Eighteenth International Conference on
Geometry, Integrability and Quantization
June 03–08, 2016, Varna, Bulgaria
Ivaïlo M. Mladenov, Guowu Meng
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2017, pp 217–224
doi: 10.7546/giq-18-2017-217-224

ON GEODESIC BIFURCATIONS

LENKA RÝPAROVÁ and JOSEF MIKEŠ

Department of Algebra and Geometry, Palacký University
77146 Olomouc, Czech Republic

Abstract. In this paper we study fundamental equations of geodesics on surfaces of revolution. We obtain examples of existence of geodesic bifurcation.

MSC: 53A05, 53B21, 53C22
Keywords: Geodesic bifurcation, surfaces of revolution

1. Introduction

The theory of geodesics began by Bernoulli and Euler. Due to this problem the variation calculus has started. Geodesics are extremals of lenght on surfaces and also on Riemannian and pseudo-Riemannian manifolds. Detailed description of this problem can be found in [1–6, 9, 13–19].

Lagrange obtained new properties of geodesics on surfaces, e.g. he proved that ideal rubber (spring) on surface has a position of the geodesic, [8].

It is well-know that there exist one and only one geodesic going throught given point in given direction. This statement is valid for surfaces where Christoffel symbols are diferentiable. The proof follows from analysis of ordinary differential equations.

On the other hand, if the Christoffel symbols are continuous, then geodesics exist for above mentioned. We demonstrate an example of connections which components are not differentiable, but geodesics have common properties, that is there do not exist bifurcations.

Further, in this paper we obtained example of surface of revolution where exist more than one geodesic going through given point in given direction, i.e., a geodesic bifurcation exists.

In [20] the authors have studied the geodesic bifurcations from a different point of view.

Mappings of surfaces of revolution which preserved geodesics were studied in [7, 10–12, 19].

2. Geodesics

Let (M, ∇) be a manifold M with affine connection ∇. In local chart (U, x) the connection ∇ is defined with its components $\Gamma^h_{ij}(x)$.

The curve $\gamma(t)$ on M is called **geodesic** if its tangent vector $\gamma'(t)$ is reccurent along it, i.e., the vector $\nabla_t \gamma'$ is parallel to tangent vector γ', therefore

$$\nabla_t \gamma' = \rho(t)\, \gamma'$$

where $\rho(t)$ is a function of parameter t, see [4, 5, 9, 13–16, 18, 19].

Locally, on the geodesic γ exists *canonical* parameter s for which satisfy

$$\nabla_s \dot{\gamma} = 0. \tag{1}$$

In this case vector field $\dot{\gamma}(s)$ is parallel along γ.

In the chart (U, x) equation (1) has following the form

$$\ddot{x}^h(s) + \sum_{i,j=1}^{n} \Gamma^h_{ij}\big(x(s)\big)\, \dot{x}^i(s)\, \dot{x}^j(s) = 0 \tag{2}$$

where $x^h = x^h(s)$ are coordinates of geodesic γ in the chart (U, x).

Often, we call geodesic those curves that satisfy equations (2), see [2] and discussion [14, pp. 88–91].

These equations (2) can be written in the following form of ordinary differential equations of first order, respective uknown functions $x^h(s)$ and $\lambda^h(s)$

$$\dot{x}^h(s) = \lambda^h(s), \qquad \dot{\lambda}^h(s) = -\sum_{i,j=1}^{n} \Gamma^h_{ij}\big(x(s)\big)\, \dot{x}^i(s)\, \dot{x}^j(s). \tag{3}$$

Here $\lambda^h(s)$ is a tangent vector of $\gamma(s)$ at the point $x^h(s)$.

Initial conditions of equations (3)

$$x^h(s) = x^h_0, \quad \text{and} \quad \lambda^h(s) = \lambda^h_0 \tag{4}$$

satisfy, that geodesic goes through point x^h_0 in direction λ^h_0 $(\neq 0)$.

From general theory of ordinary differential equations follows that equations (3) with initial conditions (4) have solution if $\Gamma^h_{ij}(x)$ are continuous functions. If functions $\Gamma^h_{ij}(x)$ are differentiable, then solution of Cauchy problem (3) and (4) is *unique*. Moreover, last mentioned applies even if functions $\Gamma^h_{ij}(x)$ satisfy Lipschitz condition.

In the following exercises we analyze solvability of equations (3) and (4).

