Eighteenth International Conference on
Geometry, Integrability and Quantization
June 03–08, 2016, Varna, Bulgaria
Ivaïlo M. Mladenov, Guowu Meng
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2017, pp 241–251
doi: 10.7546/giq-18-2017-241-251

Geometry, Integrability and Quantization XVIII

ON AUTO AND HETERO BÄCKLUND TRANSFORMATIONS FOR THE HÉNON-HEILES SYSTEMS

ANDREY V. TSIGANOV and ALEXEY P. SOZONOV

Department of Computational Physics, St.Petersburg State University
198504, St.Petersburg, Russia

Abstract. We consider a canonical transformation of parabolic coordinates on the plain associated with integrable Hénon-Heiles systems and suppose that this transformation together with some additional relations may be considered as a counterpart of the auto and hetero Bäcklund transformations.

MSC: 58J72, 70H06
Keywords: Auto and hetero Bäcklund transformations, Hénon-Heiles systems

1. Introduction

According classical definition by Darboux [5], a Bäcklund transformation between two given PDEs

$$E_1(u,x,t)=0 \quad \text{and} \quad E_2(v,y,\tau)=0$$

is a pair of relations

$$F_{1,2}(u,x,t,v,y,\tau)=0 \tag{1}$$

and some additional relations between (x,t) and (y,τ), which allow to get both equations $E_{1,2}$. The BT is called an auto-BT or a hetero-BT depending whether the two PDEs are the same or not. The hetero-BTs describe a correspondence between equations rather than a one-to-one mapping between their solutions [1, 12]. In the modern theory of partial differential equation Bäcklund transformations are seen also as a powerful tool in the discretization of PDEs [2].

A counterpart of the auto Bäcklund transformations for finite dimensional integrable systems can be seen as canonical transformation

$$(u, p_u) \to (v, p_v)\,, \qquad \{u_i, p_{u_j}\} = \{v_i, p_{v_j}\} = \delta_{ij}\,, \qquad i, j = 1, \ldots, n \qquad (2)$$

preserving the algebraic form of the Hamilton-Jacobi equations

$$H_i\left(u, \frac{\partial S}{\partial u}\right) = \alpha_i \qquad \text{and} \qquad H_i\left(v, \frac{\partial S}{\partial v}\right) = \alpha_i$$

associated with the Hamiltonians $H_1, \ldots, H_n$ [15].

The counterpart of the discretization for finite dimensional systems is also currently accepted: by viewing the new v-variables as the old u-variables, but computed at the next time step; then the Bäcklund transformation (2) defines an integrable symplectic map or discretization of the continuous model, see discussion in [8,17].

The counterpart of the hetero Bäcklund transformations for finite dimensional integrable systems has to be a canonical transformation (2), which has to relate two different systems of the Hamilton-Jacobi equations

$$H_i\left(u, \frac{\partial S}{\partial u}\right) = \alpha_i \qquad \text{and} \qquad \tilde{H}_i\left(v, \frac{\partial \tilde{S}}{\partial v}\right) = \tilde{\alpha}_i \qquad (3)$$

and has to satisfy some additional conditions. It is necessary to add these conditions to (2) and (3) in order to get non-trivial, useful theory. The question of how to do it, remains open.

One of the possible additional conditions may be found in the theory of superintegrable systems. For instance, let us consider integrals of motion for the two-dimensional harmonic oscillator

$$H_1 = p_x^2 + p_y^2 + a(x^2 + y^2)\,, \qquad H_2 = p_x^2 - p_y^2 + a(x^2 - y^2)$$

which yield the Hamilton-Jacobi equations separable in Cartesian coordinates $u = (x, y)$ on the plane. Another pair of Hamiltonians for the same harmonic oscillator

$$\tilde{H}_1 = p_x^2 + p_y^2 + a(x^2 + y^2)\,, \qquad \tilde{H}_2 = xp_y - yp_x$$

is separable in polar coordinates $v = (r, \varphi)$ on the plain. Canonical transformation of variables

$$(u, p_u) = (x, y, p_x, p_y) \to (v, p_v) = (r, \varphi, p_r, p_\varphi) \qquad (4)$$

defines a correspondence between two different systems of the Hamilton-Jacobi equations

$$H_{1,2}\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) = \alpha_{1,2} \qquad \text{and} \qquad \tilde{H}_{1,2}\left(r, \varphi, \frac{\partial \tilde{S}}{\partial r}, \frac{\partial \tilde{S}}{\partial \varphi}\right) = \tilde{\alpha}_{1,2}.$$

