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LECTURES ON GEOMETRIC QUANTIZATION

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Abstract. These lectures notes are meant as an introduction to geometric quantization. In Section 1, I begin with presentation of the historical background of quantum mechanics. I continue with discoveries in the theory of representations of Lie groups, which lead to emergence of geometric quantization as a part of pure mathematics. This presentation is very subjective, flavored by my own understanding of the role of geometric quantization in quantum mechanics and representation theory. Section 2 is devoted to a review of geometry of Hamiltonian systems. Geometric quantization is discussed in the next two sections: prequantization in Section 3 and polarization in Section 4. In particular, I discuss geometric quantization with respect to polarizations given by Kähler structure, cotangent bundle projection and completely integrable system. More advanced topics, like metaplectic structure, pairing of polarizations, and commutation of quantization and reduction, are not included.

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CONTENTS

1. Historical Background

We trace the beginning of Quantum Mechanics to Planck's work on the black body radiation. In his 1901 paper [29], Planck writes

> "If we apply Wien's displacement law in the latter form to equation (6) for the entropy S, we then find that the energy element must be proportional to the frequency, thus: $E = h\nu$."

I want to emphasize that Planck did not postulate this result but derived it.

The next giants on the scene are Niels Bohr and Arnold Sommerfeld. In his 1913 paper [6], Bohr postulates that admissible orbits of a Hamiltonian system satisfy the quantization condition

$$\int p_i \mathrm{d}q^i = nh$$

where $(q^1, ..., q^n)$ are position coordinates, $(p_1, ..., p_n)$ are conjugate momenta, h is Planck's constant, and the Einstein convention of summation over repeated indices is adopted. These conditions applied to the harmonic oscillator yield Planck's relation. Bohr's quantization of the hydrogen atom with Hamiltonian

$$H = \frac{1}{2m}p^2 - \frac{k}{|q|}$$

gives a discrete family of allowable orbits of the electron. Atoms absorb or emit radiation only when the electrons jump between allowed orbits.

In 1915 [36], Sommerfeld generalized Bohr's quantization condition to the relativistic hydrogen atom with Hamiltonian

$$H = \sqrt{m^2 - p^2} - \frac{k}{|q|}.$$

In this case motion is not periodic, but orbits lie on three-dimensional tori. Sommerfeld interpreted the integral in Bohr's quantization condition as integration over generators of the tori. The energy spectrum of the relativistic hydrogen atom obtained by Sommerfeld describes exactly the Balmer series of spectral line emissions of the hydrogen atom. The same energy spectrum is obtained from the Dirac equation with mass m and potential energy $-\frac{k}{|q|}$.

Sommerfeld's interpretation of Bohr's quantization rules is usually referred to as Bohr-Sommerfeld quantization rules. The results of Bohr and Sommerfeld are the foundation of the Bohr-Sommerfeld theory, which historians call Old Quantum Theory. This theory predicts, sometimes very accurately, quantum states of atoms and molecules but does not provide information about quantum interactions.

It is worth noting that applying to the Hamiltonian of the relativistic hydrogen atom the modified quantization conditions

$$\int p\mathrm{d}q = (n + \frac{1}{2})h$$

we obtain the energy spectrum, which is exactly the same as the spectrum obtained from the Klein-Gordon equation with mass m and potential energy $-\frac{k}{|q|}$ [15]. The relation between the correction $\frac{1}{2}$ in Bohr-Sommerfeld quantization rules and the change from spin $\frac{1}{2}$ to spin 0 is a mystery of quantum mechanics that is still unresolved [33].

The next stage of understanding of quantum physics originated with a paper of Werner Heisenberg published in 1925 [19], importance of which was described by P.A.M. Dirac in 1975 [18][1]

> "The great advance was made by Heisenberg in 1925. He made a very bold step. He had the idea that physical theory should concentrate on quantities which are very closely related to observed quantities. Now, the things you observe are only very remotely connected with the Bohr orbits. So Heisenberg said that the Bohr orbits are not very important. The things that are observed, or which are connected closely with the observed quantities, are all associated with two Bohr orbits and not with one Bohr orbit: *two* instead of *one*."

In the following year, we had two competing theories: the matrix mechanics of Max Born and Pascuale Jordan [7] and the wave mechanics of Ervin Schrödinger [30]. A unification of both theories into the present day quantum mechanics came in the work of Paul Dirac [16].

The fundamental structure of modern quantum theory was formulated by Dirac in *Principles of Quantum Mechanics* [17] in terms of operator algebras and their representations. In classical mechanics, states of the system under consideration are points of its phase space P, and dynamical variables are real-valued functions on P. In quantum mechanics, states of the system form a complex Hilbert space $\mathfrak{H}$ and dynamical variables are self-adjoint operators on $\mathfrak{H}$. There is a Poisson subalgebra A of the Poisson algebra of P, on which we can define a classical analogy given by a linear map $\boldsymbol{Q}$ associating to each $f \in A$ a self-adjoint operator $\boldsymbol{Q}_f$ on

[1] See also [26, p.261].

$\mathfrak{H}$ such that, for every $f_1, f_2 \in A$

$$[\boldsymbol{Q}_{f_1}, \boldsymbol{Q}_{f_2}] = \mathrm{i}\hbar \boldsymbol{Q}_{\{f_1, f_2\}} \tag{1}$$

where $\hbar$ is Planck's constant divided by 2π, and $\{f_1, f_2\}$ is the Poisson bracket of f_1 and f_2. Dirac referred to equation (1) as the *fundamental quantum condition.* It implies that the map $f \mapsto \frac{1}{\mathrm{i}\hbar}\boldsymbol{Q}_f$ is a homomorphism of the Poisson algebra of A into the Lie algebra of skew-adjoint operators on $\mathfrak{H}$.

Another consequence of equation (1) is that we may regard the classical mechanics as the limiting case of quantum mechanics when $\hbar \to 0$. On the other hand, if we know the classical theory of a system, we may use equation (1) to quantize our system; that is to construct a quantum theory of this system.

In the three decades that followed the publication of Dirac's *Principles of Quantum Mechanics*, there was a lot of work done on quantization of classical systems of physical importance. Also, there was feed back from the mathematics community aimed at clarification of mathematical concepts in quantum theory. Thus, quantum physics accelerated development of mathematics, in particular functional analysis and representation theory.

The foundation of geometric quantization is based on the fact, discovered independently by Kirillov, Souriau and Kostant, that every co-adjoint orbit P of a Lie group G is endowed with a symplectic form. In 1962, Aleksandr Kirillov constructed unitary representations of nilpotent Lie groups using the *orbit method*, which relied on the symplectic structure of co-adjoint orbits [21]. Kirillov also conjectured that irreducible unitary representations of compact group were in one-to-one correspondence with integral co-adjoint orbits. In 1966, Jean-Marie Souriau formulated a quantization scheme in terms of sections of a circle bundle over the phase space (P, ω) of the quantized system [37]. Souriau's *quantification géométrique* did not provide for probability amplitudes in quantum mechanics.

In 1965, Bertram Kostant outlined his theory of *geometric quantization* at the US-Japan Seminar in Differential Geometry, Kyoto. A comprehensive presentation of the first step of geometric quantization, called *prequantization*, was given in his 1970 paper [24]. Application of the complete theory to representations of solvable group appeared in a joint paper with L. Auslander published in 1971 [3]. This established the basic principle that the quantization techniques used by physicists can be adapted so that, a large class of connected Lie groups, they yield irreducible unitary representations corresponding to integral co-adjoint orbits.

