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BIFURCATION OF CLOSED GEODESICS

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Abstract. This paper is denoted to further study of geodesic bifurcation on surfaces of revolution. We demonstrate an example of bifurcation of closed geodesics on surfaces.

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1. Introduction

Geodesics are special curves that play important role in differential geometry. These curves were studied in many works, see [1–3, 5, 6, 8, 10].

We studied geodesic bifurcations in our paper [9]. Here, we describe problem about geodesics. We found and example of geodesic bifurcation on certain surface of revolution. We explain the term bifurcation as a situation when at least two different geodesics go through the given point in the given direction.

This term was also used but with a different meaning, see [11]. There, geodesic bifurcation is understood as situation when more geodesic go through given point but do not have the same tangent vector.

The result of our study is a construction of surface of revolution where exist closed geodesics which admit above described geodesic bifurcation.

2. Geodesics

Let (M, ∇) be a manifold M with affine connection ∇. In local chart (U, x) the connection ∇ is defined with its components $\Gamma_{ij}^h(x)$.

The curve $\gamma(t)$ on the manifold M is called *geodesic* if its tangent vector $\gamma'(t)$ is recurrent along it, therefore the vector $\nabla_t\gamma'$ is parallel to tangent vector $\dot\gamma$ and the following equation holds

$$\nabla_t\gamma' = \rho(t)\dot\gamma$$

where $\rho(t)$ is a function of parameter t, see also [3–7, 10].

Locally, there exists *canonical* parameter s on the curve γ which satisfies

$$\nabla_s\dot\gamma = 0 \tag{1}$$

and in this case vector field $\dot\gamma(s)$ is parallel along γ.

In some chart (U, x) equation (1) can be written as follows

$$\ddot x^h(s) + \Gamma^h_{ij}\big(x(s)\big)\,\dot x^i(s)\,\dot x^j(s) = 0 \tag{2}$$

where $x^h = x^h(s)$ are equations of geodesic γ.

Geodesic are often defined as curves that satisfy equations (2), see [6, pp. 88–91].

We can rewrite the equations (2) in the following form of ordinary differential equations of first order, respective unknown functions $x^h(s)$ and $\lambda^h(s)$

$$\dot x^h(s) = \lambda^h(s), \qquad \dot\lambda^h(s) = -\Gamma^h_{ij}\big(x(s)\big)\,\dot x^i(s)\,\dot x^j(s). \tag{3}$$

Here $\lambda^h(s)$ is a tangent vector of $\gamma(s)$ at the point $x^h(s)$.

The initial conditions of equations (3)

$$x^h(s) = x^h_0, \qquad \text{and} \qquad \lambda^h(s) = \lambda^h_0 \tag{4}$$

ensure that geodesic goes through point x^h_0 in direction λ^h_0 ($\neq 0$).

From general theory of ordinary differential equations follows that equations (3) with initial conditions (4) have solution if $\Gamma^h_{ij}(x)$ are continuous functions. Furthermore, if the functions $\Gamma^h_{ij}(x)$ are differentiable function the solution of Cauchy problem (3) and (4) is *unique*. Moreover, the condition of differentiability of the functions $\Gamma^h_{ij}(x)$ can be replaced with Lipschitz condition, thus if the functions $\Gamma^h_{ij}(x)$ satisfy Lipschitz condition the solution is also unique.

3. Geodesic Bifurcations

In [9] it was proved that on the surface of revolution $\mathcal{S}$ given by the equations

$$x = r(u)\cos v, \qquad y = r(u)\sin v, \qquad z = z(u)$$

where

$$r(u) = \frac{1}{\sqrt{1-u^{2\alpha}}}, \qquad u \in (-1, 1), \qquad v \in (-\pi, \pi)$$

there exist geodesic bifurcations for $\alpha \in (0, 1)$.

For example, through point $(0,0)$ in direction $(0,1)$ go two geodesics which equations has the following form

$$\begin{array}{lll} \text{I.} & u = 0, & v = s \\ \text{II.} & u = \big((1-\alpha)\, s\big)^{\frac{1}{1-\alpha}}, & v = s - \dfrac{\big((1-\alpha)\, s\big)^{\frac{1+\alpha}{1-\alpha}}}{1+\alpha}. \end{array}$$

The first curve is called gorge circle and the second we shall call nontrivial geodesic going through the given point in given direction. It is obvious that there exist infinity number of such geodesic since the surface $\mathcal{S}$ is a surface of revolution. Which means that through every point of the gorge circle goes another nontrivial geodesic.