This correspondence may be considered as a hetero-BT defined by the generating function

$$F = p_x r\cos\varphi + p_y r\sin\varphi$$

relations between (x, y) and (r, φ)

$$x = r\cos\varphi\,, \qquad y = r\sin\varphi$$

and an additional condition that Hamilton function $H_1 = \tilde{H}_1$ is *simultaneously separable* in u and v variables.

Canonical transformation (4) can be considered as the semi hetero-BT relating different Hamilton-Jacobi equations, which are various faces of the same superintegrable system. We know that theory of such semi hetero-BTs is a profound and very useful theory, both in classical and quantum cases [10].

The main aim of this note is to discuss a correspondence between integrable Hénon-Heiles systems proposed in [13]. This correspondence between different Hamiltonians may be considered as a counterpart of the generic hetero-BTs relating different but *simultaneously separable* in v-variables Hamilton-Jacobi equations.

2. The Jacobi Method

Let us consider some natural Hamilton function on $T^*\mathbb{R}^n$

$$H = p_1^2 + \cdots + p_n^2 + V(q_1, \ldots, q_n). \tag{5}$$

The corresponding Hamilton-Jacobi is said to be separable in a set of canonical coordinates u_i if it has the additively separated complete integral

$$S(u_1, \ldots, u_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} S_i(u_i; \alpha_1, \ldots, \alpha_n)$$

where S_i are found by quadratures as solutions of ordinary differential equations. In order to express initial physical variables (q, p) in terms of canonical variables of separation (u, p_u) we have to obtain momenta p_{u_i} from the second Jacobi equations

$$p_{u_i} = \frac{\partial S_i(u_i; \alpha_1, \ldots, \alpha_n)}{\partial u_i}\,, \qquad i = 1, \ldots, n. \tag{6}$$

Solving these equations with respect to α_i one gets integrals of motion $H_i = \alpha_i$ as functions on variables of separation (u, p_u).

According to Jacobi we can use canonical transformation $(q, p) \to (u, p_u)$ in order to construct different integrable systems simultaneously separable in the same coordinates, see pp. 198-199 in [3]

"The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied."

For instance, let us consider some natural Hamiltonian on the plane

$$H_1 = p_1^2 + p_2^2 + V(q_1, q_2)$$

separable in the parabolic coordinates $u_{1,2}$

$$\lambda - 2q_2 - \frac{q_1^2}{\lambda} = \frac{(\lambda - u_1)(\lambda - u_2)}{\lambda}. \tag{7}$$

In this case second Jacobi equations (6) may be rewritten in the following form

$$p_{u_i}^2 + U_i(u_i) = H_1 + \frac{H_2}{u_i}, \qquad i = 1, 2 \tag{8}$$

where $U_i(u_i)$ are functions defined by the potential $V(q_1, q_2)$ [9].

Adding together the separated relations (8) one gets another integrable Hamiltonian

$$\tilde{H}_1 = \frac{1}{2}\Big(p_{u_1}^2 + U_1(u_1) + p_{u_2}^2 + U_2(u_2)\Big) = H_1 + \frac{H_2}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right) \tag{9}$$

which can be considered as an integrable perturbation of H (5) because there exists second independent integrals of motion

$$\tilde{H}_2 = \Big(p_{u_1}^2 + U_1(u_1) - p_{u_2}^2 - U_2(u_2)\Big) = H_2\left(\frac{1}{u_1} - \frac{1}{u_2}\right).$$