2. Geometry of Hamiltonian Systems

2.1. Symplectic Manifolds

A differential two-form ω on a manifold P is symplectic if it is closed and non-degenerate. In other words, ω is symplectic if $d\omega = 0$ and

$$v \lrcorner \omega = 0$$

for every vector $v \in TP$, where $\lrcorner$ denotes the left interior product (contraction). A symplectic manifold is a pair (P, ω), where ω is a symplectic form on P.

For each smooth function $f(p, q)$ on a symplectic manifold (P, ω), the Hamiltonian vector field of f is the vector field X_f defined by

$$X_f \lrcorner \omega = -\mathrm{d}f.$$

Hence

$$\pounds_{X_f}\omega = X_f \lrcorner \mathrm{d}\omega + \mathrm{d}(X_f \lrcorner \omega) = 0$$

and the local one-parameter local group $\exp tX_f$ of diffeomorphisms of P generated by X_f preserves ω.

A symplectic structure ω of the phase space P of a classical system gives rise to the Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(P)$, defined by

$$\{f_1, f_2\} = X_{f_2}(f_1) \tag{2}$$

for all $f_1, f_2 \in C^\infty(P)$. The Poisson bracket is bilinear, and antisymmetric. Moreover, it satisfies the Leibniz rule

$$\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + f_2\{f_1, f_3\}$$

and the Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3\{f_1, f_2\}\} = 0.$$

The map $f \mapsto X_f$ is an antihomomorphism of the Poisson algebra of $C^\infty(P)$ to the Lie algebra $\mathfrak{X}(P)$ of smooth vector fields on P, that is

$$[X_{f_1}, X_{f_2}] = -X_{\{f_1, f_2\}}$$

for every $f_1, f_2 \in C^\infty(P)$.

The orbit of a vector field X on P through a point $p \in P$ is a maximal curve $c : t \mapsto c(t)$ in P such that $c(0) = p$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = X(f)(t)$$

for every $f \in C^\infty(P)$. Translations along integral curves of X gives rise to a local one-parameter local groups of diffeomorphisms of X denoted $\exp tX$. For every $p \in P$

$$(\exp tX)(p) = c(t)$$

where c is the maximal integral curve of X such that $c(0) = p$.
Let G be a connected Lie group and

$$\Phi : G \times P \to P : (g, p) \mapsto \Phi_g(p) = gp$$

be an action of G on P. We say that Φ is a Hamiltonian action of G on (P, ω) if there exists an Ad^*-equivariant map J from P to the dual $\mathfrak{g}^*$of the Lie algebra $\mathfrak{g}$ of G such that, for each $\xi \in \mathfrak{g}$, the action on P of the one-parameter subgroup $\exp t\xi$ of G is given by translations along the integral curves of X_{J_ξ}, where

$$J_\xi = \langle J \mid \xi \rangle.$$

The function J_ξ is called the momentum corresponding to ξ.[2] For every $f \in \mathcal{C}^\infty(P)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi^*_{\exp t\xi} f_{|t=0} = X_{J_\xi}(f)$$

which implies that the action of $\exp t\xi$ on P coincides with $\exp tX_{J_\xi}$.
A symplectic form ω on P defines a de Rham cohomology class $[\omega] \in H^2(P, \mathbb{R})$. We say that ω is integral if $[\omega] \in H^2(P, \mathbb{Z})$. In other words, ω is integral if, for every closed two-surface S in P

$$\int_S \omega = \text{integer}.$$

A symplectic manifold (P, ω) is integral if the symplectic form ω on P is integral. One of the main objectives of geometric quantization in the theory of representations of Lie groups is construction of a unitary representation of a connected Lie group from its Hamiltonian action on an integral symplectic manifold.

2.2. Examples

2.2.1. Cotangent bundles

For every manifold Q, the cotangent bundle space T^*Q of Q has a canonical symplectic form ω defined as follows. Let $\pi : T^*Q \to Q$ be the cotangent bundle projection and let $T\pi : T(T^*Q) \to TQ$ denote the derived map. For each $p \in T^*Q$, the Liouville form θ of T^*Q associates to vectors $u \in T_p(T^*Q)$, the evaluation of p on $T\pi(u)$; that is

$$\theta(u) = \langle p \mid T\pi(u) \rangle \,. \tag{3}$$

The canonical symplectic form of T^*P is the exterior differential of the Liouville form.[3] In other words,

$$\omega = \mathrm{d}\theta.$$

[2]Some authors use the French term *moment.*

[3]The form ω defined here has been traditionally used in theoretical mechanics [2], [39], [40]. However, some authors use the negative of $\mathrm{d}\theta$ as the canonical symplectic form of the cotangent bundle.

It follows that the de Rham cohomology class of the canonical symplectic form of the cotangent bundle vanishes.

If Q is the configuration space of a dynamical system, then T^*Q is the phase space of the system. For simple Hamiltonian systems the symplectic structure of the phase space is given by $\omega = \mathrm{d}\theta$. Electromagnetic interactions leads to an additional term. For example, the phase space of a relativistic particle in an electromagnetic field is $(T^*Q, \mathrm{d}\theta + \pi^*F)$, where Q is the space-time manifold and F is the electromagnetic field. Internal degrees of freedom like spin do not admit a configuration space.

2.2.2. Coadjoint orbits

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. For $\mu \in \mathfrak{g}^*$, the co-adjoint orbit of G through μ is

$$O = \{\mathrm{Ad}^*_g \mu \;;\; g \in G\}$$

where

$$\left\langle \mathrm{Ad}^*_g \mu \mid \xi \right\rangle = \left\langle \mu \mid \mathrm{Ad}_{g^{-1}} \xi \right\rangle$$

for every $\xi \in \mathfrak{g}$. Since the co-adjoint action of G is transitive on O, for each $\xi \in \mathfrak{g}$, there exists a unique vector field X^ξ on O generating the action of $\exp t\xi$ on O, and for every $\nu \in O$

$$T_\nu O = \{X^\xi(\nu) \;;\; \xi \in \mathfrak{g}\}.$$

Let $J : O \to \mathfrak{g}^*$ denote the inclusion map and let J_ξ be the restriction of J to evaluations on $\xi \in \mathfrak{g}$.In other words, for all $\nu \in \mathfrak{g}^*$ and all $\zeta \in \mathfrak{g}$

$$J(\nu) = \nu \qquad \text{and} \qquad J_\zeta(\nu) = \langle \nu \mid \xi \rangle .$$

Since $\mathrm{d}J$ has maximal rank, $X^\xi(\nu) = 0$ if and only if $\left\langle \mathrm{d}J \mid X^\xi(\nu) \right\rangle = 0$, which is equivalent to $X^\xi(J_\zeta)(\nu) = 0$ for all $\zeta \in \mathfrak{g}$. But

$$\begin{aligned} X^\xi(J_\zeta)(\nu) &= \left\langle \mathrm{d}J_\zeta \mid X^\xi(\nu) \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \mathrm{Ad}^*_{\exp t\xi} \nu \mid \zeta \right\rangle_{|t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \nu \mid \mathrm{Ad}_{\exp(-t\xi)} \zeta \right\rangle_{|t=0} = - \langle \nu \mid [\xi, \zeta] \rangle . \end{aligned} \tag{4}$$

Hence

$$X^\xi(\nu) = 0 \qquad \Longleftrightarrow \qquad \langle \nu \mid [\xi, \zeta] \rangle \qquad \text{for all} \quad \zeta \in \mathfrak{g}.$$

These authors define the Hamiltonian vector field X_f of $f \in C^\infty T^*Q$ with respect to the symplectic form $\mathrm{d}\theta$ by $X_f \lrcorner \mathrm{d}\theta = \mathrm{d}f$. Thus, the notion of the Hamiltonian vector field is the same in both conventions.

