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PROJECTIVE BIVECTOR PARAMETRIZATION OF ISOMETRIES IN LOW DIMENSIONS

DANAIL S. BREZOV

Department of Mathematics, University of Architecture, Civil Engineering and Geodesy, 1 Hristo Smirnenski Blvd., 1046 Sofia, Bulgaria

Abstract. The paper provides a pedagogical study on vectorial parameterizations first proposed by O. Rodrigues for the rotation group in $\mathbb{R}^3$ by means of the so-called Rodrigue's vector. Although his technique yields significant advantages in both theoretical and applied context, the vectorial interpretation is easily seen to be completely wrong and in order to benefit most from this otherwise fruitful approach, we put it in the proper perspective, namely, that of Clifford's geometric algebras, spin groups and projective geometry. This allows for a natural generalization and straightforward implementations in various physical models, some of which are pointed out below in the text.

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1. Rodrigues' Vector from 1840's Perspective

It may be considered a historical misfortune that the French - Jewish banker Olinde Rodrigues proposed his vector-parameter description of rotations as early as in 1840 (see [23]), three years before William Hamilton figured out quaterions, six years before Arthur Cayley came up with his famous transform and before W. K. Clifford and Sophus Lie were even born. And as it appeared so early, it did not find the appropriate context, namely that of hypercomplex numbers Lie groups and geometric algebras, and thus was not well enough understood and appreciated. From the 1840's perspective, the Rodrigues' construction could be realized via Euler's trigonometric substitution from the so-called spherical vector $\mathbf{s} = \varphi\mathbf{n}$, where φ denotes the rotation angle and $\mathbf{n}$ - the unit vector along the invariant axis (with counterclockwise orientation). Then, the Rodrigues' vector may be introduced as

$$\mathbf{c} = \tau\mathbf{n}, \qquad \tau = \tan\frac{\varphi}{2} \tag{1}$$

where $\tau \in \mathbb{RP}^1$ is usually referred to as the *scalar parameter* of the corresponding rotation. Note that the expression (1) cannot be considered a vector, but rather, a point in projective space, since the plane at infinity is included and as we discuss in the next section, it is associated with half-turns. Its advantages, on the other hand, are not hard to see. One pretty straightforward consequence of the substitution (1) is that the matrix entries of the corresponding orthogonal map are rational functions of τ. We may derive them by considering the famous Rodrigues' rotation formula

$$\mathbf{c}^\times \xrightarrow{\exp} \mathcal{R}(\mathbf{n},\varphi) = \cos\varphi\,\mathcal{I} + (1-\cos\varphi)\,\mathbf{n}\mathbf{n}^t + \sin\varphi\,\mathbf{n}^\times \tag{2}$$

where $\mathcal{I}$ denotes the identity, $\mathbf{n}\mathbf{n}^t$ stands for the parallel projector along $\mathbf{n}$ and $\mathbf{n}^\times$ is the skew-symmetric adjoint related to $\mathbf{n}$ via Hodge duality, i.e., $\mathbf{n}^\times\mathbf{a} = \mathbf{n}\times\mathbf{a}$ for an arbitrary vector $\mathbf{a} \in \mathbb{R}^3$. Then, the Euler substitution (1) yields directly

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}\cdot \tag{3}$$

The Decomposition Problem

Formula (3) has a significant advantage from both theoretical and practical point of view. For example, unlike (2) it provides exact expressions (with no transcendent functions whatsoever) and makes it possible to consider rotational motion in rational spaces in a consistent way, which seems unnatural with the exponential map, so physicists very quickly abandoned the idea. Moreover, it has been used successfully in [8] to obtain the full solution to the generalized Euler decomposition problem in $\mathrm{SO}(3)$ with arbitrary axes. The approach is based on Euler's invariant axis theorem and basically involves solving a quadratic equation with discriminant Δ depending on the configuration of axes and the compound rotation, which yields the necessary and sufficient condition for decomposability $\Delta \geq 0$. There have been earlier attempts involving Rodrigues' vector in this context but to the author's knowledge the first paper that does it in a consistent way is [20]. This idea has been developed in [8], which finally provides closed form compact explicit solutions and covers all possible cases, such as the gimbal lock singularity, the two axes decompositions and the identity transformation that lacks even in Piovan and Bullo's paper [22] from 2012, which is the first one to claim completeness. Not being able to resolve for $\tau = 0$, however, is a serious problem if one wishes to consider the infinitesimal case for example, as it has been done in [9]. Moreover, this solution naturally extends to the spin covering group $\mathrm{SU}(2)$ as well as the dual one $\mathrm{SO}(2,1)$ and its spin cover $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$, so the applications in hyperbolic geometry, classical and quantum mechanics are numerous (cf. [5]). The relatively simple compact expressions allow for analytical treatment, which has been used in [10] for obtaining different representations of the quantum-mechanical angular momentum and spherical Laplacian beyond the standard Euler angles. It is also straightforward to derive explicit formulas for transition from one parametrization to another (see [8]). Having such a convenient tool to work with, one is tempted to explore various decomposition settings that may appear useful in practical engineering problems (see for example [6]) as well as for generating convenient parameterizations of $\mathrm{SO}(3)$. For an illustration of the hyperbolic case we refer to [5], [11] with applications in special relativity and quantum mechanics.