Of course, in generic case this perturbation has no physical meaning. Auto-BTs of the initial Hamilton-Jacobi equation

$$(u_i, p_{u_i}) \quad \to \quad (v_i, p_{v_i})$$

preserve an algebraic form of the initial Hamiltonian H (5) and change the form of the second Hamiltonians

$$\tilde{H}_1 = H_1 + \frac{H_2}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right), \qquad \tilde{H}_2 = H_2\left(\frac{1}{v_1} - \frac{1}{v_2}\right). \tag{10}$$

We can try to pick out a special and maybe unique auto-BT, which gives physical meaning to the second Hamiltonian $\tilde{H}$, as some of the possible counterparts of the hetero-BTs relating different but *simultaneously separable* Hamilton-Jacobi equations.

3. Hénon-Heiles Systems

There are three integrable Hénon-Heiles systems on the plane, which can be identified with appropriate finite-dimensional reductions of the integrable fifth order KdV, Kaup-Kupershmidt and Sawada-Kotera equations [6]. An explicit integration for all these cases is discussed in [4].

We can try to get a hetero-BT for the finite-dimensional Hénon-Heiles systems taking the hetero-BT between these integrable PDEs and then applying the Fordy finite-dimensional reduction [6]. We believe that the same information may be directly extracted from the well-known Lax representation for the Hénon-Heiles system separable in parabolic coordinates.

Let us take a Lax matrix for the first Hénon-Heiles system separable in parabolic coordinates

$$L(\lambda) = \begin{pmatrix} \dfrac{p_2}{2} + \dfrac{p_1 q_1}{2\lambda} & \lambda - 2q_2 - \dfrac{q_1^2}{\lambda} \\ a\lambda^2 + 2aq_2\lambda + a(q_1^2 + 4q_2^2) + \dfrac{p_1^2}{4\lambda} & -\dfrac{p_2}{2} - \dfrac{p_1 q_1}{2\lambda} \end{pmatrix}, \qquad a \in \mathbb{R}. \tag{11}$$

Characteristic polynomial of this matrix

$$\det\Big(L(\lambda) - \mu\Big) = \mu^2 - a\lambda^3 - \frac{H_1}{4} + \frac{H_2}{\lambda}$$

contains the Hamilton function of the first Hénon-Heiles system associated with a fifth order KdV

$$H_1 = p_1^2 + p_2^2 - 16aq_2(q_1^2 + 2q_2^2) \tag{12}$$

and a second integral of motion

$$H_2 = aq_1^2(q_1^2 + 4q_2^2) + \frac{p_1(q_2 p_1 - q_1 p_2)}{2}. \tag{13}$$

The auto-BTs preserve the algebraic form of the Hamiltonians [15]. Since the characteristic polynomial is the generating function of these integrals of motion, their invariance amounts to requiring the existence of a similarity transformation for the Lax matrix

$$\hat{L} = VLV^{-1}$$

associated with the given auto-BT. The matrix V needs not to be unique because a dynamical system can have different auto BTs [8, 17].

In contrast with [7] we do not require that the transformed Lax matrix $\hat{L}(\lambda)$ have the same structure in the spectral parameter λ as the original Lax matrix. This requirement is a property of the particular BTs associated with the special translations on hyperelliptic Jacobians.

Let us consider a special, unique similarity transformation associated with matrix

$$V = \begin{pmatrix} L_{12} & 0 \\ 4\big(L_{11} - \hat{L}_{11}(\lambda)\big) & 4L_{12} \end{pmatrix} \tag{14}$$

where L_{ij} are entries of the Lax matrix (11) and

$$\hat{L}_{11}(\lambda) = \frac{p_2}{2} + \frac{p_1(\lambda - 2q_2)}{2q_1}.$$

The Lax matrix $\hat{L}(\lambda) = VLV^{-1}$ has the following properties:

