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COMPLEX AND REAL HYPERSURFACES OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

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Abstract. We studied complex and real hypersurfaces immersed in locally conformal Kähler manifolds. We have obtained conditions for the immersions, if the manifolds admit existence of such complex hypersurfaces that are orthogonal to both Lee and anti-Lee vector fields. Also we explore real hypersurfaces immersed in LCK-manifolds.

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1. Introduction

Differential geometric aspects of submanifolds of manifolds with certain structures are very fruitful fields for Riemannian geometry. Study of complex submanifolds immersed in locally conformal Kähler manifolds(for brevity, LCK-manifolds) was initiated by Vaisman in [12], and more attention was paid to the so called Generalized Hopf manifolds. Further development was made in [4]. Real hypersurfaces of LCK-manifolds was explored in [2]. We continue to study the immersions of submanifolds that a tangent space in all points of the submanifolds to be normal to Lee field.

2. Preliminaries

A Hermitian manifold (M^{2m}, J, g) is called a *locally conformal Kähler manifold* (*LCK-manifold*) if there is an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ of M^{2m} and a family

$\{\sigma_\alpha\}_{\alpha\in A}$ of C^∞ functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ so that each local metric

$$\hat{g}_\alpha = \mathrm{e}^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian. An LCK-manifold is endowed with some form ω, so called a *Lee form* which can be calculated as [1]

$$\omega = \frac{1}{m-1}\,\delta\Omega \circ J \qquad \text{or} \qquad \omega_i = -\frac{2}{n-2}\,J^\alpha_{\beta,\alpha} J^\beta_i .$$

The form should be closed

$$\mathrm{d}\omega = 0.$$

One can compute covariant derivative an almost complex structure with respect of the Levi-Civita connection of (M^{2m}, J, g) using the formulae

$$J^k_{i,j} = \frac{1}{2}\big(\delta^k_j J^\alpha_i \omega_\alpha - \omega^k J_{ij} - J^k_j \omega_i + J^k_\alpha \omega^\alpha g_{ij}\big). \tag{1}$$

Let (M^{2m}, J, g) be a complex m-dimensional Hermitian manifold, g is it's Hermitian metric, J is it's complex structure. Consider an immersion of a m-dimensional manifold $\bar{M}^k$ in M^{2m}

$$\Psi : \bar{M}^k \longrightarrow M^{2m}.$$

Let ∇ and $\bar{\nabla}$ be operators of covariant differentiations on M^{2m} and $\bar{M}^k$, respectively. Then the Gauss and Weingarten formulas are given by [1, p. 148]

$$\nabla_X Y = \bar{\nabla}_X Y + h(X, Y) \tag{2}$$

$$\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{3}$$

respectively, where X and Y are vector fields tangent to $\bar{M}^k$ and ξ normal to $\bar{M}^k$. As usual $h(X, Y)$ denotes the *second fundamental form*, $\nabla^\perp$ the linear connection induced in the normal bundle $E(\Psi)$ called the *normal connection*, and A_ξ is the *second fundamental tensor* at ξ.

Conditions for the integrability of (2) and (3), the so called equations of *Gauss*, *Codazzi*, and *Ricci* [1, p. 150] are given respectively in the explicit form by

$$\begin{aligned}
g(R(X,Y)Z,W) &= \bar{g}(\bar{R}(X,Y)Z,W) \\
&\quad - g(h(X,W),h(Y,Z)) + g(h(Y,W),h(X,Z)) \\
\{R(X,Y)Z\} &= (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \\
g(R(X,Y)\xi,\eta) &= g(R^\perp(X,Y)\xi,\eta) - g([A_\xi, A_\eta]X, Y).
\end{aligned}$$

We call $\bar{M}^k$ a *CR-submanifold* of (M^{2m}, J, g) if $\bar{M}^k$ carries a C^∞ distribution D so that

1. D is *holomorphic* (i.e., $J_x(D_x) = D_x$) for any $x \in \bar{M}^k$
2. the orthogonal complement $D^\perp$ with respect to $\bar{g} = \Psi^* g$ of D in $T(\bar{M}^k)$ is *anti-invariant* (i.e., $J_x({D^\perp}_x) \subseteq E(\Psi)_x$) for any $x \in \bar{M}^k$) [1, p. 153].