Definition 1. *The Kirillov-Kostant-Souriau form of a co-adjoint orbit O is the unique two-form ω on O such that*

$$\omega(X^{\xi}(\nu), X^{\zeta}(\nu)) = -\langle \nu \mid [\xi, \zeta] \rangle$$

for every $\xi, \zeta \in \mathfrak{g}$.

If $X^{\xi}(\nu) \lrcorner \omega = 0$, then $\langle \nu \mid [\xi, \zeta] \rangle = 0$ for all $\zeta \in \mathfrak{g}$, which implies that $X^{\xi}(\nu) = 0$. Hence, ω is non-degenerate. Moreover, equation (4) gives

$$\begin{aligned}(X^{\xi} \lrcorner \omega)(X^{\zeta}(\nu)) &= \omega(X^{\xi}(\nu), X^{\zeta}(\nu)) \\ &= -\langle \nu \mid [\xi, \zeta] \rangle = \langle \nu \mid [\zeta, \xi] \rangle = -\left\langle \mathrm{d}J_{\xi} \mid X^{\zeta}(\nu) \right\rangle\end{aligned}$$

for every $\xi, \zeta \in \mathfrak{g}$. Hence

$$X^{\xi} \lrcorner \omega = -\mathrm{d}J_{\xi}. \tag{5}$$

Further

$$\begin{aligned}(\pounds_{X^{\eta}}\omega)((X^{\xi}(\nu), X^{\zeta}(\nu))) = &\ \pounds_{X^{\eta}}(\omega(X^{\xi}(\nu), X^{\zeta}(\nu))) - \omega((\pounds_{X^{\eta}}X^{\xi})(\nu), X^{\zeta}(\nu)) \\ &-\omega(X^{\xi}(\nu), (\pounds_{X^{\eta}}X^{\zeta})](\nu)) \\ = &-\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathrm{Ad}^{*}_{\exp(t\eta)}\nu \mid [\xi, \zeta] \right\rangle_{|t=0} \\ &-\omega([X^{\eta}, X^{\xi}](\nu), X^{\zeta}(\nu)) - \omega(X^{\xi}(\nu), [X^{\eta}, X^{\zeta}](\nu)) \\ = &\ \langle \nu \mid [\eta, [\xi, \zeta]] \rangle - \langle \nu \mid [[\eta, \xi], \zeta] \rangle - \langle \nu \mid [\xi, [\eta, \zeta]] \rangle = 0\end{aligned}$$

for every $\eta, \xi, \zeta \in \mathfrak{g}$. Hence

$$\pounds_{X^{\eta}}\omega = 0 \tag{6}$$

for every $\eta \in \mathfrak{g}$, which implies that ω is invariant under the co-adjoint action of G on O. Equations (5) and (6) yield

$$X^{\eta} \lrcorner \mathrm{d}\omega = \pounds_{X^{\eta}}\omega - \mathrm{d}(X^{\eta} \lrcorner \omega) = 0$$

for every $\eta \in \mathfrak{g}$. Hence, $\mathrm{d}\omega = 0$,and omega is symplectic. It follows from equation (5) that the inclusion map $J : O \to g^{*}$ is the momentum map for the co-adjoint action of G on O, and $X^{\xi} = X_{J_{\xi}}$ for every $\xi \in \mathfrak{g}$.

2.2.3. Coadjoint orbits of $\mathrm{SO}(3)$

Co-adjoint orbits of $\mathrm{SO}(3)$ are spheres

$$S^{2}_{r} = \{x \in \mathbb{R}^{3};\, x^{2} = r^{2}\}.$$

For a fixed $r > 0$, let $\boldsymbol{s} = x_{|S_{r}}$ denote the restriction of x to S^{2}_{r}. The Kirillov-Kostant-Souriau ω form on S^{2}_{r} can be written as

$$\omega = -\frac{1}{2}r^{-2}\sum_{i,j,k}\varepsilon_{ijk}s^{i}\mathrm{d}s^{j} \wedge \mathrm{d}s^{k} = \frac{1}{r}\mathrm{vol}_{S^{2}_{r}} \tag{7}$$

where s^1, s^2, s^3 are components of the spin vector $\boldsymbol{s}$, ε_{ijk} is the completely anti-symmetric tensor with $\varepsilon_{123} = 1$, and $\mathrm{vol}_{S_r^2}$ is the standard area form on S_r^2 with $\int_{S_r^2} \mathrm{vol}_{S_r^2} = 4\pi r^2$, for detailed computations see [12].

For a non-relativistic particle with spin $\boldsymbol{s}$, interpreted as internal angular momentum with fixed length $|\boldsymbol{s}| = r$, the phase space is $(T^*\mathbb{R}^3 \times S_r^2, \mathrm{d}\theta + \omega)$, where $\mathrm{d}\theta$ is the canonical symplectic form of $T^*\mathbb{R}$ and ω is the Kirillov-Kostant-Souriau form given by equation (7). Geometric quantization of this system leads to the Pauli theory of spin [28].

2.3. Reduction of Symmetries

Consider a Hamiltonian action $\Phi : G \times P \to P : (g, p) \mapsto gp$ of a connected Lie group G on a symplectic manifold (P, ω) with a momentum map $J : P \to \mathfrak{g}^*$. Suppose that we need to solve equations of motion for a Hamiltonian system on (P, ω) with a G-invariant Hamiltonian $H \in C^\infty(P)$. Given a point $p_0 \in P$, we want to find the integral curve $t \mapsto c(t)$ of X_H through p_0. In other words, $c(0) = p_0$ and, for every $f \in C^\infty(P)$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) = X_H(f)(c(t)). \tag{8}$$

Since the action Φ of G preserves ω and H, it follows that it preserves X_H. Therefore, the curve $t \mapsto gc(t)$ is an integral curve of X_H through gp_0.

Let P/G denote the space of G-orbits in P and $\rho : P \to P/G : p \to Gp$ denote the orbit map. The projection $t \mapsto \rho(c(t))$ is a curve in P/G such that $\rho(c(0)) = Gp_0$. If we know $\rho(c(t))$, we can find $c(t)$ as follows. First, lift $\rho(c(t))$ to a curve $c_1(t)$ through p_0. In other words, $t \mapsto c_1(t)$ is a curve in P such that $c_1(0) = 0$ and $\rho(c_1(t)) = \rho(c(t))$. This implies that there exists a curve $t \mapsto g(t)$ in G such that, for every t in the domain of c

$$g(t)c_1(t) = c(t) \tag{9}$$

Substituting equation (9) into equation (8), yields

$$\frac{\mathrm{d}}{\mathrm{d}t} f(g(t)c_1(t)) = X_H(f)(g(t)c_1(t)) \tag{10}$$

for every $f \in C^\infty(P)$. Equation (10) is a first order differential equation for the curve $t \mapsto g(t)$ with initial condition $g(0) = \mathrm{identity}$.

The G-invariance of the Hamiltonian H implies that $X_H J_\xi = -X_{J_\xi} H = 0$ for every $\xi \in \mathfrak{g}$. Hence, the momenta J_ξ are constant along integral curves of X_H. In particular, if $\mu = J(p_0) \in \mathfrak{g}^*$, then $J(c(t)) = \mu$ for all t, and the curve $t \mapsto g(t)$ has to have the stability group $G_\mu = \{g \in G \ ; \ \mathrm{ad}_g^* \mu = \mu\}$ of μ.