Rodrigues' Vector in Literature

For apparent reasons, at first Rodrigues' original paper [23] did not seem to attract the deserved attention, apart from a study by Gibbs [17]. More than a century later it was discovered by the Belarusian mathematician F. Fedorov, who wrote a wonderful book about its applications in special relativity [16] that inspired some more thorough study on its practical aspects, such as [18] and [19]. However, since the book was written in Russian and has not been translated to English yet, its impact on the western literature is quite modest - this includes only a few surveys, e.g. [3, 21], as well as a brief mention in some books on rigid body mechanics. Moreover, although it provides plenty of results, Fedorov's brute force approach is insufficient to reveal the whole elegance and universality of the vector-parameter, so in order to enjoy these features fully, we propose a modern algebraic perspective.

2. Hamilton's Contribution

In October 16-th 1843 William Hamilton created one of the most famous graffiti in the history of mathematics carving in the stone of a bridge in Dublin his famous quaternion relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

This charming act of vandalism had a tremendous impact on modern mathematics and mathematical physics. For quite a while quaternions were a preferred tool in mechanics and electrodynamics, but several decades later Gibbs' vector calculus won the battle. Nevertheless, the quaternion description of rotations remains quite popular in theoretical physics and recently it draws more attention in the context of computer graphics and virtual reality. Moreover, it gave major insight for the fields of hypercomplex calculus, quantum mechanics, integrable systems and Clifford's geometric algebras that appeared several decades after Hamilton's discovery.

There are plenty of surveys on quaternions, some more abstract, others - oriented mostly towards the applications. Here we only revise the standard spin cover construction that is later going to be compared to the projective bivector approach. To begin with, let us introduce the notation $\mathbb{H} \ni \zeta = (\zeta_\circ, \boldsymbol{\zeta})$ where $\zeta_\circ \in \mathbb{R}$ and $\boldsymbol{\zeta} = \zeta_1\mathbf{i}+\zeta_3\mathbf{j}+\zeta_3\mathbf{k}$ are referred to as the *real* and *imaginary* parts of ζ, respectively. Moreover, we may clearly identify vectors in $\mathbb{R}^3$ with *pure quaternions* ($\zeta_\circ = 0$)

$$\mathbf{x} \in \mathbb{R}^3 \quad \longrightarrow \quad \mathbf{X} = x_1\,\mathbf{i} + x_2\,\mathbf{j} + x_3\,\mathbf{k} \in \mathbb{H}.$$

Next, introducing quaternion conjugation $\zeta \to \bar{\zeta} = (\zeta_\circ, -\zeta)$ and the corresponding norm $|\,\zeta\,|^2 = \zeta\bar{\zeta}$, we may represent elements of $\mathrm{SU}(2) \cong \mathbb{S}^3$ with unit quaternions

$$\mathbb{S}^3 = \{\zeta \in \mathbb{H};\ |\zeta|^2 = 1\}.$$

The adjoint action of $\mathrm{SU}(2) \cong \mathbb{S}^3$ in its algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$

$$\mathrm{Ad}_\zeta :\ \mathbf{X} \ \longrightarrow\ \zeta\,\mathbf{X}\,\bar{\zeta}$$

preserves the metric and orientation, thus representing a $\mathrm{SO}(3)$ transformation

$$\mathcal{R}(\zeta) = (\zeta_\circ^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\boldsymbol{\zeta}\boldsymbol{\zeta}^t + 2\zeta_\circ\boldsymbol{\zeta}^\times$$

and the famous *Rodrigues'* rotation formula follows with the substitution

$$\zeta_\circ = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\,\mathbf{n}.$$

Then, central projection onto the hyperplane $\zeta_\circ = 1$ yields the vector-parameter[1]

$$\mathbf{c} = \frac{\boldsymbol{\zeta}}{\zeta_\circ} = \tau\mathbf{n} \in \mathbb{RP}^3, \qquad \tau = \tan\frac{\varphi}{2}\cdot \tag{4}$$

Since this projective construction is invariant under multiplication by a non-zero scalar, one is only restricted to invertible quaternions $\zeta \in \mathbb{H}^\times \cong \mathbb{H}/\{0\}$ rather than unit ones $\zeta \in \mathbb{S}^3$. Moreover, due to homogeneity, quaternion multiplication

$$(\xi_\circ, \boldsymbol{\xi}) \otimes (\zeta_\circ, \boldsymbol{\zeta}) \ \stackrel{\vee}{\rightarrow}\ (\xi_\circ\zeta_\circ - \boldsymbol{\xi}\cdot\boldsymbol{\zeta},\ \xi_\circ\boldsymbol{\zeta} + \zeta_\circ\boldsymbol{\xi} + \boldsymbol{\xi}\times\boldsymbol{\zeta})$$

projects nicely to $\mathbb{RP}^3$ giving rise to the vector-parameter composition law

$$\langle\,\mathbf{c}_2, \mathbf{c}_1\rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2\times\mathbf{c}_1}{1 - \mathbf{c}_2\cdot\mathbf{c}_1} \tag{5}$$

that is related to the usual matrix realization via $\mathcal{R}(\langle\,\mathbf{c}_2, \mathbf{c}_1\rangle) = \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$ and suggests an alternative representation of $\mathrm{SO}(3)$ with

$$\langle\,\mathbf{c},\,0\,\rangle = \langle\,0,\,\mathbf{c}\,\rangle = \mathbf{c}, \qquad\qquad \langle\,\mathbf{c},\,-\mathbf{c}\,\rangle = 0.$$