Let $(\bar{M}^k, D)$ be a $CR-$submanifold of the Hermitian manifold M_0^{2m}. Set $p = \dim_{\mathbb{C}} D_x$ and $p = \dim_{\mathbb{R}} D_x^{\perp}$, for any $x \in \bar{M}^k$ so that $2p + q = k$. If $q = 0$ then $\bar{M}^k$ is a *complex submanifold*, i.e., it is a complex manifold and Ψ is a holomorphic immersion. If $p = 0$ then $\bar{M}^k$ is an *anti-invariant submanifold* (i.e., $J_x(T_x(\bar{M}^k)) \subseteq E(\Psi)_x$ for any $x \in \bar{M}^k$). A $CR-$submanifold $(\bar{M}^k, D)$ is *proper* if $p \neq 0$ and $q \neq 0$. Also $(\bar{M}^k, D)$ is *generic* if $q = 2m - k$ (i.e., $J_x(T_x(\bar{M}^k)) = E(\Psi)_x$ for any $x \in \bar{M}^k$). A submanifold $\bar{M}^k$ of the complex manifold (M^{2m}, J) is *totally real* if

$$T_x(\bar{M}^k) \cap J_x(T_x(\bar{M}^k)) = \{0\}$$

for any $x \in \bar{M}^k$.

3. Complex Hypersurfaces of LCK-Manifolds

Let submanifold $\bar{M}^k$ is immersed in LCK-manifold M^{2m}

$$\Psi : \ \bar{M}^k \longrightarrow M^{2m}$$

so that $k = 2p$ and for any $x \in \bar{M}^{2p}$

$$J_x(T_x(\bar{M}^{2p})) = T_x(\bar{M}^{2p}). \tag{4}$$

Let $\bar{M}^{2p}$ be represented by

$$x^\alpha = x^\alpha(y^1, \ldots, y^{2p}) \tag{5}$$

where $\alpha = 1, \ldots, 2m$ and y^i, $i = 1, \ldots, 2p$, are local coordinate systems respectively on M^{2m} and on $\bar{M}^{2p}$. Then the tangent subspace of $\bar{M}^{2p}$ at each point $x = x(y)$ is spanned by vectors

$$B_i^\alpha = \partial_i x^\alpha. \tag{6}$$

If a tensor $g_{\alpha\beta}$ is a Riemannian metric of M^{2m} then induced metric of $\bar{M}^{2p}$ take the form

$$\bar{g}_{ij} = B_i^\alpha g_{\alpha\beta} B_j^\beta. \tag{7}$$

We can define a tensor

$$B_\alpha^i = B_j^\beta \bar{g}^{ij} g_{\alpha\beta} \tag{8}$$

where $\bar{g}^{ij}$ is a tensor whose matrix is inverse to the matrix of the induced metric tensor $\bar{g}_{ij}$. Then

$$B_\alpha^i B_j^\alpha = \delta_j^i \tag{9}$$

whereas the operator

$$P_\alpha^\beta = B_\alpha^i B_i^\beta \tag{10}$$

defined on $T_x(M^{2m})$, is a projector on the tangent space $T_x(\bar{M}^{2p})$. Introducing

$$\bar{J}_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta \tag{11}$$

and taking into account (10) and (11) we obtain

$$\bar{J}_i^j \bar{J}_j^k = -\delta_j^k.$$

This means that the affinor $\bar{J}_i^j$, defined on the bundle $T(\bar{M}^{2p})$ is an almost complex structure on the manifold $\bar{M}^{2p}$. On account of (7) and (11) we get also

$$\bar{J}_k^j \bar{g}_{ij} \bar{J}_l^i = \bar{g}_{ij}$$

and on this basis to conclude that the metric (7) is Hermitian. One can calculate the Christoffel symbols for connection $\bar{\nabla}$ with respect to metric $\bar{g}_{ij}$ by formulas

$$\bar{\Gamma}_{jk}^h = B_\alpha^h \big(B_j^\beta B_k^\gamma \Gamma_{\beta\gamma}^\alpha + \partial_j B_k^\alpha \big).$$

In the local coordinates (5) the Gauss and Weingarten formulas (2) and (3) can be written as

$$\bar{\nabla}_j B_i^\alpha = H_{ij(x)} C_{(x)}^\alpha, \qquad \bar{\nabla}_j C_{(x)}^\alpha = -H_{j(x)}^i B_i^\alpha + L_{j(xy)} C_{(y)}^\alpha. \tag{12}$$

Here $H_{ij(x)}$ are the second fundamental tensors of $\bar{M}^{2p}$ with respect to mutually orthogonal unit normals $C_{(x)}^\alpha$ to $\bar{M}^{2p}$, $x = 2p+1, \ldots, 2m$, and $L_{j(xy)} = \big(\bar{\nabla}_j C_{(x)}^\alpha\big) C_{(y)}^\beta g_{\alpha\beta}$.