In the discussion above, we have split the problem of finding integral curves X_H into two steps. The first step, called reduction, consists of finding the projection of an integral curve of X_H to the orbit space P/G. The second step, called reconstruction, consists of solving equation (10).

2.3.1. Regular reduction

In order to discuss methods of finding projections of integral curve of X_H to the orbit space P/G we must make further assumptions on the action of G on P. Suppose first that the action of G on P is free and proper. Then, the orbit space P/G is a manifold and the orbit map $\rho : P \to P/G$ is a submersion. The ring $C^\infty(P/G)$ of smooth functions on P/G is isomorphic to the ring $C^\infty(P)^G$ of smooth G-invariant functions on P. Since the symplectic form ω on P is G-invariant, it follows that the Poisson bracket (2) is G-invariant. Thus, if $f_1, f_2 \in C^\infty(P)^G$ then $\{f_1, f_2\} \in C^\infty(P)$. Thus, $C^\infty(P)^G$ has the structure of a Poisson algebra, which implies that the orbit space P/G is a Poisson manifold and, for every $\bar{f}_1, \bar{f}_2 \in C^\infty(P/G)$, the Poisson bracket $\{\bar{f}_1, \bar{f}_2\} \in C^\infty(P)$ satisfies the condition

$$\rho^*\{\bar{f}_1, \bar{f}_2\} = \{\rho^*\bar{f}_1, \rho^*\bar{f}_2\}. \tag{11}$$

There exists a symplectic form $\bar{\omega}$ on P/G such that for every $\bar{f} \in C^\infty(P/G)$, the Hamiltonian vector field $X_{\bar{f}}$, defined with respect to the symplectic form $\bar{\omega}$, is ρ-related to the Hamiltonian vector field $X_{\rho^* f}$ of $\rho^* f$. In other words

$$X_{\bar{f}} \circ \rho = T\rho \circ X_{\rho^* f}$$

where $T\rho : TP \to T(P/G)$ is the tangent map of $\rho : P \to P/G$. This implies that, for every $\bar{f}_1, \bar{f}_2 \in C^\infty(P/G)$

$$\{\bar{f}_1, \bar{f}_2\} = X_{\bar{f}_2}\bar{f}_1.$$

For $H \in C^\infty(P)^G$, we denote by $\bar{H} \in C^\infty(P/G)$ the push-forward of H by ρ. In other words, $H = \rho^*\bar{H}$. If $c(t)$ is an integral curve of X_H through $p_0 \in P$, then $\rho(c(t))$ is an integral curve of $X_{\bar{H}}$ through $\rho(p_0)$.

The regular reduction was introduced by Meyer [27] and Marsden and Weinstein [25]. It is also known as the Marsden-Weinstein reduction.

2.3.2. Singular reduction

Suppose now that the action of G on P is not free but it is proper. In this case, the orbit space P/G is a stratified subcartesian differential space with the differential structure $C^\infty(P/G)$ isomorphic to $C^\infty(P)^G$. As in the case of manifolds, geometry of differential spaces can be studied in terms of their smooth but one has to be careful not to jump to conclusions. For example, a global derivation of $C^\infty(P/G)$ need not generate a local one-parameter local group of diffeomorphisms of P/G. Therefore, vector fields on a P/G are defined as global derivations of $C^\infty(P/G)$ that generate local one-parameter local group of diffeomorphisms of P/G. With this definition, orbits of any family of vector fields on P/G are smooth manifolds immersed in P/G. In particular, strata of the stratification of P/G are orbits of the family of all vector fields on P/G [35].

As before, $C^\infty(P/G)$ inherits from $C^\infty(P)^G$ the structure of a Poisson algebra. For each $\bar{f}_0 \in C^\infty(P/G)$, the derivation

$$X_{\bar{f}_0} : C^\infty(P/G) \to C^\infty(P/G) : \bar{f} \mapsto \{\bar{f}, \bar{f}_0\}$$

generates a local one-parameter local group of diffeomorphisms of P/G and we refer to it as the Poisson vector field of $\bar{f}_0$. Orbits of the family of all Poisson vector fields on P/G are smooth symplectic manifolds immersed in strata of the stratification of P/G.

As before, for $H \in C^\infty(P)^G$, we denote by $\bar{H} \in C^\infty(P/G)$ the push-forward of H by ρ. If $c(t)$ is an integral curve of X_H through $p_0 \in P$, then $\rho(c(t))$ is an integral curve of $X_{\bar{H}}$ through $\rho(p_0)$.

The technique of singular reduction, in terms of the Poisson algebra structure, was initiated by Cushman [9], and later formalized by Arms, Cushman and Gotay [1]. The role of Sikorski's theory of differential spaces in singular reduction was first described by Cushman and Śniatycki [11].

3. Prequantization

Let $\lambda : L \to P$ be a complex line bundle. A connection on L is given by a covariant derivative operator ∇, which associates to each section σ of L and each vector field X on P a section $\nabla_X \sigma$ of L such that

$$\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma \qquad \text{and} \qquad \nabla_{fX}\sigma = f\nabla_X\sigma$$

for every $f \in \mathcal{C}^\infty(P)$. For every section σ of L, $f \in \mathcal{C}^\infty(P)$ and $X_1, X_2 \in \mathfrak{X}(P)$,

$$(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})(f\sigma) = f(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})\sigma.$$

Hence, there is a two-form α on P such that

$$(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})\sigma = 2\pi \mathrm{i}\alpha(X_1, X_2)\sigma. \tag{12}$$

The form α is the pull-back by the section σ of the curvature form of the connection ∇.

Theorem 2. *The de Rham cohomology class $[\alpha]$ of the curvature form α of a connection on a complex line bundle $\lambda : L \to P$ is in $H^2(P, \mathbb{Z})$, that is, for every compact oriented two-dimensional submanifold M of P*

$$\int_M \alpha \in \mathbb{Z}.$$

Moreover, for every form α on P with $[\alpha] \in H^2(P, \mathbb{Z})$, there exists a complex line bundle $\lambda : L \to P$ with connection ∇ such that α is the curvature of ∇.

Equivalence classes of complex line bundles with connection with curvature form α are parametrized by $H^1(P, \mathbb{Z})$.

Proof: See [24]. ∎

A Hermitian form $\langle \cdot \mid \cdot \rangle$ on L is connection invariant if, for every pair of sections σ_1, σ_2 of L and every vector field X on P

$$X(\langle \sigma_1 \mid \sigma_2 \rangle) = \langle \nabla_X \sigma_1 \mid \sigma_2 \rangle + \langle \sigma_1 \mid \nabla_X \sigma_2 \rangle.$$

For every line bundle with connection, there exists a connection invariant Hermitian form defined up to a constant factor.

Quantization of a symplectic manifold is defined in terms of an additional free parameter h. In quantum mechanics, h is Planck's constant. However, in the quasi-classical approximation, we consider limits of various expressions as $h \to 0$. In the theory of representations of Lie groups, the value of h is usually taken to be $-\mathrm{i}$.

Definition 3. *A symplectic manifold (P, ω) is quantizable if there exists a complex line bundle L over P with a connection ∇ and a connection invariant Hermitian form such that the curvature of ∇ is $-\frac{1}{h}\omega$.*

Remark 4 (Prequantization Condition)**.** *By* Theorem 2 *(P, ω) is quantizable if and only if the de Rham cohomology class $[\frac{1}{h}\omega]$ is integral. In other words, (P, ω) is quantizable if and only if, for every compact oriented two-dimensional submanifold M of P*

$$\int_M \omega = nh$$

for some integer n that depends on M.