The correspondence with the usual spin cover is given by

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathbb{H}^\times/\mathbb{R}^\times$$

[1]Similarly, stereographic projection yields the *Wiener-Milenkovic conformal vector* (see [3]).

and Euler's trigonometric substitution allows for taking a shortcut in the sequence

$$\mathbb{H}^\times \xrightarrow{\mathbb{R}^+} \mathrm{SU}(2) \xrightarrow{\mathbb{Z}_2} \mathrm{SO}(3)$$

thus avoiding the complications of introducing unit spheres appearing in the standard spin cover choosing instead to construct $\mathbb{RP}^3$ as the space of rays through the origin in $\mathbb{R}^4$. This yields the group multiplication (5) via the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^\times \otimes \mathbb{H}^\times & \xrightarrow{\ \vee\ } & \mathbb{H}^\times \\ \big\downarrow{\scriptstyle \pi\otimes\pi} & & \big\downarrow{\scriptstyle \pi} \\ \mathrm{SO}(3)\otimes\mathrm{SO}(3) & \xrightarrow{\langle\cdot,\cdot\rangle} & \mathrm{SO}(3) \end{array} \tag{6}$$

where $\vee$ denotes the Clifford multiplication of quaternions and π is the projection defining the vector-parameter (4). This construction yields directly the associative composition law (5) expressed for an arbitrary number of elements $\zeta_j \in \mathbb{H}^\times$ as

$$\langle \zeta_k, \zeta_{k-1}, \dots, \zeta_1 \rangle = \frac{\Im\,(\zeta_k \vee \zeta_{k-1} \vee \cdots \vee \zeta_1)}{\Re\,(\zeta_k \vee \zeta_{k-1} \vee \cdots \vee \zeta_1)} \tag{7}$$

which allows for certain interpretations and generalizations. Before we discuss this, however, we need to consider its relation to the usual matrix representation from the proper perspective, i.e., define it in an invariant, extendable manner too. The way to do so is by means of the well-known (now, but definitely not in 1840) Cayley's transform, of which formula (3) certainly happens to be a particular case.

The Plane at Infinity

Our projective construction yields $\mathbf{c}$ as an element of $\mathbb{RP}^3$ endowed with an additional group structure (5). The only cases, in which the term "vector" is suitable are when $\mathbf{c}_{1,2}$ are either infinitesimal or (in the complex or hyperbolic setting) restricted to an isotropic line. One more reason we cannot refer to $\mathbf{c}$ as vector is that its magnitude becomes infinite when $\zeta_\circ = 0$, i.e., $\varphi = \pi$, in which case the rotation is a symmetric involution and thus, determined only by its invariant axis. The set of all axes in $\mathbb{R}^3$ constitutes the *plane at infinity* $\mathbb{RP}^2$ embedded in $\mathrm{SO}(3) \cong \mathbb{RP}^3$. Although thoroughly discussed in [8] and [21], this issue can easily be dealt with by simply considering the limit $\mathbf{c}^2 \to \infty$, in which formula (3) clearly converges to

$$\mathcal{O}(\mathbf{n}) = \mathcal{R}(\mathbf{n}, \pi) = 2\mathbf{n}\mathbf{n}^t - \mathcal{I} \tag{8}$$

that we refer to as a *half-turn* about $\mathbf{n}$. Similarly, if we let $\hat{\mathbf{c}}_k$ denote the unit vector in the $\mathbf{c}_k$-direction, the composition law (5) yields directly

$$\langle\, \mathbf{c}_2, \mathbf{c}_1 \rangle \xrightarrow{\mathbf{c}_1^2,\,\mathbf{c}_2^2 \to \infty} \frac{\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2}{\hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_2}$$

which illustrates a well-known result in elementary geometry. Furthermore, two generic rotations represented by $\mathbf{c}_{1,2} \in \mathbb{RP}^3$ add up to a half-turn if and only if $\mathbf{c}_1 \cdot \mathbf{c}_2 = -1$ while $\mathbf{c}_1 + \mathbf{c}_2 \neq 0$ as formula (5) shows. Beside the limiting procedure described above, one may also lift up back to the spin cover $\mathrm{SU}(2)$ as

$$\zeta_\circ^\pm = \pm(1+\mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^\pm = \zeta_\circ^\pm \boldsymbol{\zeta} \tag{9}$$