1. first off-diagonal element of the Lax matrix
$$\hat{L}_{12}(\lambda) = \frac{(\lambda - u_1)(\lambda - u_2)}{4\lambda}$$
yields initial parabolic coordinates on the plane (7);
2. second off-diagonal element
$$\hat{L}_{21} = 4a(\lambda - v_1)(\lambda - v_2) \tag{15}$$
$$= 4a\lambda^2 + \frac{(8aq_1^2 q_2 - p_1^2)\lambda}{q_1^2} + 4a(q_1^2 + 4q_2^2) + \frac{2p_1(p_1 q_2 - p_2 q_1)}{q_1^2}$$
has only two commuting and functionally independent zeroes $v_{1,2}$;
3. the conjugated momenta for u and v variables are the values of the diagonal element
$$p_{u_i} = \hat{L}_{11}(\lambda = u_i)\,, \qquad p_{v_i} = \hat{L}_{11}(\lambda = v_i)\,, \qquad i = 1, 2.$$

In generic case such 2×2 Lax matrices $\hat{L}(\lambda)$ exist only if the genus of hyperelliptic curve defined by equation

$$\det\Big(L(\lambda) - \mu\Big) = 0$$

is no more a number of degrees of freedom. The corresponding transformation of the classical r-matrix is discussed in [13].

In (u, p_u) and (v, p_v) variables entries of the transformed Lax matrix $\hat{L}$ have the following form:

$$\begin{aligned}
\hat{L}_{11} &= \frac{\lambda - u_2}{u_1 - u_2} p_{u_1} + \frac{\lambda - u_1}{u_2 - u_1} p_{u_2} = \frac{\lambda - v_2}{v_1 - v_2} p_{v_1} + \frac{\lambda - v_1}{v_2 - v_1} p_{v_2} \\
\hat{L}_{12} &= \frac{(\lambda - u_1)(\lambda - u_2)}{4\lambda} \\
&= \frac{\lambda^2 + \lambda(v_1 + v_2) + v_1^2 + v_1 v_2 + v_2^2}{4\lambda} - \frac{(p_{v_1} - p_{v_2})^2}{4a(v_1 - v_2)^2} \\
\hat{L}_{21} &= 4a\big(\lambda^2 + \lambda(u_1 + u_2) + u_1^2 + u_1 u_2 + u_2^2\big) - \frac{4\lambda(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^2} - \frac{4(p_{u_1}^2 - p_{u_2}^2)}{u_1 - u_2} \\
&= 4a(\lambda - v_1)(\lambda - v_2).
\end{aligned}$$

Thus, we have two families of variables of separation for the first Hénon-Heiles system and canonical transformation between them:

$$\begin{aligned} u_{1,2} &= -\frac{v_1+v_2}{2} + \frac{(p_{v_1}-p_{v_2})^2 \pm \sqrt{A}}{2a(v_1-v_2)^2} \\ p_{u_{1,2}} &= \frac{(p_{v_1}-p_{v_2})\big((p_{v_1}-p_{v_2})^2 \pm \sqrt{A}\big)}{2a(v_1-v_2)^3} - \frac{p_{v_1}(v_1+3v_2)-p_{v_2}(v_2+3v_1)}{2(v_1-v_2)} \end{aligned} \tag{16}$$

where

$$\begin{aligned} A = (p_{v_1}-p_{v_2})^4 + 2a(v_1-v_2)^2(p_{v_1}-p_{v_2})\big(p_{v_1}(v_1-3v_2)-p_{v_2}(v_2-3v_1)\big) \\ - a^2(3v_1^2+2v_1v_2+3v_2^2)(v_1-v_2)^4. \end{aligned}$$

We can directly prove that Hamiltonains $H_{1,2}$ (12-13) have the same algebraic form in (u, p_u) and (v, p_v) variables using this explicit canonical transformation.

Definition 1. *The auto-BT for the first Hénon-Heiles system is a correspondence between two equivalent systems of the Hamilton-Jacobi equations*

$$H_{1,2}\left(\lambda, \frac{\partial S}{\partial \lambda}\right) = \alpha_{1,2}\,, \qquad \lambda = u, v$$

where variables (u, p_u) and (v, p_v) are related by canonical transformation (16) and Hamiltonians $H_{1,2}$ are defined by the following equations

$$\Phi(\lambda,\mu) = \mu^2 - a\lambda^3 = \frac{H_1}{4} - \frac{H_2}{\lambda}\,, \qquad \lambda = u_{1,2}, v_{1,2}, \quad \mu = p_{u_{1,2}}, p_{v_{1,2}}. \tag{17}$$

This auto Bäcklund transformation changes coordinates on an algebraic invariant manifold defined by $H_{1,2}$ without changing the manifold itself [8].