Equation (12) implies that, for each section σ of a prequantization line bundle L and every pair X_1, X_2 of vector fields on P

$$(\nabla_{X_1} \nabla_{X_2} - \nabla_{X_2} \nabla_{X_1} - \nabla_{[X_1, X_2]})\sigma = -\frac{\mathrm{i}}{\hbar}\omega(X_1, X_2)\sigma \tag{13}$$

where $\hbar = \frac{h}{2\pi}\cdot$ which we adopt in the following.

If σ is a non-zero local section of L, the covariant derivative $\nabla_X \sigma$ is proportional to σ, and there is a complex-valued one-form θ on the domain of σ in P such that $\nabla_X \sigma = -\mathrm{i}\hbar^{-1} \langle \theta \mid X \rangle \sigma$ for every vector field X. Hence

$$\nabla \sigma = -\mathrm{i}\hbar^{-1} \theta \otimes \sigma.$$

The one-form θ is called the pull-back by σ of the connection form of ∇. Equation (13) implies that

$$\mathrm{d}\theta = \omega_{|\text{domain } \sigma}.$$

A function $f \in \mathcal{C}^\infty(P)$ generates a local one-parameter group $\exp tX_f$ of local symplectomorphisms of (P, ω). The Hamiltonian vector field X_f on f can be lifted to a unique vector field $\widehat{X}_f$ on L such that $\exp t\widehat{X}_f$ is a lift of $\exp tX_f$ that preserves the connection ∇. For each $p \in P$ and non-zero $z \in L_p$, the horizontal component of $\widehat{X}_f(z)$ is the horizontal lift of X_f at z, while the vertical component of $\widehat{X}_f(z)$ is proportional to $f(p)$. If $X_f(p) = 0$ then $\exp t\widehat{X}_f$ acts on the fibre L_p by multiplication by $\mathrm{e}^{-2\pi\mathrm{i}f(p)}$. For each $\sigma \in S^\infty(L)$ we set

$$\boldsymbol{P}_f\sigma = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}(\exp t\widehat{X}_f \circ \sigma \circ \exp(-tX_f))_{|t=0}. \tag{14}$$

Direct computation yields

$$\boldsymbol{P}_f\sigma = (-\mathrm{i}\hbar\nabla_{X_f} + f)\sigma. \tag{15}$$

We refer to $\boldsymbol{P}_f$ as the prequantization operator corresponding to f.

Proposition 5. *For each $f_1, f_2 \in \mathcal{C}^\infty(P)$ and $\sigma \in S^\infty(L)$*

$$[\boldsymbol{P}_{f_1}, \boldsymbol{P}_{f_2}] = \mathrm{i}\hbar\boldsymbol{P}_{\{f_1,f_2\}}. \tag{16}$$

Proof: See [24]. ■

The map

$$\boldsymbol{P} : \mathcal{C}^\infty(P) \times S^\infty(L) \to S^\infty(L) : (f,\sigma) \mapsto \boldsymbol{P}_f\,\sigma$$

is called the prequantization map.

Corollary 6. *The map $\mathcal{C}^\infty(P) \times S^\infty(L) \to S^\infty(L) : (f,\sigma) \mapsto \frac{\mathrm{i}}{\hbar}\boldsymbol{P}_f\,\sigma$ is a representation of the Lie algebra structure of $\mathcal{C}^\infty(P)$ on $S^\infty(L)$.*

The space $S_0^\infty(L)$ of compactly supported smooth sections of L has a Hermitian scalar product

$$(\sigma_1 \mid \sigma_2) = \int_P \langle\sigma_1 \mid \sigma_2\rangle\omega^n \tag{17}$$

where $n = \frac{1}{2}\dim P$. For each $f \in \mathcal{C}^\infty(P)$, the prequantization operator $\boldsymbol{P}_f$ is symmetric with respect to the scalar product (17). If the Hamiltonian vector field X_f of f is complete, then $\boldsymbol{P}_f$ is self-adjoint on the Hilbert space $\mathfrak{H}$ obtained by the completion of $S_0^\infty(L)$ with respect to the norm given by (17). Equation (16) gives the usual commutation relations imposed in quantum mechanics. However, prequantization does not correspond to the quantum theory, because interpretation of $(\sigma \mid \sigma)(p)$ as the probability density of localizing the state σ at a point $p \in P$ fails to satisfy Heisenberg's Uncertainty Principle.

3.1. Prequantization Representation a Lie Group

Suppose that we have a Hamiltonian action of a connected Lie group G on (P, ω) with an equivariant momentum map $J : P \to \mathfrak{g}^*$. Since the map $\xi \mapsto J_\xi$ is a homomorphism of $\mathfrak{g}$ to the Poisson algebra $\mathcal{C}^\infty(P)$, the map $\xi \mapsto (\mathrm{i}/\hbar)\boldsymbol{P}_{J_\xi}$ is a linear representation of the Lie algebra $\mathfrak{g}$ on the space $S^\infty(L)$, which we call the prequantization representation of $\mathfrak{g}$. Since Hamiltonian vector fields X_{J_ξ} are complete, each operator $(\mathrm{i}/\hbar)P_{J_\xi}$ is skew-adjoint on the Hilbert space $\mathcal{H}$ obtained by the completion of $S_0^\infty(L)$ with respect to the norm given by (17). Recall that the action of $\mathfrak{g}$ on L is given by vector fields $\widehat{X}_{J_\xi}$ on L, see equation (14). We assume that this action integrates to an action of G on L that covers the action of G on P. We refer to this action as the prequantization action of G on P. This assumption implies that the prequantization representation of $\mathfrak{g}$ described above integrates to a representation of G. That is, we have a linear representation

$$\boldsymbol{R} : G \times S^\infty(L) \to S^\infty(L) : (g, \sigma) \mapsto \boldsymbol{R}_g \sigma$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R}_{\exp t\xi}\sigma)_{|t=0} = (\mathrm{i}/\hbar)\boldsymbol{P}_{J_\xi}\sigma$$

for each $\xi \in \mathfrak{g}$. The linear representation $\boldsymbol{R}$ induces a unitary representation

$$\boldsymbol{U} : G \times \mathfrak{H} \to S^\infty(L) : (g, \sigma) \mapsto \boldsymbol{U}_g \sigma$$

such that $\boldsymbol{U}_g\sigma = \boldsymbol{R}_g\sigma$ for each $\sigma \in S^\infty(L) \cap \mathfrak{H}$. We refer to $\boldsymbol{R}$ and $\boldsymbol{U}$ as prequantization representations of G. In general, the unitary prequantization representation $\boldsymbol{U}$ fails to be irreducible.

3.2. Prequantization Representations of $\mathrm{SO}(3)$

3.2.1. Quantization of spin

In Section 2.2.3, we described the symplectic structure of co-adjoint orbits of $\mathrm{SO}(3)$. They are spheres $S_r^2 \subset \mathbb{R}^3$ with the Kirillov-Kostant-Souriau form

$$\omega_r = -\frac{1}{2}r^{-2}\sum_{i,j,k}\varepsilon_{ijk}s^i\mathrm{d}s^j \wedge \mathrm{d}s^k = \frac{1}{r}\mathrm{vol}_{S_r^2}$$

which satisfy The symplectic manifold (S_r^2, ω_r) satisfies the Prequantization Condition (Remark 4) if

$$\int_{S_r^2}\omega = \int_{S_r^2}\frac{1}{r}\mathrm{vol}_{S_r^2} = 4\pi r = nh$$

or

$$r = \frac{nh}{4\pi} = n\frac{\hbar}{2}\cdot$$

Thus, the length r of the spin vector $\boldsymbol{s}$ has to be an integral multiple of $\frac{\hbar}{2}$. For even n, prequantization of (S^2_r, ω) gives the prequantization representation of $\mathrm{SO}(3)$.