which in the case of a half-turn takes the form $\zeta_\circ^\pm = 0$ and $\boldsymbol{\zeta}^\pm = \pm\mathbf{n}$, i.e., each point in $\mathbb{RP}^2$ defines a pair of a pure quaternions mapped to the same half-turn. Finally, let us note that obtaining the invariant axis from the symmetric matrix of a half-turn is a standard eigenvector problem: $\mathbf{n}$ corresponds to the simple eigenvalue $\lambda = 1$, while for a generic rotation ($\varphi \neq 0, k\pi$) one may instead use the formula

$$\mathbf{c}^\times = \frac{\mathcal{R} - \mathcal{R}^t}{1 + \mathrm{Tr}\mathcal{R}} \tag{10}$$

and then derive the Rodrigues' vector-parameter $\mathbf{c}$ by means of the Hodge duality.

3. Cayley's Transform

One way to define the Cayley transform in the extended complex plane $\mathbb{CP}^1 \cong \mathbb{S}^2$ is as a linear-fractional map from the imaginary line to the unit circle given by

$$z \in \mathrm{i}\,\mathbb{R} \cup \{\infty\} \quad \xrightarrow{\mathrm{Cay}} \quad w = \frac{1+z}{1-z} \in \mathbb{S}^1 \tag{11}$$

so in order to express its image in the form $w = \mathrm{e}^{\mathrm{i}\varphi}$, one needs to have $z = \mathrm{i}\tau$ with $\tau = \tan\dfrac{\varphi}{2}$ and the result follows from basic trigonometry. Therefore, the inverse

$$w \in \mathbb{S}^1 \quad \xrightarrow{\mathrm{Cay}^{-1}} \quad \frac{w-1}{w+1} \in \mathrm{i}\,\mathbb{R} \cup \{\infty\}$$

has an obvious trigonometric interpretation as well. Next, if we let $\mathcal{A}$ represent a linear operator with no unit eigenvalues, then $(\mathcal{I} - \mathcal{A})^{-1}$ is well-defined and commutes with $\mathcal{I} + \mathcal{A}$, so the Cayley map (11) may also be extended to linear operators with the above property and written in the form

$$\mathrm{Cay}: \mathcal{A} \quad \longrightarrow \quad \mathcal{R} = \frac{\mathcal{I}+\mathcal{A}}{\mathcal{I}-\mathcal{A}}, \qquad \mathcal{R} \quad \xrightarrow{\mathrm{Cay}^{-1}} \quad \mathcal{A} = \frac{\mathcal{R}-\mathcal{I}}{\mathcal{R}+\mathcal{I}} \cdot \tag{12}$$

In particular, one can easily see that Cay maps skew-symmetric operators to orthogonal ones and skew-hermitian to unitary. This property surely generalizes to non-trivial signatures as well, but it is not correct to say that (12) is a map from the Lie algebra $\mathfrak{so}(p,q)$ to the corresponding group $\mathrm{SO}(p,q)$ alternative to the usual matrix exponent as the composition law in the domain is not a Lie bracket. Instead, one has a fractional expression that generalizes formula (5) in a certain way as we show below. Let us also note that the expressions (12) are clearly expandable (by analytic continuation) in a geometric Volterra series, which is reduced to a finite sum due to the recursive relations imposed by Hamilton-Cayley's theorem. In particular, our construction uses $\mathcal{A} = \mathbf{c}^{\times}$ whose characteristic equation yields

$$\left(\mathbf{c}^{\times}\right)^{n+2} = -\mathbf{c}^2 \left(\mathbf{c}^{\times}\right)^n, \qquad n \in \mathbb{N}.$$

Then, the Cayley transform maps it to an orthogonal operator in the form

$$\mathrm{Cay}(\mathbf{c}^{\times}) = \mathcal{R}(\mathbf{c}) = \frac{\mathcal{I} + \mathbf{c}^{\times}}{\mathcal{I} - \mathbf{c}^{\times}} = \mathcal{I} + \frac{2}{1+\mathbf{c}^2}\left(\mathcal{I} + \mathbf{c}^{\times}\right)\mathbf{c}^{\times} \tag{13}$$

that is easily seen to coincide with (3). Like the exponential map, this formalism works also in the hyperbolic and complex cases, as well as in higher dimensions, at least for plane (pseudio-)rotations, as we thoroughly explain in the next section.