We can convert this special, unique auto-BT to some analogue of the hetero-BT by adding one more relation. Namely, substituting roots of the off-diagonal element $\hat{L}_{2,1}$ (15) into the definition (10) one gets the Hamilton function for the second integrable Hénon-Heiles system associated with the Kaup-Kupershmidt equation

$$\tilde{H}_1 = p_1^2 + p_2^2 - 2aq_2(3q_1^2 + 16q_2^2) \tag{18}$$

up to rescaling $p_1 \to \sqrt{2}p_1$ and $q_1 \to q_1/\sqrt{2}$.

After canonical transformation

$$(q,p) \to (Q,P)\,, \qquad P_{1,2} = \frac{p_{v_1} \pm p_{v_2}}{\sqrt{2}}\,, \quad Q_{1,2} = \frac{v_1 \pm v_2}{\sqrt{2}} \tag{19}$$

the same Hamiltonian

$$\tilde{H}_1 = P_1^2 + P_2^2 - 2aQ_2(3Q_1^2 + Q_2^2) \tag{20}$$

defines a third integrable Hénon-Heiles system associated with the the Sawada-Kotera equation. According [6] canonical transformation (19) is a counterpart of the gauge equivalence of the Sawada-Kotera and Kaup-Kupershmidt equations.

So, all the Hénon-Heiles systems on the plane are *simultaneously separable* in v-variables, and we suppose that this fact allows us to define some natural counterpart of the hetero-BT.

Definition 2. *For the Hénon-Heiles systems on the plane (12) and (18,20) an analogue of the hetero-BT is the correspondence between two different systems of the Hamilton-Jacobi equations*

$$H_{1,2}\left(u, \frac{\partial S}{\partial u}\right) = \alpha_{1,2} \qquad \textit{and} \qquad \tilde{H}_{1,2}\left(v, \frac{\partial \tilde{S}}{\partial v}\right) = \tilde{\alpha}_{1,2}$$

where variables (u, p_u) and (v, p_v) are related by canonical transformation (16) and Hamiltonians are defined by the following equations

$$\Phi(\lambda, \mu) = \mu^2 - a\lambda^3 = \frac{H_1}{4} - \frac{H_2}{\lambda}, \qquad \lambda = u_{1,2}, v_{1,2}, \quad \mu = p_{u_{1,2}}, p_{v_{1,2}}$$

and

$$\tilde{H}_{1,2} = \Phi(v_1, p_{v_1}) \pm \Phi(v_2, p_{v_2}).$$

This analogue of the hetero-BT relates different algebraic invariant manifolds associated with Hamiltonians $H_{1,2}$ and $\tilde{H}_{1,2}$ similar to the well-studied relations between different invariant manifolds in the theory of superintegrable systems.

For the first Hénon-Heiles system on the plane (12) we can consider parabolic variables $(u_{1,2}, p_{u_{1,2}})$ as coordinates on the Jacobian variety defined by equations (17). In order to get integrals of motion for the second or third Hénon-Heiles systems (18,20) we have to take linear combinations of these equations and to make simultaneously the special shift of the coordinates $(u, p_u) \to (v, p_v)$ on the Jacobian variety.

4. Integrable Hamiltonian with Velocity Dependent Potential

It is well-known that Hamilton-Jacobi equation is separable in parabolic coordinates $u_{1,2}$ if the Hamilton function has the form

$$H = p_1^2 + p_2^2 + V_N(q_1, q_2), \qquad V_N = 4a \sum_{k=0}^{[N/2]} 2^{1-2k} \left(\begin{array}{c} N-k \\ k \end{array} \right) q_1^{2k} q_2^{N-2k}$$

where the positive integer N enumerates the members of the hierarchy.