3.2.2. Prequantization line bundle

First, we need to construct a prequantization line bundle of (S^2_r, ω) under the assumption that n is even, and we write $n = 2s$, where $s \in \mathbb{N}$.

Let $V_\pm$ be complements of the south pole and the north pole in S^2_r, respectively that is

$$V_\pm = \{J \in S^2_r \ ; \ s^3 \pm r > 0\}.$$

We introduce in V_+ and V_- complex functions

$$z_\pm = \frac{s^1 \mp \mathrm{i}s^2}{r \pm s^3} \tag{18}$$

respectively. In $V_+ \cap V_-$

$$z_+ z_- = 1$$

and the functions z_+ and z_- define a complex structure on S^2_r.

Solving equation (18) for the spin vector $\boldsymbol{s}$ we obtain

$$\begin{aligned} s^1 &= r(z_\pm + \bar{z}_\pm)(1 + z_\pm \bar{z}_\pm)^{-1} \\ s^2 &= \pm \mathrm{i}r(z_\pm - \bar{z}_\pm)(1 + z_\pm \bar{z}_\pm)^{-1} \\ s^3 &= \pm r(1 - z_\pm \bar{z}_\pm)(1 + z_\pm \bar{z}_\pm)^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} \omega_{|V_\pm} &= -2\mathrm{i}r(1 + z_\pm \bar{z}_\pm)^{-2} \mathrm{d}\bar{z}_\pm \wedge \mathrm{d}z_\pm \\ \theta_\pm &= -2\mathrm{i}r(1 + z_\pm \bar{z}_\pm)^{-1} \bar{z}_\pm \mathrm{d}z_\pm \end{aligned}$$

and

$$\theta_+ - \theta_- = \mathrm{id}(\log z_-^{2r}) = \mathrm{i}\hbar \mathrm{d}(\log z_-^{2r/\hbar}) = \mathrm{i}\hbar \mathrm{d}(\log z_-^n).$$

Since n is an integer, the transition function z_-^n is globally defined and single-valued

Consider an equivalence relation $\sim$ on $(\mathbb{C} \times V_+ \times \{+\}) \cup (\mathbb{C} \times V_- \times \{-\})$ defined by

$$(c, x, \alpha) \sim (c', x', \alpha')$$

if (i) $(c, x, \alpha) = (c', x', \alpha')$ or (ii) $\alpha = +$, $\alpha' = -$, $x = x' \in V_+ \cap V_-$, and $c = z_-(x)^n c'$, or (iii) $\alpha = -$, $\alpha' = +$, $x = x' \in V_+ \cap V_-$, and $c' = z_-(x)^n c$. The space

$$L = [(\mathbb{C} \times V_+ \times \{+\}) \ \cup \ (\mathbb{C} \times V_- \times \{-\})]/ \sim$$

of $\sim$-equivalence classes is a complex line bundle over S^2_r. The projection map λ assigns to each equivalence class $[(c,x,\alpha)] \in L$ the point $x \in P$. The restrictions of L to $V_\pm$ are trivial, with trivializing sections

$$\sigma_\pm : V_\pm \to L : x \mapsto [(1,x,\pm)].$$

For $x \in V_+ \cap V_-$

$$\sigma_+(x) = z_-(x)^n \sigma_-(x).$$

We define connections $\nabla_\pm$ on $L_{|V_\pm}$ by

$$\nabla_\pm \sigma_\pm = -\mathrm{i}\hbar^{-1}\theta_\pm \otimes \sigma_\pm.$$

In $V_+ \cap V_-$

$$\begin{aligned}
\nabla_-\sigma_+ &= \nabla_-(z_-^n\lambda_-) = \mathrm{d}z_-^n \otimes \sigma_- + z_-^n\nabla_-\sigma_- = \mathrm{d}z_-^n \otimes \sigma_- - \mathrm{i}\hbar^{-1}z_-^n\theta_- \otimes \sigma_- \\
&= \mathrm{d}z_-^n \otimes \sigma_- + \mathrm{i}\hbar^{-1}z_-^n(\theta_+ - \theta_- - \theta_+) \otimes \sigma_- \\
&= \mathrm{d}z_-^n \otimes \sigma_- + \mathrm{i}\hbar^{-1}z_-^n(\theta_+ - \theta_-) \otimes \sigma_- - \mathrm{i}\hbar^{-1}z_-^n\theta_+ \otimes \sigma_- \\
&= \mathrm{d}z_-^n \otimes \sigma_- + \mathrm{i}\hbar^{-1}z_-^n(\mathrm{i}\hbar\mathrm{d}(\log z_-^n)) \otimes \sigma_- - \mathrm{i}\hbar^{-1}\theta_+ \otimes z_-^n\sigma_- \\
&= \nabla_+\sigma_+.
\end{aligned}$$

Similarly, $\nabla_+\sigma_- = \nabla_-\sigma_-$. Hence, there exists a unique connection ∇ on L that restricts to $\nabla_\pm$ on $L_{|V_\pm}$. By construction, the curvature of ∇ is $\frac{-1}{h}\omega$, as required. If $\langle\cdot,\cdot\rangle$ is a connection invariant Hermitian form on L, then

$$\begin{aligned}
\mathrm{d}\langle\sigma_\pm,\sigma_\pm\rangle &= -\mathrm{i}\hbar^{-1}(\theta_\pm - \bar{\theta}_\pm)\langle\sigma_\pm,\sigma_\pm\rangle \\
&= -\mathrm{i}\hbar^{-1}(-2\mathrm{i}r(1+z_\pm\bar{z}_\pm)^{-1}\bar{z}_\pm\mathrm{d}z_\pm - 2\mathrm{i}r(1+z_\pm\bar{z}_\pm)^{-1}z_\pm\mathrm{d}\bar{z}_\pm)\langle\sigma_\pm,\sigma_\pm\rangle \\
&= -2\hbar^{-1}r(1+z_\pm\bar{z}_\pm)^{-1}(\bar{z}_\pm\mathrm{d}z_\pm + z_\pm\mathrm{d}\bar{z}_\pm)\langle\sigma_\pm,\sigma_\pm\rangle \\
&= -2\hbar^{-1}r(1+z_\pm\bar{z}_\pm)^{-1}\mathrm{d}(1+z_\pm\bar{z}_\pm)\langle\sigma_\pm,\sigma_\pm\rangle \\
&= -2\hbar^{-1}r\mathrm{d}\log(1+z_\pm\bar{z}_\pm)\langle\sigma_\pm,\sigma_\pm\rangle.
\end{aligned}$$

Since $r = n\frac{\hbar}{2} = s\hbar$, it follows that

$$\mathrm{d}\langle\sigma_\pm,\sigma_\pm\rangle = -2s\mathrm{d}\log(1+z_\pm\bar{z}_\pm)\langle\sigma_\pm,\sigma_\pm\rangle$$

or

$$\mathrm{d}\log\langle\sigma_\pm,\sigma_\pm\rangle = -2s\mathrm{d}\log(1+z_\pm\bar{z}_\pm) = \mathrm{d}\log(1+z_\pm\bar{z}_\pm)^{-2s}.$$

Therefore, we may choose

$$\langle\sigma_\pm,\sigma_\pm\rangle = (1+z_\pm\bar{z}_\pm)^{-2s} = \frac{1}{(1+z_\pm\bar{z}_\pm)^{2s}}.$$

In this way, we have constructed a complex line bundle L over S^2_r with connection ∇ such that the curvature of ∇ is $\frac{-1}{h}\omega$, and with a connection invariant Hermitian form $\langle\cdot,\cdot\rangle$ on L.