3. Example of Existence of Geodesic with Continuous Connection

We give an example of affine connection ∇, for which components Γ^h_{ij} are only continuous and only one geodesic passes through given point x^h_0 in given direction λ^h_0.

Let ∇ be an affine connection which is defined in coordinate neighbourhood (U, x) by following continuous components

$$\Gamma^h_{hh} = f^h(x^h), \qquad h = 1, \dots, n \tag{5}$$

the other components are vanishing. Here and after we do not use Einstein summation convention.

We verify, that geodesics in space with above mentioned connection ∇ have usual properties, i.e., only one geodesic goes through given point in given direction.

The system of equations (2) of geodesic $\gamma(s)$ has the following form

$$\ddot{x}^h(s) + f^h\big(x^h(s)\big) \cdot \big(\dot{x}^h(s)\big)^2 = 0, \qquad h = 1, \dots, n \tag{6}$$

where s is a canonical parametr of the geodesic.

First, we solve problem of existence of geodesic which goes through given point $x^h_0 \in U$ in given direction λ^h_0, i.e., we find solution of equations (6) for the Cauchy initial conditions (4).

We show that the equations (6) with Cauchy initial conditions (4) have only one solution.

Let us rewrite equations (6) into the following forms

$$\frac{\ddot{x}^h}{\dot{x}^h} = -f^h(x^h)\,\dot{x}^h$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\ln|\dot{x}^h|) = -f^h(x^h)\,\dot{x}^h$$

and

$$\ln|\dot{x}^h(s)| = -\int_{x^h_0}^{x^h(s)} f^h(\tau)\,\mathrm{d}\tau + \ln|C^h_0|.$$

Apparently,

$$\dot{x}^h(s) = C^h_0 \exp\left(-\int_{x^h_0}^{x^h(s)} f^h(\tau)\,\mathrm{d}\tau\right).$$

For $s = 0$, we obtain $C_0^h = \lambda_0^h$, and

$$\dot{x}^h(s) = \lambda_0^h \exp\left(-\int_{x_0^h}^{x^h(s)} f^h(\tau)\,\mathrm{d}\tau\right). \tag{7}$$

This solution is also acceptable for $\lambda_0^h = 0$, i.e., for trivial solutions.

Rewrite (7) in following form

$$\exp\left(\int_{x_0^h}^{x^h(s)} f^h(\tau)\,\mathrm{d}\tau\right)\dot{x}^h(s) = \lambda_0^h. \tag{8}$$

We integrate (8) with respect to s and get

$$\int_{x_0^h}^{x^h(s)} \exp\left(\int_{x_0^h}^{w} f^h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}w = \lambda_0^h\, s + {}^*C_0^h. \tag{9}$$

If $x^h(0) = x_0^h$ then from (9) follows ${}^*C_0^h = 0$.

Finally, we obtain an implicit equation of solution of the geodesic

$$\int_{x_0^h}^{x^h(s)} \exp\left(\int_{x_0^h}^{w} f^h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}w = \lambda_0^h\, s.$$

This solution is unique for the Cauchy problem (6) and (4). It is unique in the whole neighbourhood U and exists in certain neighbourhood of point x_0^h. Let us remind that $f^h(x)$ are not differentiable.

This statement coresponds with more general theory. From this follows that we can find torse and curvature tensors. Their components in coordinates (U, x) have following form

$$S_{ij}^h(x) = \Gamma_{ij}^h(x) - \Gamma_{ji}^h(x)$$

and

$$R_{ijk}^h = \partial_j \Gamma_{ik}^h - \partial_k \Gamma_{ij}^h + \sum_{\alpha=1}^{n} (\Gamma_{ik}^\alpha\, \Gamma_{\alpha j}^h - \Gamma_{ij}^\alpha\, \Gamma_{\alpha k}^h). \tag{10}$$

Elementary, for (5) we obtained that S and R are vanishing and space is locally affine.

In our case, all components of R exist even if functions $f^h(x^h)$ are not differentiable. That is because in formulas (10) appear derivatives $\partial_k f^h$, where $k \neq h$, and because functions f^h do not depend on x^k then $\partial_k f^h = 0$.

In locally affine manifold there exists affine coordinate system where components of affine connection are vanishing and from that follows linear form of geodesic equations. Property, that through one point in given direction goes one and the only one straight line, is basic.