We also proved that every geodesic curve γ of any surface of revolution has to satisfy the following conditions

$$\dot{u} = \sqrt{1 - \frac{{C_1}^2}{f(u)}}, \qquad \dot{v} = \frac{C_1}{f(u)} \tag{5}$$

where $C_1 \in \mathbb{R}$.

In the following part we construct a surface on which closed geodesics exist.

4. Bifurcation of Closed Geodesics

The construction of such surface is based on previous results. We take a certain neighbourhood along the gorge circle of the surface $\mathcal{S}$ where bifurcations exist. Now we suppose there exist geodesic γ starting in the point $(0,0)$ going in direction $(0,1)$. Then we suppose that this curve has its "end" also on the gorge circle and is also part of some nontrivial geodesic (This is because we are constructing surface of revolution so it has to be symmetrical). Our goal is to connect those ends with the curve that would form the surface.

Since we have some starting part of the curve γ we can calculate functions $\dot{u}$ and $\dot{v}$ in known point β, where $\beta > 0$. We denote canonical parameter corresponding to the this point as s_β then

$$\dot{u}(s_\beta) = \dot{u}_\beta \qquad \text{and} \qquad \dot{v}(s_\beta) = \dot{v}_\beta.$$

Therefore, after substitution to the equations (5) we get

$$\dot{u}_\beta = \sqrt{1 - \frac{C_1^2}{f(u_\beta)}}, \qquad \dot{v}_\beta = \frac{C_1}{f(u_\beta)}.$$

Let us consider a certain point B, the *extremal* point, on the geodesic curve γ in which the tangent vector is the same as in the starting point but does not have to be unitary. Furthermore, we suppose that the point B lies, again, on the gorge circle where bifurcations exist. Simply, we take another neighbourhood along the gorge circle of the surface $\mathcal{S}$ (mentioned in the previous section) and move it so the point B lies on the second gorge circle. Similarly, we denote function corresponding to the point B as follows

$$\dot{u}(s_B) = \dot{u}_B \qquad \text{and} \qquad \dot{v}(s_B) = \dot{v}_B.$$

Again, after direct substitution to the equations (5) we obtain

$$\dot{u}_B = \sqrt{1 - \frac{C_1^2}{f(u_B)}}, \qquad \dot{v}_B = \frac{C_1}{f(u_B)}. \tag{6}$$

As we mentioned above the tangent vector in the point B should have same direction as vector $(0,1)$ thus

$$\dot{u}_B \equiv 0 = \sqrt{1 - \frac{C_1^2}{f(u_B)}}.$$

From it follows that

$$f(u_B) = C_1^2$$

where C_1 is a real constant, see previous section. We verify if this result satisfies second equation from (6)

$$\dot{v}_B = \frac{C_1}{f(u_B)} = \frac{C_1}{C_1^2} = C_1.$$

Trivially, we see that the vector $(0, C_1)$ has the same direction as vector $(0,1)$. The situation is described in the Fig. 1.

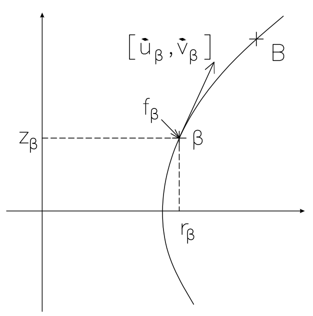

Figure 1. The graphical representation.

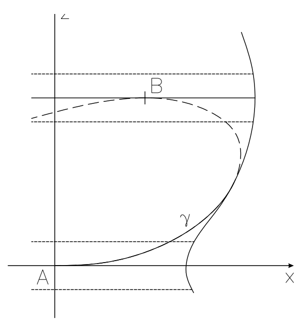

Figure 2. Global situation.

Finally, we have two points A and B of the geodesic γ and certain neighbourhood around these points where bifurcations exist. The rest of the geodesic is constructed as smooth curve that has to satisfy equations (5). The situation in global scale is demonstrated in the Fig. 2.

5. Conclusion

In the paper we constructed a surface of revolution where closed geodesics exist. In addition, these geodesic admit geodesic bifurcation in certain points. Bifurcation in this case means that through one point in given direction go more than one geodesic.

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