The covariant derivative of the almost complex structure (11) with respect to the connection $\bar{\nabla}_x$ is

$$\bar{\nabla}_k \bar{J}_i^j = \big(\bar{\nabla}_k J_\beta^\alpha\big) B_\alpha^j B_i^\beta + J_\beta^\alpha \big(\bar{\nabla}_k B_\alpha^j\big) B_i^\beta + J_\beta^\alpha B_\alpha^j \bar{\nabla}_k B_i^\beta.$$

On account of (12), we obtain

$$\bar{\nabla}_k \bar{J}_i^j = B_k^\gamma \big(\nabla_\gamma J_\beta^\alpha\big) B_\alpha^j B_i^\beta + J_\beta^\alpha H_{k(x)}^j C_{\alpha(x)} B_i^\beta + J_\beta^\alpha B_\alpha^j H_{ki(x)} C_{(x)}^\beta.$$

Since immersion of the manifold $\bar{M}^{2p}$ is the complex one hence (4) and the second and third items in left-hand side are equal to zero. Finally we obtain

$$\bar{\nabla}_k \bar{J}_i^j = B_k^\gamma B_\alpha^j B_i^\beta \nabla_\gamma J_\beta^\alpha. \tag{13}$$

Let us calculate the Nijenhuis tensor for the manifold $\bar{M}^{2p}$

$$\bar{N}_{ij}^h = \bar{J}_i^t (\bar{\nabla}_t \bar{J}_j^h - \bar{\nabla}_j \bar{J}_t^h) - \bar{J}_j^t (\bar{\nabla}_t J_i^h - \bar{\nabla}_i \bar{J}_t^h).$$

Substituting (11) and (13) into above equations we have

$$\begin{aligned} \bar{N}_{ij}^h &= J_\beta^\alpha B_\alpha^t B_i^\beta (B_t^\gamma B_\delta^h B_j^\lambda \nabla_\gamma J_\lambda^\delta - B_j^\lambda B_\delta^h B_t^\gamma \nabla_\lambda J_\gamma^\delta) \\ &\quad - J_\lambda^\alpha B_\alpha^t B_j^\lambda (B_t^\gamma B_\delta^h B_i^\beta \nabla_\gamma J_\beta^\delta - B_i^\beta B_\delta^h B_t^\gamma \nabla_\beta J_\gamma^\delta) \\ &= B_i^\beta B_j^\lambda B_\delta^h B_\alpha^t B_t^\gamma \big(J_\beta^\alpha (\nabla_\gamma J_\lambda^\delta - \nabla_\lambda J_\gamma^\delta) - J_\lambda^\alpha (\nabla_\gamma J_\beta^\delta - \nabla_\beta J_\gamma^\delta) \big). \end{aligned}$$

It is known, that the operator defined in (10) is a projector on the tangent space $T_x(\bar{M}^{2p})$. Then we have

$$\bar{N}_{ij}^h = B_i^\beta B_j^\lambda B_\delta^h \big(J_\beta^\alpha (\nabla_\alpha J_\lambda^\delta - \nabla_\lambda J_\alpha^\delta) - J_\lambda^\alpha (\nabla_\alpha J_\beta^\delta - \nabla_\beta J_\alpha^\delta) \big)$$

or

$$\bar{N}_{ij}^h = B_i^\beta B_j^\lambda B_\delta^h N_{\beta\lambda}^\delta.$$

Here $N_{\beta\lambda}^\delta$ is the Nijenhuis tensor for the manifold M^{2m}. Since M^{2m} is Hermitian $N_{\beta\lambda}^\delta \equiv 0$. Hence $\bar{M}^{2p}$ is Hermitian too.

Let us take into account that the M^{2m} is an LCK-manifold. Let us substitute (1) into (13) to obtain

$$\begin{aligned} \bar{\nabla}_k \bar{J}_i^j &= B_k^\gamma B_\alpha^j B_i^\beta \nabla_\gamma J_\beta^\alpha \\ &= B_k^\gamma B_\alpha^j B_i^\beta \frac{1}{2}\big(\delta_\gamma^\alpha J_\beta^\delta \omega_\delta - \omega^\alpha J_{\beta\gamma} - J_\gamma^\alpha \omega_\beta + J_\delta^\alpha \omega^\delta g_{\beta\gamma}\big). \end{aligned} \tag{14}$$

By virtue of the definitions(6) – (11) and using (14) we obtain

$$\bar{\nabla}_k \bar{J}_i^j = \frac{1}{2}\big(\delta_k^j \bar{J}_i^p \bar{\omega}_p - \bar{\omega}^j \bar{J}_{ik} - \bar{J}_k^j \bar{\omega}_i + \bar{J}_p^j \bar{\omega}^p \bar{g}_{ik}\big) \tag{15}$$

where $\bar{\omega}_i = B_i^\gamma \omega_\gamma$. Hence, from (15) and according to

$$\mathrm{d}\Psi^*\omega = \Psi^*\mathrm{d}\omega = 0$$

the theorem below follows.