At $N=3$ one gets the Hénon-Heiles system (12), at $N=4$ the next member of hierarchy is a "(1:12:16)" system with the following Hamiltonian

$$H = p_1^2 + p_2^2 - 4a\left(q_1^4 + 12q_1^2q_2^2 + 16q_2^4\right). \tag{21}$$

The corresponding Lax matrix is equal to

$$L(\lambda) = \begin{pmatrix} \dfrac{p_2}{2} + \dfrac{p_1q_1}{2\lambda} & \lambda - 2q_2 - \dfrac{q_1^2}{\lambda} \\ \begin{array}{c} a\lambda^3 + 2aq_2\lambda^2 + a(q_1^2+4q_2^2)\lambda + \\ 4aq_2(q_1^2+2q_2^2) + \dfrac{p_1^2}{4\lambda} \end{array} & -\dfrac{p_2}{2} - \dfrac{p_1q_1}{2\lambda} \end{pmatrix}. \tag{22}$$

After similarity transformation of $L(\lambda)$ with matrix V (14), where

$$\hat{L}_{11}(\lambda) = \sqrt{a}\,\lambda^2 - \frac{4\sqrt{a}q_2q_1 - p_1}{2q_1}\,\lambda - \frac{2\sqrt{a}q_1^3 + 2p_1q_2 - p_2q_1}{2q_1}$$

one gets the transformed Lax matrix with two off-diagonal elements $\hat{L}_{12}(\lambda)$ and $\hat{L}_{21}(\lambda)$, which yield two families of variables of separation.

As above first coordinates are parabolic coordinates $u_{1,2}$, whereas second coordinates $v_{1,2}$ are zeroes of the polynomial

$$\begin{aligned} \hat{L}_{21} &= \frac{4(4aq_1q_2 - \sqrt{a}p_1)}{q_1}\lambda^2 + \frac{8aq_1^2(q_1^2+2q_2^2) + 4\sqrt{a}q_1(2p_1q_2 - p_2q_1) - p_1^2}{q_1^2}\lambda \\ &\quad + \frac{16aq_1^2q_2(q_1^2+2q_2^2) + 2p_1(p_1q_2 - p_2q_1)}{q_1^2} \\ &= \frac{4(4aq_1q_2 - \sqrt{a}p_1)}{q_1}(\lambda - v_1)(\lambda - v_2). \end{aligned}$$

Substituting roots of this polynomial into the definition (10) one gets integrable Hamiltonian with velocity dependent potential

$$\tilde{H} = \frac{p_1^2}{2} + p_2^2 + 4\sqrt{a}p_1q_1q_2 - 2\sqrt{a}p_2q_1^2 - 8aq_2^2(5q_1^2 + 8q_2^2).$$

Using canonical transformation we can rewrite this Hamiltonian in a more symmetric form

$$\tilde{H} = p_1^2 + p_2^2 - 3\sqrt{a}p_2q_1^2 + 2a(q_1^4 - 12q_1^2q_2^2 - 32q_2^4). \tag{23}$$

The corresponding second integral of motion is fourth order polynomial in momenta

$$\begin{aligned} \tilde{H}_2 &= p_1^4 + 4q_1^4\left(q_1^4 - 8q_1^2q_2^2 - 112q_2^4\right)a^2 + 4q_1^3\left(64p_1q_2^3 - p_2q_1^3 - 12p_2q_1q_2^2\right)a^{3/2} \\ &\quad + q_1^2\left(4p_1^2q_1^2 - 48p_1^2q_2^2 + 32p_1p_2q_1q_2 + p_2^2q_1^2\right)a - 6a^{1/2}p_1^2p_2q_1^2 \end{aligned}$$

which also can be obtained from the Lax matrix $\hat{L}(\lambda)$. Of course, this integrable system on the plane (23) could be obtained in the framework of different theories, see [6, 11, 16] and references within.

Canonical transformation (19) allows us to identify a Hamilton function with velocity dependent potential (23) and Hamilton function

$$\hat{H} = P_1^2 + P_2^2 - a(Q_1^4 + 6Q_1^2Q_2^2 + Q_2^4)$$

similar to the relation between second and third Hénon-Heiles systems.

