of (non degenerate) Yamabe metrics. Moreover, the set of the Yamabe metrics with finite energy is compact.

5. The Best Constants of Sobolev and Aubin's Theorem

There is another interesting problem related on the multiplicity problem of Yamabe solutions: to determine the best constant of the Sobolev inclusions. Aubin [1] has shown that for a positive number $A \geq K^2(n,2) = \dfrac{4\omega_n^{-\frac{2}{n}}}{n(n-2)}$ on the compact Riemannian manifold (M, g) of dimension $n > 2$, where ω_n denotes the volume of the standard sphere, there exists a constant B such that, for all $u \in H_1^2(M, g)$, we obtain the following Sobolev inequality

$$\|u\|^2_{L^{\frac{2n}{n-2}}} \leq A\|\nabla u\|^2_{L^2} + B\|u\|^2_{L^2}\,. \tag{11}$$

The existence of the second constant B seems play a role in the Yamabe multiplicity problem and in the Nirenberg problem. Hebey-Vaugon [7] have calculated the best small constant B denoted by $C(M, g)$. For this constant precisely, inequality (10) is verified by any function $u \in H_1^2(M, g)$, on the quotient manifold of the standard sphere S^n/G, where G is a cyclic group of isometries on the sphere. Moreover, one has estimated the best constant for a conformally flat manifolds. More precisely for $S^1 \times S^{n-1}$ endowed with a Riemannian metric product, they find

$$C(S^1(T) \times S^{n-1}) \leq \frac{4\omega_n^{-\frac{2}{n}}}{n(n-2)} T^{-2} + \frac{n-2}{n} \omega_n^{-\frac{2}{n}}$$

where T is the lenght of the circle. Concerning the first constant A, notice that this one may attain the value $K^2(n, 2)$ if the manifold (M, g) has constant sectional curvature.

Consider now the following equation

$$4\frac{n-1}{n-2} \Delta_{g_0} u + R_{g_0} u = R_{g_0} u^{\frac{n+2}{n-2}} \tag{12}$$

on the product $S^1(T) \times S^{n-1}$. R_{g_0} is the scalar curvature of the $(n-1)-$ dimensional standard sphere S^{n-1}. S^1 is the circle of length T and $u \equiv 1$ being a trivial solution of (12). According to the implicit function theorem, if

$$T = 2\pi \sqrt{\frac{n-1}{R_{g_0}}}$$

equation (12) is not locally invertible on $S^1(T) \times S^{n-1}$ near the trivial solution $u \equiv 1$ and

$$\lambda = \left(\frac{2\pi}{T}\right)^2 = \frac{R_{g_0}}{n-1}$$

is an eigenvalue, which yet verify the inequality

$$\lambda < n \left(\frac{\omega_n}{\int \mathrm{d}v}\right)^{\frac{2}{n}} .$$

Moreover, by Proposition 1 the (unique) solution $u \equiv 1$ minimizes the functional

$$J(u) = \frac{4\frac{n-1}{n-2} \int\limits_M |\nabla u|^2 \, \mathrm{d}v + \int\limits_M R(x) u^2 \, \mathrm{d}v}{\left(\int\limits_M u^{\frac{2n}{n-2}} \, \mathrm{d}v\right)^{\frac{n-2}{n}}} \quad \text{and} \quad J(1) < \mu(S^n, g_0) .$$

The constant $K^2(n,2)$ obviously is not attained, and there exists $u \in H_1^2(M,g)$ such that

$$\|u\|^2_{L^{\frac{2n}{n-2}}} > K^2(n,2)\left(\|\nabla u\|^2_{L^2} + \frac{(n-2)^2}{4}\|u\|^2_{L^2}\right).$$

The following complete and improve Theorem 11 of Aubin [1].

Theorem 1. *Let (M_n, g) be a Riemannian compact manifold ($n \geq 3$), and consider the differential equation on M_n*

$$\Delta\phi + K\phi = K(1+f)\phi^{\frac{n+2}{n-2}},$$

where K is a positive number and f is a C^∞ function on M_n. If, for $K = K_0$ and $f \equiv 0$ this equation is not locally invertible on $\phi_0 = 1$ then $\lambda = \dfrac{4K_0}{n-2}$ is an eigenvalue of the Laplace operator. Moreover, if the eigenvalue verify $\lambda \geq n\left(\dfrac{\omega_n}{\int \mathrm{d}v}\right)^{\frac{2}{n}}$, then the Yamabe problem have at least two solutions. Otherwise, M_n is conformal to the standard sphere.

This theorem may be proved by using the same techniques as in [1].

6. Stability of the Yamabe Metrics and Remarks

(i) Consider the stability problem of the pseudo-cylindric metrics $g_c^j = (u_c^j)^{\frac{4}{n-2}} g_0$, we want to precise that among the k metrics given by the above theorem, only the solution u_T with fundamental period T is stable. Notice that the limiting period is the fundamental period of the associated linearized equation of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u + (n-2)u\,.$$

The other $(k-1)$ solutions which have fundamental period $\dfrac{T}{j}$ with $j = 2, \ldots, k-1$, are unstable. By crossing the point $(T_1, u_{T_1}(t))$, there is an "exchange of stability" between the two curves of solutions. The trivial one becomes unstable after crossing this bifurcation point. Therefore, if we suppose

$T > T_1$ the trivial solution $u \equiv \alpha$ does not minimize the Yamabe functional

$$J(u) = \frac{4\frac{n-1}{n-2}\int\limits_M |\nabla u|^2 \, \mathrm{d}v + \int\limits_M Ru^2 \, \mathrm{d}v}{\left(\int\limits_M u^{\frac{2n}{n-2}} \, \mathrm{d}v\right)^{\frac{n-2}{n}}} .$$

In fact, the infimum is attained by the solution u_T having the fundamental period T. If we get $T \leq T_1$, the (unique) solution $u \equiv \alpha$ minimizes the functional. To see that, it is enough to calculate its value. Indeed, let V denotes the Riemannian volume of $(S^1(T) \times S^{n-1}, g_0)$, we get

$$J(u \equiv \alpha) = (n-1)(n-2)\left(\int \mathrm{d}v\right)^{\frac{2}{n}}$$

where

$$\alpha = \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}} \quad \text{and} \quad \left(\int \mathrm{d}v\right) = T \operatorname{vol}(S^{n-1}, \mathrm{d}\xi^2) .$$

If we assume

$$T \leq \frac{2\pi}{\sqrt{n-2}} ,$$

then we get obviously

$$J(u \equiv \alpha) < n(n-1)[\operatorname{vol}(S^n)]^{\frac{2}{n}} .$$

Furthermore, notice that only the non trivial solution u_T tends to $u_0(t) = (\cosh t)^{-\frac{n-2}{2}}$, when T becomes ∞. The minimum of the Yamabe functional μ_T has to tend to $\mu(S^n, g_0) = n(n-1)[\operatorname{vol}(S^n)]^{2/n}$, when T tends to ∞. Actually, Schoen proves that when $T \to +\infty$ our problem possesses an increasing number of solutions [11].

(ii) The multiplicity of the Yamabe problem may occur for other locally conformally flat manifolds. Consider the real projective space: among the pseudo-cylindric metrics $g = u^{\frac{4}{n-2}} g_0$, we examine those satisfying the condition $u(t, \xi) = u(-t, -\xi)$; any such solution is independent of ξ. Thus we obtain solutions to the Yamabe problem on the punctured real projective space $\mathbb{RP}^n \setminus \{p\}$ by identifying any solution on this space with a solution on $S^n \setminus \{p, -p\}$, invariant under the antipodal reflection.

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