3.2.3. Prequantization representation of $\mathrm{SO}(3)$

Every section $\sigma : S^2_r \to L$ can be expressed in terms of trivializing sections $\sigma_\pm$ as $\sigma_{|V_\pm} = \psi_\pm \sigma_\pm$, where $\psi_\pm$ are functions on $V_\pm$. For $x \in V_+ \cap V_-$, equation $\sigma_+(x) = z_-(x)^n \sigma_-(x)$ and $\psi_+(x)\sigma_+(x) = \psi_-(x)\sigma_-(x)$ imply

$$\psi_+(x) = z_-^{-n}(x)\psi_-(x) = z_-^{-2s}(x)\psi_-(x).$$

For each $f \in C^\infty(S^2)$, the prequantization operator gives

$$\boldsymbol{P}_f \sigma_\pm = (-\mathrm{i}\hbar \nabla_{\pm X_f} + f)\sigma_\pm = -\langle \theta_\pm \mid X_f \rangle \sigma_\pm + f\sigma_\pm.$$

Hence

$$\boldsymbol{P}_f \sigma_{|V_\pm} = \boldsymbol{P}_f(\psi_\pm\sigma_\pm) = (-\mathrm{i}\hbar X_f \psi_\pm + f - \langle \theta_\pm \mid X_f \rangle)\sigma_\pm.$$

By Corollary 6 the map $\mathcal{C}^\infty(S^2_r) \times S^\infty(L) \to S^\infty(L) : (f, \sigma) \mapsto \frac{\mathrm{i}}{\hbar}\boldsymbol{P}_f\ \sigma$ is a representation of the Lie algebra structure of $\mathcal{C}^\infty(S^2_r)$ on $S^\infty(L)$.

Note that elements of the Lie algebra $\mathfrak{so}(3)$ of $\mathrm{SO}(3)$ correspond to functions on S^2_r that are linear in the components s^1, s^2 and s^3 of the spin vector $\boldsymbol{s}$. The Hamiltonian vector fields of s^1, s^2 and s^3, expressed in terms of the functions $z_\pm$ and $\bar{z}_\pm$ are

$$\begin{aligned}
X_{s^1|V_\pm} &= -\frac{\mathrm{i}}{2}\Big((z_\pm^2 - 1)\frac{\partial}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial}{\partial \bar{z}_\pm}\Big)\\
X_{s^2|V_\pm} &= \pm\frac{1}{2}\Big((z_\pm^2 + 1)\frac{\partial}{\partial z_\pm} + (\bar{z}_\pm^2 + 1)\frac{\partial}{\partial \bar{z}_\pm}\Big)\\
X_{s^3|V_\pm} &= \pm\mathrm{i}\Big(\bar{z}_\pm\frac{\partial}{\partial \bar{z}_\pm} - z_\pm\frac{\partial}{\partial z_\pm}\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
\boldsymbol{P}_{s^1}(\psi_\pm\sigma_\pm) &= -\frac{\hbar}{2}\Big((z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\Big)\sigma_\pm\\
&\quad + \Big(r(z_\pm + \bar{z}_\pm)(1 + z_\pm\bar{z}_\pm)^{-1}\Big)\psi_\pm\sigma_\pm\\
&\quad - \Big\langle -2\mathrm{i}r(1 + z_\pm\bar{z}_\pm)^{-1}\bar{z}_\pm \mathrm{d}z_\pm \,\Big|\, -\frac{\mathrm{i}}{2}\Big((z_\pm^2 - 1)\frac{\partial}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial}{\partial \bar{z}_\pm}\Big)\Big\rangle\psi_\pm\sigma_\pm\\
&= -\frac{\hbar}{2}\Big((z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\Big)\sigma_\pm + s\hbar\frac{z_\pm + \bar{z}_\pm}{1 + z_\pm\bar{z}_\pm}\psi_\pm\sigma_\pm\\
&\quad + s\hbar\frac{z_\pm^2 - 1}{1 + z_\pm\bar{z}_\pm}\bar{z}_\pm\psi_\pm\sigma_\pm\\
&= -\frac{\hbar}{2}\Big((z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\Big)\sigma_\pm + s\hbar\frac{z_\pm + z_\pm^2\bar{z}_\pm}{1 + z_\pm\bar{z}_\pm}\psi_\pm\sigma_\pm\\
&= -\frac{\hbar}{2}\Big((z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\Big)\sigma_\pm + s\hbar z_\pm\psi_\pm\sigma_\pm.
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{P}_{s^2}(\psi_\pm\sigma_\pm) &= (-\mathrm{i}\hbar)\Big(\pm\frac{1}{2}\Big)\Big((z_\pm^2+1)\frac{\partial\psi_\pm}{\partial z_\pm}+(\bar z_\pm^2+1)\frac{\partial\psi_\pm}{\partial\bar z_\pm}\Big)\sigma_\pm \\
&\quad +[\pm\mathrm{i}r(z_\pm-\bar z_\pm)(1+z_\pm\bar z_\pm)^{-1}]\psi_\pm\sigma_\pm \\
&\quad -\Big\langle -2\mathrm{i}r(1+z_\pm\bar z_\pm)^{-1}\bar z_\pm \mathrm{d}z_\pm \mid \pm\frac{1}{2}\Big((z_\pm^2+1)\frac{\partial}{\partial z_\pm}\Big)\Big\rangle\psi_\pm\sigma_\pm \\
&= \mp\frac{\mathrm{i}\hbar}{2}\Big((z_\pm^2+1)\frac{\partial\psi_\pm}{\partial z_\pm}+(\bar z_\pm^2+1)\frac{\partial\psi_\pm}{\partial\bar z_\pm}\Big)\sigma_\pm \pm \mathrm{i}s\hbar\frac{z_\pm-\bar z_\pm}{1+z_\pm\bar z_\pm}\psi_\pm\sigma_\pm \\
&\quad \pm\mathrm{i}s\hbar\frac{\bar z_\pm(z_\pm^2+1)}{1+z_\pm\bar z_\pm}\psi_\pm\sigma_\pm \\
&= \mp\frac{\mathrm{i}\hbar}{2}\Big((z_\pm^2+1)\frac{\partial\psi_\pm}{\partial z_\pm}+(\bar z_\pm^2+1)\frac{\partial\psi_\pm}{\partial\bar z_\pm}\Big)\sigma_\pm \pm \mathrm{i}s\hbar z_\pm\psi_\pm\sigma_\pm
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P}_{s^3}(\psi_\pm\sigma_\pm) &= (-\mathrm{i}\hbar)(\pm\mathrm{i})\Big(\bar z_\pm\frac{\partial\psi_\pm}{\partial\bar z_\pm}-z_\pm\frac{\partial\psi_\pm}{\partial z_\pm}\Big)\sigma_\pm \\
&\quad +[\pm r(1-z_\pm\bar z_\pm)(1+z_\pm\bar z_\pm)^{-1}]\psi_\pm\sigma_\pm \\
&\quad -\Big\langle -2\mathrm{i}r(1+z_\pm\bar z_\pm)^{-1}\bar z_\pm \mathrm{d}z_\pm \mid \pm\mathrm{i}\Big(\bar z_\pm\frac{\partial}{\partial\bar z_\pm}-z_\pm\frac{\partial}{\partial z_\pm}\Big)\Big\rangle\psi_\pm\sigma_\pm \\
&= \pm\hbar\Big(\bar z_\pm\frac{\partial\psi_\pm}{\partial\bar z_\pm}-z_\pm\frac{\partial\psi_\pm}{\partial z_\pm}\Big)\sigma_\pm \pm s\hbar\frac{1-z_\pm\bar z_\pm}{1+z_\pm\bar z_\pm}\psi_\pm\sigma_\pm \\
&\quad \pm 2s\hbar\frac{\bar z_\pm z_\pm}{1+z_\pm\bar z_\pm}\psi_\pm\sigma_\pm \\
&= \pm\hbar\Big(\bar z_\pm\frac{\partial\psi_\pm}{\partial\bar z_\pm}-z_\pm\frac{\partial\psi_\pm}{\partial z_\pm}\Big)\sigma_\pm \pm s\hbar\psi_\pm\sigma_\pm.
\end{aligned}
$$