Canonical transformation $(u, p_u) \to (v, p_v)$ is the special auto-BT for the "(1:12:16)" system, which can be considered as a hetero-BT relating two different Hamilton-Jacobi equations associated with Hamiltonians H (21) and $\widetilde{H}$ (23), respectively.

5. Conlusion

The problem of finding separation coordinates for the Hamilton-Jacobi equations is highly non-trivial. The problem was originally formulated by Jacobi when he invented elliptic coordinates and successfully applied them to solve several important mechanical problems with quadratic integrals of motion in momenta.

We suppose that after suitable Bäcklund transformations standard elliptic, parabolic etc. coordinates turn into variables of separation for physically interesting integrable systems with higher order integrals of motion. For example, in this note we have constructed a canonical transformation of the standard parabolic coordinates, which yields variables of separation for the three integrable Hénon-Heiles systems.

Moreover, we believe that information about such suitable Bäcklund transformations and the corresponding integrable systems is incorporated into the Lax matrices associated with these elliptic, parabolic etc. coordinates. In order to prove it we obtained integrals of motion, variables of separation and separated relations for some new integrable system with velocity dependent potential and fourth order integral of motion in momenta. In similar manner we can construct various simultaneously separable integrable systems associated with other curvilinear coordinates on the Riemannian manifolds of constant curvature, see examples in [13, 14].

References

[1] Ablowitz M. and Segur H., *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia 1981.

[2] Bobenko A. and Suris Yu., *Discrete Differential Geometry: Integrable Structure*, Graduate Studies in Mathematics, **98** AMS, Providence (2008).

[3] Jacobi C., *Vorlesungen über Dynamik*, Georg Reimer, Berlin 1866.

[4] Conte R., Musette M. and Verhoeven C., *Explicit Integration of the Hénon-Heiles Hamiltonians*, J. Nonlinear Math. Phys. **12** (2005) 212–227.

[5] Darboux G., *Leçons Sur La Théorie Générale Des Surfaces Et Les Applications Géométriques Du Calcul Infinitésimal*, Vol. III, Gauthier-Villars, Paris 1894.

[6] Fordy A., *The Hénon-Heiles System Revisited*, Phys. D **52** (1991) 204–210.

[7] Kuznetsov V. and Sklyanin E., *On Bäcklund Transformations for Many-Body Systems*, J. Phys. A **31** (1998) 2241–225.

[8] Kuznetsov V. and Vanhaecke P., *Bäcklund Transformations for Finite-Dimensional Integrable Systems: a Geometric Approach*, J. Geom. Phys. **44** (2002) 1–40.

[9] Landau L.D., Lifshitz E.M., *Mechanics*, A Course of Theoretical Physics, Pergamon Press, 1969.

[10] Miller W., Post S. and Winternitz P., *Classical and Quantum Superintegrability with Applications*, J. Phys. A: Math. and Theor. **46** no. 42 (2013) 423001.

[11] Pucacco G., *On Integrable Hamiltonians with Velocity Dependent Potentials*, Celestial Mech. and Dyn. Astronomy **90** no. 1-2 (2004) 109–123.

[12] Rogers C. and Schief W., *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Volume 30, Cambridge University Press, Cambridge 2002.

[13] Tsiganov A., *Simultaneous Separation for the Neumann and Chaplygin Systems*, Reg. Chaotic Dyn. **20** (2015) 74–93,

[14] Tsiganov A., *On the Chaplygin System on the Sphere with Velocity Dependent Potential*, Journal of Geometry and Physics **92** (2015) 94–99.

[15] Wojciechowski S., *The Analogue of the Bäcklund Transformation for Integrable Many-Body Systems*, J. Phys. A: Math. Gen. **15** (1982) 653–657.

[16] Yehia H. and Hussein A., *New Families of Integrable Two-Dimensional Systems with Quartic Second Integrals*, arXiv 1308:1442v1.

[17] Zullo F., *Bäcklund Transformations and Hamiltonian Flows*, J. Phys. A: Math. and Theor. **46** (2013) 145203.