Theorem 1. *If a complex submanifold $\bar{M}^{2p}$ is immersed in a LCK-manifold M^{2m} then the immersed manifold $\bar{M}^{2p}$ is a LCK-manifold. Moreover if the Lee field $B = \omega^\#$ defined in M^{2m} is normal to $\bar{M}^{2p}$, then the immersed $\bar{M}^{2p}$ is a Kähler one.*

Similar results were presented in [4] but it is important to explore the immersions also with regard to the position of $\Psi(\bar{M}^{2p})$ with respect to the Lee field of M^{2m}. There are limitations. For instance, we have [9]

Theorem 2. *Let M^k be k-dimensional $(k \geq 2)$ CR-submanifold of a Vaisman manifold M^{2m}. If the anti-Lee field $A = -JB = -J\omega^\#$ is normal to M^k then M^k is an anti-invariant submanifold of M^{2m} $(k \leq m)$. Consequently, a Vaisman manifold admits no proper CR-submanifolds so that $\bar{A} = \Psi_* A = 0$. In particular, there are no proper CR-submanifolds of a Vaisman manifold with $B \in D^\perp$. Also, there are no complex submanifolds of a Vaisman manifold normal to the Lee field $B = \omega^\#$.*

We are concerned with finding conditions under which LCK-manifold M^{2m} admit immersion of complex submanifolds. Then we obtain the following theorem.

Theorem 3. *The LCK-manifold M^{2m} admit immersion of complex hypersurface $\bar{M}^{2m-2}$ so that the Lee field $B = \omega^\#$ and the anti-Lee field $A = -JB = -J\omega^\#$*

are normal to the hypersurface $\bar{M}^{2m-2}$ if and only if the Lee form of M^{2m} satisfies the condition

$$\Phi_4(\nabla_X \omega(Y)) = \frac{\|\omega\|^2}{2} g(X,Y).$$

Here Φ_4 is the fourth Obata's projector

$$\Phi_4(\omega_{i,j}) = \frac{1}{2}(\delta_i^a \delta_j^b + J_i^a J_j^b)\omega_{a,b}.$$

Proof: *Necessity.* Let us consider an LCK-manifold M^{2m}. Let $\theta = \omega \circ J$ and $A = -JB$ be respectively the anti-Lee form and the anti-Lee vector field. Then, we can rewrite (1) as

$$\nabla_X(J)Y = \frac{1}{2}\big(\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\big)$$

and hence we get

$$\nabla_X A = -J\nabla B + \frac{1}{2}\big(\|\omega\|^2 JX + \omega(X) - \theta(X)B\big)$$

for any $X \in T(M^{2m})$. Let M^{2m-2} be a complex hypersurface of an M^{2m}. If $B \in E(\Psi)$, then $A \in E(\Psi)$ since the immersion is analytic one. Moreover, if $X, Y \in T(M^{2m-2})$, then $[X,Y] \in T(M^{2m-2})$ according to the classical Frobenius theorem. Hence

$$\begin{aligned} 0 = g([X,Y],A) &= g(\nabla_X Y, A) - g(\nabla_Y X, A) \\ &= -g(Y, \nabla_X A) + g(X, \nabla_Y A) \\ &= g(Y, J\nabla_X B) - g(X, J\nabla_Y B) + \|\omega\|^2 \Omega(X,Y). \end{aligned} \tag{16}$$

Rewriting (16) in the local coordinates, we obtain

$$\omega_{t,j} J_i^t - \omega_{t,i} J_j^t - \|\omega\|^2 J_{ij} = 0. \tag{17}$$

Next, by multiplying equation (17) with J_k^j we get

$$\omega_{t,j} J_i^t J_k^j + \omega_{k,i} - \|\omega\|^2 g_{ik} = 0. \tag{18}$$

We can rewrite finally (18) as

$$2\Phi_4(\omega_{i,j}) - \|\omega\|^2 g_{ij} = 0$$

where Φ_4 is the fourth Obata's projector [5]. For instance, applying the operator to the tensor Q_{ij}^h means

$$\Phi_4(Q_{ij}^h) = \frac{1}{2}(\delta_i^a \delta_j^b + J_i^a J_j^b) Q_{ab}^h.$$