Rigid Body Mechanics

For a rigid body rotating about a fixed center, determined by the smooth time flow $\mathcal{R}(t) = \mathrm{Cay}(\mathbf{c}^{\times}(t)) \in \mathrm{SO}(3)$ one may define the angular velocity vectors $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ in the body and the inertial frame, respectively as

$$\boldsymbol{\Omega}^{\times} = \dot{\mathcal{R}}\mathcal{R}^t, \qquad \boldsymbol{\omega}^{\times} = \mathcal{R}^t\dot{\mathcal{R}}$$

where $\dot{\mathcal{R}}$ denotes the time derivative and $\mathcal{R}^t$ stands for the usual matrix transposition. Now, via straightforward differentiation in the Cayley representation (13) one easily obtains the expressions (see [19] and [21])

$$\boldsymbol{\Omega} = \frac{2}{1+\mathbf{c}^2}\left(\mathcal{I} + \mathbf{c}^{\times}\right)\dot{\mathbf{c}}, \qquad \boldsymbol{\omega} = \frac{2}{1+\mathbf{c}^2}\left(\mathcal{I} - \mathbf{c}^{\times}\right)\dot{\mathbf{c}}$$

while direct matrix inversion yields

$$\dot{\mathbf{c}} = \frac{1}{2}\left(\mathcal{I} + \mathbf{c}\mathbf{c}^t - \mathbf{c}^{\times}\right)\boldsymbol{\Omega} = \frac{1}{2}\left(\mathcal{I} + \mathbf{c}\mathbf{c}^t + \mathbf{c}^{\times}\right)\boldsymbol{\omega}. \tag{14}$$

Since vectorial parametrization is conveniently related to various decompositions as discussed above, in many situations the Riccati equations (14) are more useful for the description of kinematical problems than other known representations.

4. Modern Clifford Perspective

Clifford's geometric algebras provide both efficient and esthetic approach to a wide variety of research areas - from geometry and group theory, to quantum mechanics, VR and computer vision. The algebra $C\ell_{p,q}$ is built upon the associative Clifford product, which for vectors may be defined as (we refer to [1] for a pedagogical study)

$$\mathbf{x} \vee \mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,q} \subset C\ell_{p,q}(\mathbb{R}) \tag{15}$$

where the first (symmetric) part is given by the dot product for the corresponding metric and the second (skew-symmetric) one is the exterior wedge product. It is not hard to see that (15) generalizes the multiplication in $\mathcal{C}\ell_0 \cong \mathbb{R}$, $\mathcal{C}\ell_{0,1} \cong \mathbb{C}$ and $\mathcal{C}\ell_{0,2} \cong \mathbb{H}$. Although it shares a mutual basis of multi-vectors with the Gassmann algebra, $\mathcal{C}\ell_{p,q}$ is $\mathbb{Z}_2$-graded and yet, there are grade projectors $\langle \cdots \rangle_k$ onto linear subspaces of k-vectors $\{\mathbf{u}_1 \vee \mathbf{u}_2 \cdots \vee \mathbf{u}_k\}$. Moreover, as a factor of the tensor algebra, $\mathcal{C}\ell_{p,q}$ is unital and the inverse is defined via conjugation (see [1] for details) just like in the case of complex numbers and quaternions, although there are zero divisors in the general setting. To make the relation to isometries in $\mathbb{R}^{p,q}$ apparent, we consider the Clifford group of all invertible elements $\Gamma_{p,q} = \mathcal{C}\ell^{\times}_{p,q}$ and let $\Gamma^{\circ}_{p,q}$ be its even subgroup (consisting of even grade multi-vectors). Then, the restriction to elements of $\Gamma^{\circ}_{p,q}$ with unit norm yields the spin group $\mathrm{Spin}(p,q)$, which is a double cover of $\mathrm{SO}(p,q)$ acting on Clifford multi-vectors via conjugation. The advantage of the vectorial parametrization is that it surpasses this cumbersome procedure and projects $\Gamma^{\circ}_{p,q}$ directly to $\mathrm{SO}(p,q)$ simplifying the Clifford product and maintaining rationality via the Cayley map (12). A straightforward extension suggests the projection $\Gamma^{\circ}_{p,q} \xrightarrow{\mathbb{R}^{\times}} \mathrm{G}_{p,q} \cong \mathrm{SO}(p,q)$ and the corresponding composition law generalizing (7) derived from the commutative diagram

$$\begin{tikzcd} \Gamma^{\circ} \otimes \Gamma^{\circ} \arrow[r, "\vee"] \arrow[d, "\pi\otimes\pi"] & \Gamma^{\circ} \arrow[d, "\pi"] \\ \mathrm{G} \otimes \mathrm{G} \arrow[r, "\langle\cdot,\cdot\rangle"] & \mathrm{G} \end{tikzcd} \tag{16}$$

in the form

$$\langle \mathbf{c}_k, \mathbf{c}_{k-1}, \ldots, \mathbf{c}_1 \rangle = \langle \zeta_k \vee \zeta_{k-1} \vee \cdots \zeta_1 \rangle_0^{-1} \sum_{k=1}^{[\frac{p+q}{2}]} \langle \zeta_k \vee \zeta_{k-1} \vee \cdots \zeta_1 \rangle_{2k}.$$