4. Example of Geodesic Bifurcation

Bifurcation of geodesic is studied in [20]. There, bifurcation is described as situation where (different) geodesics go through one point and have different tangent vectors. We show bifurcation of geodesics on surfaces of revolution, where two different geodesics go through the *same* point and have the *same* tangent vector.

Let $\mathcal{S}_2$ be a surface of revolution given by the equations

$$x = r(u)\cos v, \quad y = r(u)\sin v, \quad z = z(u) \tag{11}$$

where v is a parameter from $(-\pi, \pi)$, $u \in I \subset \mathbb{R}$ and $I = \langle u_1, u_2 \rangle$.

In these equations we exclude meridian corresponding to coordinate $v = \pi$. Naturally, we also exclude “poles" which correspond to $r(u) = 0$.

Rotational surface $\mathcal{S}$ given by equations (11) has following metric form

$$\mathrm{d}s^2 = \left({r'}^2(u) + {z'}^2(u)\right)\mathrm{d}u^2 + {r'}^2(u)\,\mathrm{d}v^2.$$

Let us choose parameter u as a lenght parameter of forming curve $(r(u), 0, z(u))$ then ${r'}^2(u) + {z'}^2(u) = 1$. In this case, the metric on the surface $\mathcal{S}$ is simplified to

$$\mathrm{d}s^2 = \mathrm{d}u^2 + r^2(u)\,\mathrm{d}v^2.$$

If we introduce the function $f(u) \equiv r^2(u)$, then the metric get the following form

$$\mathrm{d}s^2 = \mathrm{d}u^2 + f(u)\,\mathrm{d}v^2 \tag{12}$$

i.e., nonzero components of metric and inverse tensors are

$$g_{11} = g^{11} = 1 \qquad \text{and} \qquad g_{22} = \left(g^{22}\right)^{-1} = f(u).$$

Non-vanishing Christoffel symbols of first kind are

$$\Gamma_{122} = \Gamma_{212} = \frac{1}{2}\,f'(u) \qquad \text{and} \qquad \Gamma_{221} = -\frac{1}{2}\,f'(u)$$

and nonzero Christoffel symbols of second kind are

$$\Gamma^1_{22} = -\frac{1}{2}\,f'(u) \qquad \text{and} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\,\frac{f'(u)}{f(u)}.$$

Further, let $u \equiv x^1$ and $v \equiv x^2$. The equations (2) of geodesics on surface $\mathcal{S}$ can be written in following form

$$\ddot{u} = \tfrac{1}{2}\,f'(u)\,\dot{v}^2, \qquad \ddot{v} = -\frac{f'(u)}{f(u)}\,\dot{u}\,\dot{v}. \tag{13}$$

Because s is parameter of lenght, tangent vector of these geodesics is unitary, i.e., first integral applies

$$\dot{u}^2 + f(u)\,\dot{v}^2 = 1. \tag{14}$$

Trivially, we verify that u-coordinate curves ($u = s$, $v = \text{const}$, i.e., "meridian") are geodesic. In general, same does not apply for the v-coordinates, v-curves are geodesic if and only if $f'(u) = 0$ (they are also called *gorge circles*).

Further, let us study geodesics, which are none of the mentioned above. Suppose that $v(s) \neq 0$, i.e., $\dot{v}(s) \neq 0$. Then we can rewrite second equation of (13) in form

$$\frac{\ddot{v}}{\dot{v}} = -\frac{f'(u)}{f(u)}\,\dot{u}.$$

After modification and integration by s we get

$$\dot{v} = \frac{C_1}{f(u)}, \quad C_1 \in \mathbb{R} \tag{15}$$

By using (15), from (14) we get $\dot{u}^2 + f(u)\,\dfrac{{C_1}^2}{f^2(u)} = 1$ therefore

$$\dot{u} = \sqrt{1 - \frac{{C_1}^2}{f(u)}}. \tag{16}$$

Finally the equations (15) and (16) determine system of differential equations of first order.

Now we construct example of rotational surface $\mathcal{S}$, where above mentioned bifurcation exists.

Example

Let $\mathcal{S}$ be a surface of revolution with functions

$$r(u) = \frac{1}{\sqrt{1 - u^{2\alpha}}} \quad \left(\Rightarrow f(u) = \frac{1}{1 - u^{2\alpha}}\right), \quad u \in (-1, 1).$$

The function r must to be differentiable so the Christoffel symbols exist and equations of geodesics can be written. On the other hand, the Christoffel symbols can not satisfy the Lipschitz condition and, of course, can not be differentiable (there would be an unique solution and bifurcation would not exist).