Hence

$$\Phi_4(\omega_{i,j}) = \frac{\|\omega\|^2}{2} g_{ij}. \tag{19}$$

Sufficiency. The tangent bundle $T(M^{2m})$ should satisfy the system since the bundle is normal to both Lee field B and anti-Lee field A, i.e.,

$$\omega = 0, \qquad \theta = 0. \tag{20}$$

According to the Frobenius theorem the system (20) is completely integrable if and only if both the Lee-form and the anti-Lee form identically satisfy the conditions

$$\begin{array}{ll} 1) & \mathrm{d}\omega \wedge \omega \wedge \theta = 0 \\ 2) & \mathrm{d}\theta \wedge \omega \wedge \theta = 0. \end{array} \tag{21}$$

Identity (21_1) is satisfied since M^{2m} is LCK-manifold, hence $\mathrm{d}\omega = 0$. We have to explore (21_2). Let us take the exterior differential of the anti-Lee form $\theta = \omega \circ J$ [14, p. 6]

$$\mathrm{d}\theta = \frac{1}{2}\big(\nabla_k(\omega_i J^i_j) - \nabla_j(\omega_i J^i_k)\big)\mathrm{d}x^k \wedge \mathrm{d}x^j$$
$$= \frac{1}{2}\big(\omega_{i,k} J^i_j + \omega_i J^i_{j,k} - \omega_{i,j} J^i_k - \omega_i J^i_{k,j}\big)\mathrm{d}x^k \wedge \mathrm{d}x^j.$$

According to (1) we obtain

$$\begin{aligned} \mathrm{d}\theta &= \big(\omega_{i,k} J^i_j + \omega_i J^i_{j,k}\big)\mathrm{d}x^k \wedge \mathrm{d}x^j \\ &= \big(\omega_{i,k} J^i_j + \frac{1}{2}\omega_k J^t_j \omega_t - \frac{1}{2}\|\omega\|^2 J_{jk} - \frac{1}{2}\omega_t J^t_k \omega_j\big)\mathrm{d}x^k \wedge \mathrm{d}x^j \\ &= \frac{1}{2}\big(\omega_{i,k} J^i_j - \omega_{i,j} J^i_k - \|\omega\|^2 J_{jk} + \omega_k J^t_j \omega_t - \omega_t J^t_k \omega_j\big)\mathrm{d}x^k \wedge \mathrm{d}x^j. \end{aligned}$$

We have also

$$\begin{aligned} \mathrm{d}\theta \wedge \omega \wedge \theta &= \frac{1}{2}\big(\omega_{i,k} J^i_j - \omega_{i,j} J^i_k - \|\omega\|^2 J_{jk} \\ &\quad + \omega_k J^t_j \omega_t - \omega_t J^t_k \omega_j\big)\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \omega_l dx^l \wedge \omega_s J^s_h \mathrm{d}x^h \\ &= \frac{1}{2}\big(\omega_{i,k} J^i_j - \omega_{i,j} J^i_k - \|\omega\|^2 J_{jk}\big)\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \omega_l \mathrm{d}x^l \wedge \omega_s J^s_h \mathrm{d}x^h \end{aligned} \tag{22}$$

since the equation

$$\frac{1}{2}\big(\omega_k J^t_j \omega_t - \omega_t J^t_k \omega_j\big)\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \omega_l \mathrm{d}x^l \wedge \omega_s J^s_h \mathrm{d}x^h = 0$$

is identically satisfied. Hence the equation

$$\omega_{i,k} J^i_j - \omega_{i,j} J^i_k - \|\omega\|^2 J_{jk} = 0$$

gives us a sufficient condition that the identity $\mathrm{d}\theta \wedge \omega \wedge \theta = 0$ is satisfied. This condition coincides with (17) which is equivalent to (19). Hence (21_2) is satisfied too. Sufficiency is proved as well. ■

4. Real Hypersurfaces of LCK-Manifolds

Let us recall some necessary definitions. Let M^{2m-1} be a $2m-1$-dimensional manifold and f, ξ, η be a tensor field of type $(1,1)$, a vector field and one-form on M^{2m-1} respectively. If f, ξ and η satisfy the conditions

$$1)\quad \eta(\xi) = 1, \qquad 2)\quad f^2X = -X + \eta(X) \tag{23}$$

for any vector field $X \in \mathfrak{X}(M^{2m-1})$, then M^{2m-1} is said to have an *almost contact structure* (f, ξ, η) and is called an *almost contact manifold* [15, p. 252]. From (23) we have

$$1)\quad f\xi = 0, \qquad 2)\quad \eta(fX) = 0, \qquad 3)\quad \operatorname{rank} f = 2m-2. \tag{24}$$

If an almost contact manifold M^{2m-1} admits a Riemannian metric tensor field g such that

$$\begin{aligned} &1)\quad \eta(X) = g(\xi, X) \\ &2)\quad g(fX, fY) = g(X,Y) - \eta(X)\eta(Y) \end{aligned} \tag{25}$$

then M^{2m-1} is said to have an *almost contact metric structure* (*almost Grayan structure*) (f, ξ, η, g) and is called an *almost contact metric manifold* [15, p. 254]. One has the following important theorem [11].