However, it makes sense only when the above sum is homogeneous, e.g. for quaternions $\mathbb{H} = \mathcal{C}\ell^{\circ}_3$ and split-quaternions $\mathbb{H}' = \mathcal{C}\ell^{\circ}_{2,1}$, where the even subalgebra consists of scalars and bi-vectors. Introducing complex coefficients $\mathbb{H}^{\mathbb{C}} \cong \mathcal{C}\ell^{\circ}_{1,3} \cong \mathcal{C}\ell_3$ one obtains the vector-parameter of the Lorentz group as thoroughly explained in [12] and [16]. Similarly, due to the well-known Lie algebra isomorphism $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$ one may also define a local vector-parameter for $\mathrm{O}(4,\mathbb{C})$, which Fedorov refers to as the complex Lorentz group in [16]. This yields a wide variety of real forms, e.g. $\mathrm{SO}(4)$, $\mathrm{SO}(2,2)$ and $\mathrm{SO}^*(4)$ that have also been discussed in [7] in the context of decomposition techniques. Certainly, the simplest example $\mathrm{SO}^+(1,2) \cong \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)/\mathbb{Z}_2$ may be obtained by duality (starting with split-quaternions) as shown in [5]. In this case the main results - formula (3) and (5) remain valid if one takes into account that the metric with signature $(1,2)$ determines both the dot and cross products, as well as the parallel projectors. Note that at the same time the analogue of formula (2) in the hyperbolic case has different versions depending on the geometric type of the invariant axis (see [5] for details).

The Plücker Embedding and Dimensional Induction

As argued by Bogush and Fedorov in [4], plane rotations in higher dimensions may be represented in a similar way. The vectorial interpretation, however, needs to be replaced with a tensorial one as the invariant axis is not unique but the corresponding plane is. Thus, the natural way to operate in higher dimensions is by considering bivectors $\theta = \mathbf{u} \wedge \mathbf{v}$ associated with oriented rotation planes, with magnitude equal to $\tau = \tan\dfrac{\varphi}{2}$, which may be derived from the scalar product associated with the Killing form in the matrix representation. Then, the cross product is replaced with the usual matrix commutator, i.e., one has for the composition law[2]

$$\langle\Theta_2,\Theta_1\rangle = \frac{\Theta_1 + \Theta_2 + [\Theta_2,\Theta_1]}{1-(\Theta_1,\Theta_2)} \tag{17}$$

where the (projective) bivectors θ_1, θ_2 need to satisfy the Plücker relations

$$\theta_i \wedge \theta_j = 0 \tag{18}$$

that guarantee first (for $i = j$) that both transformations are simple (planar) and second ($i \neq j$) that their invariant planes intersect over a line, thus giving rise to a three-dimensional subspace, in which an irreducible representation of $\mathrm{SO}_3 \subset \mathrm{SO}_n$ takes place. The latter is naturally realized by means of the Cayley map

$$\mathrm{Cay}: \quad \Theta \quad \to \quad \mathcal{R} = \frac{\mathcal{I}+\Theta}{\mathcal{I}-\Theta} = \mathcal{I} + \frac{2}{1+|\Theta|^2}\left(\mathcal{I}+\Theta\right)\Theta.$$

This construction works for the complex groups and therefore, for all their real forms, e.g. in the case of the Lorentz group $\mathrm{SO}^+(3,1)$, where vectorial parametrization is still possible, one may write the Plücker relations in the form $\Im\left(\mathbf{c}_i\cdot\mathbf{c}_j\right) = 0$ and the signature of the metric in the invariant subspace obtained in this manner determines the corresponding Wigner little group, whose representation is realized in it. In [12] one may find a thorough geometric study on the higher-dimensional induction procedure described above (see also [14] for slightly different approach).

Dualization and Screws

One typical central extension of the ring of scalars that has proven quite useful in mechanics is the introduction of *dual numbers* in the form

$$x, h \in \mathbb{R} \quad \longrightarrow \quad \underline{x} = x + \varepsilon h \in \mathbb{R}[\varepsilon], \qquad \varepsilon^2 = 0$$

which provides a convenient representation for the theory of screws (see [13], [24] for details) describing rotational and translational motion of rigid bodies in a unified framework. In particular, it is not difficult to see from the above definition

[2]Here and below we let Θ_k denote the matrix representation of the bi-vector θ_k (see [12]).

that

$$f(x+\varepsilon h) = f(x) + \varepsilon f'(x)h \tag{19}$$

and the construction generalizes to vectors, tensors, quaternions and thus, vector-parameters, e.g. in [24] the dual Rodrigues' vector is given, based on (19), as

$$\underline{\mathbf{c}} = \Big(\tau + \varepsilon(1+\tau^2)\frac{\psi}{2}\Big)\underline{\mathbf{n}} \tag{20}$$

where $\underline{\mathbf{n}} = \mathbf{n} + \varepsilon\mathbf{m}$ is referred to as the *dual axis* and $\underline{\varphi} = \varphi + \varepsilon\psi$ is the *dual angle* with $\varphi = 2\arctan\tau$, just as before. In our approach, however, one would begin with invertible dual quaternions $\mathbb{H}^\times[\varepsilon]$ and then factor out the action of $\mathbb{R}^\times[\varepsilon]$. This definition would naturally extend to the case of complex dual quaternions, thus parameterizing non-homogeneous proper Lorentz transformations in $\mathbb{R}^{3,1}$. Certainly, the construction may be obtained equivalently by starting from the conformal picture and then restricting to the light cone, which is a more rigorous way of doing that. However, the definition of vector-parameters for $\mathrm{SO}(4,2)$ is rather cumbersome. On the other hand, the above extension (dualization of $\mathbb{R}$) yields directly