Theorem 1. *On above mentioned surface of revolution $\mathcal{S}$ exist geodesic bifurcations for $\alpha \in (0, 1)$.*

Proof: The statement can be proved by existence of geodesics given by the equations:

$$\begin{aligned} &I. \qquad u = 0, \quad v = s \\ &II. \quad\ u = \big((1-\alpha)\,s\big)^{\frac{1}{1-\alpha}}, \quad v = s - \frac{\big((1-\alpha)\,s\big)^{\frac{1+\alpha}{1-\alpha}}}{1+\alpha}. \end{aligned} \tag{17}$$

We can verify that curves given by the equations (16) are geodesics by direct substitution to fundamental equations (13).

These two geodesics go through same point $(0, 0)$ and have same tangent vector $(0, 1)$. The consequence is that through this point in this direction goes infinite number of geodesics and the gorge circle (mentioned above) is one of them. ■

Remark 2. If we set

$$f(u) = -\frac{1}{1 - u^{2\alpha}}$$

then metric will be indefinite and the equations (17) describe a geodesic bifurcation on a pseudo-Riemannian space.

Because spaces with the metric form (12) admit a nontrivial geodesic mappings [12–14, 19], then projective corresponding spaces preserve above mentioned geodesic bifurcations.

Acknowledgements

The paper was supported by the project IGA PrF 2014016 Palacký University Olomouc.

References

[1] Berger M., *Lectures on Geodesics in Riemannian Geometry*, Bombay 1965.

[2] Boeckx E., Vanhecke L. and Kowalski O. *Riemannian Manifolds of Conullity Two*, World Scientific, Singapore 1996.

[3] Busemann H., *The Geometry of Geodesics*, Academic Press, New York 1955.

[4] Eisenhart L., *Riemannian Geometry*, Princeton Univ. Press, Princeton 1926.

[5] Eisenhart L., *Non-Riemannian Geometry*, Princeton Univ. Press, Princeton 1927.

[6] Fomenko A. and Mishchenko A., *A Short Course in Differential Geometry and Topology*, Cambridge Sci. Publ., Cambridge 2009.

[7] Hinterleitner I., *On Global Geodesic Mappings of Ellipsoids*, AIP Conf. Proc. **1460** (2012) 180–184.

[8] Lagrange J., Legendre A.-M. and Jacobi C., *Treatises on the Calculus of Variations*, Part II, W. Engelmann, Leipzig 1894.

[9] Levi-Civita T., *Sulle Transformationi Delle Equazioni Dinamiche*, Ann. Mat. Milano **24** (1886) 255–300.

[10] Mikeš J., *On the Existence of n-Dimensional Compact Riemannian Spaces Admitting Nontrivial Global Projective Transformations*, Sov. Math. Dokl. **39** (1989) 315–317.

[11] Mikeš J., *On Existence of Nontrivial Global Geodesic Mappings on n-Dimensional Compact Surfaces of Revolution*, In: Diff. Geometry and its Appl. Int. Conf., Brno, Czechosl. 1989, World Sci., Singapore 1990, pp 129–137.

[12] Mikeš J., *Geodesic Mappings of Affine-Connected and Riemannian Spaces*, J. Math. Scientific **78** (1996) 311–333.

[13] Mikeš J., Vanžurová A. and Hinterleitner I., *Geodesic Mappings and Some Generalizations*, Palacky Univ. Press, Olomouc 2009.

[14] Mikeš J. et al, *Differential Geometry of Special Mappings*, Palacky Univ. Press, Olomouc 2015.

[15] Norden A., *Spaces with an Affine Connection*, Nauka, Moscow 1976.

[16] Petrov A. *Einstein Spaces*, Pergamon Press, New York 1969.

[17] Postnikov M., *The Variational Theory of Geodesics*, Philadelphia 1967.

[18] Rashevskii P. *Riemannian Geometry and Tensor Analysis*, Nauka, Moscow 1964.

[19] Sinyukov N., *Geodesic Mappings of Riemannian Spaces*, Nauka, Moscow 1979.

[20] Thielhelm H., Vais A. and Wolter F.-E., *Geodesic Bifurcation on Smooth Surfaces*, The Visual Computer **31** (2015) 187–204.