Theorem 4. *A hypersurface $\bar{M}^{2m-1}$ in an almost complex manifold M^{2m} has an almost contact structure.*

We explore the case when a hypersurface $\bar{M}^{2m-1}$ is the maximal integral submanifold of the distribution defined by the equation

$$\omega = 0$$

where ω is the Lee form of the LCK-manifold M^{2m}. It is obvious that the above said equation is locally integrable since the form ω is closed. The immersion $\Psi : \bar{M}^{2m-1} \longrightarrow M^{2m}$ is locally represented by the functions

$$x^\alpha = x^\alpha(y^1, \ldots, y^{2m-1})$$

where $\alpha = 1, \ldots, 2m$, and y^i, $i = 1, \ldots, 2m-1$ is a coordinate system in $\bar{M}^{2p}$. We put

$$B_i^\alpha = \partial_i x^\alpha$$

which span the tangent hyperplane of $\bar{M}^{2m-1}$ at each point $x = x(y)$. The equations (7) – (10) are also satisfied, but we have to take into account that $i = 1, \ldots, 2m-1$. Gauss and Weingarten equations for a hypersurface can be written in the form

$$\bar{\nabla}_j B_i^\alpha = H_{ij} C^\alpha, \qquad \bar{\nabla}_j C^\alpha = -H_j^i B_i^\alpha \tag{26}$$

where H_{ij} are the second fundamental tensor of $\bar{M}^{2m-1}$ respect to normal C^α. It is obvious that $C^\alpha = \frac{1}{\|\omega\|}\omega^\alpha$. According to the latter, and to (26) we obtain

$$\bar{\nabla}_i \omega^\alpha = \partial_i(\|\omega\|)C^\alpha - \|\omega\| B_t^\alpha H_i^t. \tag{27}$$

Therefore

$$H_{ki} = -\frac{1}{\|\omega\|} B_k^\alpha \omega_{\alpha,\beta} B_i^\beta, \qquad \partial_i(\|\omega\|) = B_i^\alpha \omega_{\alpha,\beta} C^\beta. \tag{28}$$

Let us define the (1,1) tensor field f, the covariant vector field η and the contravariant vector field ξ in $\bar{M}^{2m-1}$ as follows

$$\begin{array}{ll} 1) & f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta \\ 2) & \eta_k = \frac{1}{\|\omega\|} B_k^\beta J_\beta^\alpha \omega_\alpha \\ 3) & \xi^k = -\frac{1}{\|\omega\|} B_\beta^k J_\alpha^\beta \omega^\alpha. \end{array} \tag{29}$$

We see by (29) that the Riemannian metric $\bar{g}_{ij} = B_i^\alpha g_{\alpha\beta} B_j^\beta$ induced on $\bar{M}^{2m-1}$ satisfies (23), (24) and (25). Hence (29) and $\bar{g}$ form on $\bar{M}^{2m-1}$ an almost contact metric structure $(f, \xi, \eta, \bar{g})$.

The Nijenhuis tensor N of f-structure [15, p. 386] is given by

$$N_f(X, Y) \stackrel{\text{def}}{=} [fX, fY] - f[X, fY] - f[fX, Y] + f^2[X, Y]. \tag{30}$$

In the local coordinates the tensor (30) can be written in the form

$$N_{ij}^k = f_{j,t}^k f_i^t - f_{j,i}^t f_t^k - f_{i,t}^k f_j^t + f_{i,j}^t f_t^k.$$

Differentiating covariantly (23) we have

$$f_{t,j}^k f_i^t + f_t^k f_{i,j}^t = \eta_{i,j}\xi^k + \eta_i \bar{\nabla}_j \xi^k.$$

Hence

$$N_{ij}^k = f_i^t(f_{j,t}^k - f_{t,j}^k) - f_j^t(f_{i,t}^k - f_{t,j}^k) + \xi^k(\eta_{i,j} - \eta_{j,i}) + \eta_j \nabla_i \xi^k - \eta_i \nabla_j \xi^k.$$