$$\underline{\mathbf{c}} = \frac{\underline{\boldsymbol{\zeta}}}{\underline{\zeta}_\circ} = \zeta_\circ^{-1}\underline{\boldsymbol{\zeta}}\Big(1 - \varepsilon\frac{\xi_\circ}{\zeta_\circ}\Big) = \frac{\boldsymbol{\zeta}}{\zeta_\circ} + \frac{\varepsilon}{\zeta_\circ}\Big(\boldsymbol{\xi} - \frac{\xi_\circ}{\zeta_\circ}\boldsymbol{\zeta}\Big) = \mathbf{c} + \varepsilon\mathbf{d} \tag{21}$$

where we denote

$$\underline{\zeta}_\circ = \cos\frac{\underline{\varphi}}{2} = \cos\frac{\varphi}{2} - \varepsilon\frac{\psi}{2}\sin\frac{\varphi}{2} = \zeta_\circ + \varepsilon\xi_\circ$$

and respectively

$$\underline{\boldsymbol{\zeta}} = \sin\frac{\underline{\varphi}}{2}\underline{\mathbf{n}} = \sin\frac{\varphi}{2}\mathbf{n} + \varepsilon\Big(\frac{\psi}{2}\cos\frac{\varphi}{2}\mathbf{n} + \sin\frac{\varphi}{2}\mathbf{m}\Big) = \boldsymbol{\zeta} + \varepsilon\boldsymbol{\xi}.$$

It is not hard to see that formula (21) then yields (20) since one clearly has

$$\mathbf{d} = \tau\mathbf{m} + \frac{\psi}{2}(1+\tau^2)\mathbf{n}$$

but it also yields the composition law for dual vector-parameters (5) in the form

$$\langle\underline{\mathbf{c}}_2, \underline{\mathbf{c}}_1\rangle = \langle\mathbf{c}_2, \mathbf{c}_1\rangle\,(1 + \varepsilon\lambda_\circ) + \varepsilon\boldsymbol{\lambda} \tag{22}$$

with the notation

$$\lambda_\circ(\underline{\mathbf{c}}_1, \underline{\mathbf{c}}_2) = \frac{\mathbf{c}_2\cdot\mathbf{d}_1 + \mathbf{d}_2\cdot\mathbf{c}_1}{1 - \mathbf{c}_2\cdot\mathbf{c}_1}, \qquad \boldsymbol{\lambda}(\underline{\mathbf{c}}_1, \underline{\mathbf{c}}_2) = \frac{\mathbf{d}_2 + \mathbf{d}_1 + \mathbf{c}_2\times\mathbf{d}_1 + \mathbf{d}_2\times\mathbf{c}_1}{1 - \mathbf{c}_2\cdot\mathbf{c}_1}.$$

Similarly, one derives for the Cayley transform (13) in the dual case the expression

$$\mathrm{Cay}(\underline{\mathbf{c}}) = \big(\mathcal{I} + \underline{\mathbf{c}}^\times\big)\big(\mathcal{I} - \underline{\mathbf{c}}^\times\big)^{-1} = \mathrm{Cay}(\mathbf{c})(1 + \varepsilon\mu) + \varepsilon\mu, \quad \mu(\underline{\mathbf{c}}) = \mathbf{d}^\times\big(1 - \mathbf{c}^\times\big)^{-1}.$$

Note that if we choose a definition, in which the inverse is a left factor, this affects the expression for μ correspondingly and for a helical (screw) motion $\mathbf{c}\times\mathbf{d} = 0$ both definition naturally coincide. It is not difficult to obtain also an analogue of formula (3), which would be valid for hyperbolic and complex vector-parameters, hence successfully describing non-homogeneous Lorentz transformations as well.

Further Generalizations

Following Fedorov [16] (at least partially), we begin by introducing the vector-parameter construction for $\mathrm{SO}(3)$ and then extend it to the proper Lorentz group

$$\mathrm{SO}^+(3,1) \cong \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(3,\mathbb{C})$$

via complexification. Similarly, one may choose to initially consider the complex case and then complexify once more, ending up with bicomplex-valued quaternions $\mathbb{H}^{\mathbb{C}} \to \mathbb{H}^{\mathbb{B}}$, where $\mathbb{B} \cong \mathbb{C}^{\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{C}$ appears as the complex Clifford algebra $\mathcal{C}\ell_1(\mathbb{C})$. The splitting provided by its zero divisors then yields $\mathbb{H}^{\mathbb{B}} \cong \mathbb{H}^{\mathbb{C}} \oplus \mathbb{H}^{\mathbb{C}}$ so one may actually define a local parametrization of the so-called complex Lorentz group $\mathrm{O}(4,\mathbb{C})$. Unfortunately, one cannot go any further without sacrificing some of the nice properties discussed above. However, we may still perform a naive attempt to proceed with the Cayley-Dickson doubling process extending the ring of scalars $\mathbb{K}$ to $\mathbb{H}$ or $\mathbb{O}$. Moreover, an old result due to Vahlen, later promoted by Ahlfors [2], says that all classical Möbius groups may be represented by means of linear-fractional transformations with coefficients in some Clifford algebra. In particular, for the Hurwitz algebras one has $\mathrm{PSL}(2,\mathbb{K}) \cong \mathrm{SO}^+(\dim_{\mathbb{R}} \mathbb{K} + 1, 1)$, e.g. $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}^+(2,1)$ acts as a conformal groups on the projective real line $\mathbb{RP}^1 \cong \mathbb{S}^1$. Similarly, $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}^+(3,1)$ is associated with the complex case $\mathbb{CP}^1 \cong \mathbb{S}^2$ and finally, for the quaternionic one we have (see [15] for more details)