If the tensor N_{ij}^k satisfies to the condition

$$n_{ij}^k = N_{ij}^k + \xi^k(\eta_{j,i} - \eta_{i,j}) = 0 \tag{31}$$

then the almost contact structure is said to be *normal* [6]. Following [10] we get

$$n_{ij}^k = f_i^t(f_{j,t}^k - f_{t,j}^k) - f_j^t(f_{i,t}^k - f_{t,j}^k) + \eta_j \nabla_i \xi^k - \eta_i \nabla_j \xi^k.$$

Substituting (29) into (4) we obtain

$$\begin{aligned} n_{ij}^k &= B_\alpha^t J_\beta^\alpha B_i^\beta \big(\bar\nabla_t(B_\gamma^k J_\delta^\gamma B_j^\delta) - \bar\nabla_j(B_\gamma^k J_\delta^\gamma B_t^\delta)\big) \\ &\quad - B_\alpha^t J_\beta^\alpha B_j^\beta \big(\bar\nabla_t(B_\gamma^k J_\delta^\gamma B_i^\delta) - \bar\nabla_i(B_\gamma^k J_\delta^\gamma B_t^\delta)\big) \\ &\quad - \frac{1}{\|\omega\|} B_i^\alpha J_\alpha^\beta \omega_\beta \bar\nabla_j\big(\frac{1}{\|\omega\|} B_\gamma^k J_\delta^\gamma \omega^\delta\big) \\ &\quad + \frac{1}{\|\omega\|} B_j^\alpha J_\alpha^\beta \omega_\beta \bar\nabla_i\big(\frac{1}{\|\omega\|} B_\gamma^k J_\delta^\gamma \omega^\delta\big). \end{aligned}$$

Taking into account (26) – (29) we have

$$\begin{aligned} n_{ij}^k &= f_i^t H_t^k \eta_j + \frac{1}{2}\|\omega\| f_i^k \eta_j - f_j^t H_t^k \eta_i - \frac{1}{2}\|\omega\| f_j^k \eta_i \\ &\quad - \eta_i \frac{\partial_j(\|\omega\|)}{\|\omega\|}\xi^k + \frac{1}{2}\|\omega\| f_j^k \eta_i + \eta_i \frac{\partial_j(\|\omega\|)}{\|\omega\|}\xi^k + f_j^t H_t^k \eta_i \\ &\quad + \eta_j \frac{\partial_i(\|\omega\|)}{\|\omega\|}\xi^k - \frac{1}{2}\|\omega\| f_i^k \eta_j - \eta_j \frac{\partial_i(\|\omega\|)}{\|\omega\|}\xi^k - f_i^t H_t^k \eta_j. \end{aligned}$$

We see also that the condition

$$n_{ij}^k = N_{ij}^k + \xi^k(\eta_{j,i} - \eta_{i,j}) = 0$$

is identically satisfied on $\bar M^{2m-1}$. This means that the theorem is true.

Theorem 5. *If a hypersurface $\bar M^{2m-1}$ of a LCK-manifold M^{2m} is an integral manifold of the distribution defined by the equation*

$$\omega = 0$$

where ω is Lee form of the LCK-manifold M^{2m} then the induced by the immersion almost contact structure

$$\begin{aligned} &1)\quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta \\ &2)\quad \eta_k = \tfrac{1}{\|\omega\|} B_k^\beta J_\beta^\alpha \omega_\alpha \\ &3)\quad \xi^k = -\tfrac{1}{\|\omega\|} B_\beta^k J_\alpha^\beta \omega^\alpha \end{aligned}$$

is a normal *one.*

Let us consider the case when ω satisfies the condition (19). Then from (22) it follows that $\mathrm{d}\theta \wedge \omega \wedge \theta = 0$ which means

$$\mathrm{d}\theta = \gamma_1 \wedge \omega + \gamma_2 \wedge \theta$$

where γ_1 and γ_2 are some one-forms. According to conditions $\Psi^*\omega = 0$ and $\eta = \frac{1}{\|\omega\|}\Psi^*\theta$ we have

$$\begin{aligned} \mathrm{d}\eta &= \mathrm{d}\big(\tfrac{1}{\|\omega\|}\Psi^*\theta\big) = \mathrm{d}\big(\tfrac{1}{\|\omega\|}\big) \wedge \Psi^*\theta + \tfrac{1}{\|\omega\|}\Psi^*\mathrm{d}\theta \\ &= \mathrm{d}\big(\tfrac{1}{\|\omega\|}\big) \wedge \Psi^*\theta + \tfrac{1}{\|\omega\|}\Psi^*(\gamma_1 \wedge \omega + \gamma_2 \wedge \theta) \\ &= \mathrm{d}\big(\tfrac{1}{\|\omega\|}\big) \wedge \Psi^*\theta + \tfrac{1}{\|\omega\|}\Psi^*\gamma_1 \wedge \Psi^*\omega + \tfrac{1}{\|\omega\|}\Psi^*\gamma_2 \wedge \Psi^*\theta \\ &= (\Psi^*\gamma_2 - \mathrm{d}(\ln\|\omega\|)) \wedge \eta. \end{aligned}$$

It follows that

$$\mathrm{d}\eta \wedge \eta = 0. \tag{32}$$

Since an M^{2m} is LCK-manifold hence its fundamental form satisfies the condition $\mathrm{d}\Omega = \omega \wedge \Omega$. Hence on the manifold $\bar{M}^{2m-1}$ we have

$$\mathrm{d}\bar{\Omega} = \mathrm{d}\Psi^*\Omega = \Psi^*\mathrm{d}\Omega = \Psi^*(\omega \wedge \Omega) = \Psi^*\omega \wedge \Psi^*\Omega = 0 \tag{33}$$

since $\Psi^*\omega = 0$. The condition of normality (31) is also satisfied. The theorem follows immediately from (31), (32) and (33).

Theorem 6. *Let the hypersurface $\bar{M}^{2m-1}$ of an LCK-manifold M^{2m} is an integral manifold of the distribution defined by the equation $\omega = 0$, where ω is Lee form of the LCK-manifold M^{2m} that satisfies the condition* (19). *Then the induced by the immersion almost contact structure*

$$\begin{aligned} &1) \quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta \\ &2) \quad \eta_k = \tfrac{1}{\|\omega\|} B_k^\beta J_\beta^\alpha \omega_\alpha \\ &3) \quad \xi^k = -\tfrac{1}{\|\omega\|} B_\beta^k J_\alpha^\beta \omega^\alpha \end{aligned}$$

is a normal almost contact metric structure for which the conditions

$$1)\ \mathrm{d}\eta \wedge \eta = 0, \qquad 2)\ \mathrm{d}\bar{\Omega} = 0, \qquad 3)\ n_{ij}^k = 0$$

are fulfilled.

Let us consider another case when an M^{2m} is Vaisman manifold which means that Lee form satisfies the condition $\omega_{i,j} = 0$. According to (22) we have

$$\mathrm{d}\theta = \frac{1}{2}\|\omega\|^2 J_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k$$

or

$$\mathrm{d}\theta(X, Y) = \|\omega\|^2\Omega(X, Y).$$

Taking into account that for a Vaisman manifold $\|\omega\| = \mathrm{const}$ it follows

$$\begin{aligned} \mathrm{d}\eta(X, Y) &= \frac{1}{\|\omega\|}\mathrm{d}\Psi^*\theta(X, Y) = \frac{1}{\|\omega\|}\Psi^*\mathrm{d}\theta(X, Y) \\ &= \|\omega\|\Psi^*\Omega(X, Y) = \|\omega\|\bar{\Omega}(X, Y). \end{aligned} \tag{34}$$

From (28), Theorem 5 and (34) it follows that

Theorem 7. *If a hypersurface $\bar{M}^{2m-1}$ of a LCK-manifold M^{2m} is an integral manifold of the distribution defined by the equation $\omega = 0$, where ω is Lee form of the LCK-manifold M^{2m} that satisfies the condition $\nabla_X \omega(Y) = 0$.*
Then the induced by the immersion almost contact structure

$$\begin{array}{ll} 1) & f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta \\ 2) & \eta_k = \frac{1}{\|\omega\|} B_k^\beta J_\beta^\alpha \omega_\alpha \\ 3) & \xi^k = -\frac{1}{\|\omega\|} B_\beta^k J_\alpha^\beta \omega^\alpha \end{array}$$

is a c-Sasakian structure, $c = \|\omega\|$. Moreover $\bar{M}^{2m-1}$ is a totally geodesic hypersurface in M^{2m}.

Similar results were obtained by Vaisman [13]. Moreover he had proved that if M^{2m} is conformally flat manifold then $\bar{M}^{2m-1}$ is a constant curvature manifold.
But we have proved normality of almost contact metric structure in $\bar{M}^{2m-1}$ which satisfies the condition $\omega = 0$ in LCK-manifold M^{2m}, for common case.
Taking into account that LCK-manifolds with Lee form which satisfies the condition

$$\Phi_4(\nabla\omega(X,Y)) = \frac{\|\omega\|^2}{2} g(X,Y)$$

have very particular properties, and we can refer to such LCK-manifolds as *Pseudo-Vaisman manifolds*.

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