$$\mathrm{PSL}(2,\mathbb{H}) \cong \mathrm{SO}^+(5,1).$$

A straightforward approach suggests using the vector-parameter construction for SL_2 and then simply allow the components of the corresponding (split-)quaternion to take values in $\mathbb{H}$. Here, however, the central projection (4) would not be relevant as the kernel of the homomorphism $\mathrm{GL}(2,\mathbb{H}) \to \mathrm{SL}(2,\mathbb{H})$ is just $\mathbb{R}^\times$, rather than $\mathbb{H}^\times$, so one may only divide by a non-vanishing real scalar, e.g. the norm of $\zeta_\circ$, which leaves a three-dimensional phase related to the quaternionic Hopf fibration

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^7 \longrightarrow \mathbb{S}^4.$$

Similarly, defining the vector-parameter as $\mathbf{c} = |\zeta_\circ|^{-1}\zeta$, instead of $\mathbf{c} = \zeta_\circ^{-1}\boldsymbol{\zeta}$ for the remaining Hurwitz algebras, one ends up with a realization of the Hopf bundles

$$\mathbb{S}^0 \longrightarrow \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \qquad \mathbb{S}^1 \longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2, \qquad \mathbb{S}^7 \longrightarrow \mathbb{S}^{15} \longrightarrow \mathbb{S}^8$$

but it is unnecessary in the commutative case. With this modification, one may use a basis of $\mathfrak{sl}_2$ and an analogue of formula (5) for the product $\mathrm{SL}_2 \otimes \mathbb{H}^\times$ in the form

$$\langle(\sigma_2,\mathbf{c}_2),(\sigma_1,\mathbf{c}_1)\rangle = \left(\frac{\sigma_2\sigma_1 - \mathbf{c}_2\cdot\mathbf{c}_1}{|\sigma_2\sigma_1 - \mathbf{c}_2\cdot\mathbf{c}_1|},\ \frac{\sigma_2\mathbf{c}_1 + \mathbf{c}_2\sigma_1 + \mathbf{c}_2\times\mathbf{c}_1}{|\sigma_2\sigma_1 - \mathbf{c}_2\cdot\mathbf{c}_1|}\right) \tag{23}$$

where $\sigma_{1,2} \in \mathrm{SU}(2)$ are the quaternionic phases, $\mathbf{c} \in \mathbb{H}^\times/\mathbb{R}^\times$ and the product between the components is Clifford multiplication. Then, one obtains the group action via linear-fractional transformations on the projective quaternion line

$$\mathcal{A} = \sigma\mathcal{I} + \sum_{k=1}^{3} c_k \mathbf{e}_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longrightarrow \quad (az+b)(cz+d)^{-1}$$

where the $\mathbf{e}_k$'s span the basis of $\mathfrak{sl}_2$ and $z \in \mathbb{HP}^1$. Although the above construction is not as simple as the one provided by formula (5) and (13), it covers a wide variety of conformal groups used in mathematics and physics, such as $\mathrm{SO}(4,2)$, $\mathrm{SO}(4,1)$ or $\mathrm{SO}(3,2)$. One may go even a step beyond the Clifford framework considering the last of the Hurwitz algebras $\mathbb{O}$ that suggests by analogy the correspondence

$$\mathrm{PSL}(2,\mathbb{O}) \cong \mathrm{SO}^+(9,1)$$

and then, introducing various conjugations and real forms, derive relations like

$$\mathrm{SU}(2,\mathbb{H}) \cong \mathrm{Spin}(5), \qquad \mathrm{SU}(2,\mathbb{H}') \cong \mathrm{Spin}(3,2), \qquad \mathrm{SU}(2,\mathbb{O}) \cong \mathrm{Spin}(9).$$

However, the octonions impose certain obstacles due to the loss of associativity. For a detailed study on this matter and its many connections to the classical and exceptional Lie groups we refer to [15]. A thorough investigation of the properties and possible applications of quaternionic or octonionic vector-parameters, on the other hand, goes beyond the scope of the text and demands extensive volume.

Concluding Remarks

This brief paper is meant to serve a double purpose: on the one hand, to popularize and promote the not so well-known Rodrigues' vector-parameter construction, while on the other, to put it in the perspective of Clifford's geometric algebras, which allows for both a comprehensive study and farther reaching generalizations.

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