Note that the functions s^1, s^2 and s^3 on S_r^2 are the momenta corresponding to a basis (ξ_1,ξ_2,ξ_3) of $\mathfrak{so}(3)$. In other words, $s^i = J_{\xi_i} = \langle J \mid \xi_i\rangle$, where $J : S_r^2 \to \mathfrak{so}(3)^*$ is the inclusion map. The map $\xi \mapsto (\mathrm{i}/\hbar)\boldsymbol{P}_{J_\xi}$ is a representation of $\mathfrak{so}(3)$ by skew-hermitian operators on the Hilbert space $\mathfrak{H}$ of square integrable sections of L. This representation integrates to a unitary representation $\boldsymbol{U}$ of $\mathrm{SO}(3)$ on $\mathfrak{H}$, called the prequantization representation of $\mathrm{SO}(3)$.

4. Polarization

On the quantum theory side, prequantization fails to satisfy Heisenberg's Uncertainty Principle. On the representation theory side, the prequantization representation of a connected compact Lie group, e.g. $\mathrm{SO}(3)$, is unitary but not irreducible. Since unitary representations of a compact Lie group G decompose into a direct sum of irreducible unitary representations of G, we may think of imposing an additional condition of our quantization process in order to single out the desired irreducible unitary representation. This condition is given by a choice of polarization, which is a geometric quantization analogue of Dirac's complete family of commuting observables.

A complex distribution $F \subset T^{\mathbb{C}}P = \mathbb{C} \otimes TP$ on a symplectic manifold (P, ω) is Lagrangian if, for each $p \in P$, the restriction of the symplectic form ω to the subspace $F_p \subset T_p^{\mathbb{C}}P$ vanishes identically, and $\mathrm{rank}_{\mathbb{C}} F = \frac{1}{2} \dim P$. If F is a complex distribution on P, we denote its complex conjugate by $\overline{F}$. Let

$$D = F \cap \overline{F} \cap TP \qquad \text{and} \qquad E = (F + \overline{F}) \cap TP.$$

A polarization of (P, ω) is an involutive complex Lagrangian distribution F such that D and E are involutive distributions on P. The polarization F is said to be *strongly admissible* if the spaces P/D and P/E of integral manifolds of D and P, respectively, are quotients manifolds of P and the natural projection $P/D \to P/E$ is a submersion. A polarization F is *positive* if $\mathrm{i}\omega(w, \bar{w}) \geq 0$ for every $w \in F$. A positive polarization F is semi-definite if $\omega(w, \bar{w}) = 0$ for $w \in F$ implies that $w \in D^{\mathbb{C}}$.

Let F be a polarization of a symplectic manifold (P, ω). We denote by $\mathcal{C}^\infty(P)^0_F$ be the space of smooth complex valued functions on P that are constant along F, that is

$$\mathcal{C}^\infty(P)^0_F = \{f \in \mathcal{C}^\infty(P) \otimes \mathbb{C} \; ; \; uf = 0 \text{ for all } u \in F\}.$$

If F is strongly admissible then it is locally spanned by Hamiltonian vector fields of functions in $\mathcal{C}^\infty(P)^0_F$.

Let $\mathcal{C}^\infty_F(P)$ denote the space of functions on P whose Hamiltonian vector fields preserve F. In other words, $f \in \mathcal{C}^\infty_F(P)$ if, for every $h \in \mathcal{C}^\infty(P)^0_F$, the Poisson bracket $\{f, h\} \in \mathcal{C}^\infty(P)^0_F$. If $f_1, f_2 \in \mathcal{C}^\infty_F(P)$ and $h \in \mathcal{C}^\infty(P)^0_F$ then the Jacobi identity implies that

$$\{\{f_1, f_2\}, h\} = -\{f_2, \{f_1, h\}\} + \{f_1, \{f_2, h\}\} \in C^\infty(P)^0_F.$$

Hence, the ring $\mathcal{C}^\infty_F(P)$ is a Poisson subalgebra of $\mathcal{C}^\infty(P)$.

Let $\mathcal{S}^\infty_F(L)$ denote the space of smooth sections of L that are covariantly constant along F, namely

$$\mathcal{S}^\infty_F(L) = \{\sigma \in \mathcal{S}^\infty(L) \; ; \; \nabla_u \sigma = 0 \text{ for all } u \in F\}.$$

We shall refer to $\mathcal{S}^\infty_F(L)$ as the space of polarized sections. For each $h \in \mathcal{C}^\infty(P)^0_F$, $f \in \mathcal{C}^\infty_F(P)$ and $\sigma \in \mathcal{S}^\infty_F(L)$, we have $\nabla_{X_h}(\boldsymbol{Q}_f \sigma) = 0$. Thus, for every $f \in \mathcal{C}^\infty_F(P)$, the prequantization operator $\boldsymbol{P}_f$ maps $\mathcal{S}^\infty_F(L)$ to itself.

Definition 7. *The quantization map $\boldsymbol{Q}$ relative to a polarization F is the restriction of the prequantization map*

$$\boldsymbol{P} : \mathcal{C}^{\infty}(P) \times \mathcal{S}^{\infty}(L) \to \mathcal{S}^{\infty}(L) : (f, \sigma) \mapsto \boldsymbol{P}_f \sigma = (\mathrm{i}\hbar \nabla_{X_f} + f)\sigma$$

to domain $\mathcal{C}^{\infty}_F(P) \times \mathcal{S}^{\infty}_F(L) \subset \mathcal{C}^{\infty}(P) \times \mathcal{S}^{\infty}(L)$ and codomain $\mathcal{S}^{\infty}_F(L) \subset \mathcal{S}^{\infty}(L)$. In other words

$$\boldsymbol{Q} : \mathcal{C}^{\infty}_F(P) \times \mathcal{S}^{\infty}_F(L) \to \mathcal{S}^{\infty}_F(L) : (f, \sigma) \mapsto \boldsymbol{Q}_f \sigma = (\mathrm{i}\hbar \nabla_{X_f} + f)\sigma. \tag{19}$$

Assume that the action $\Phi : G \times P \to P$ preserves the polarization F. Hence, for each $\xi \in \mathfrak{g}$, the momentum J_ξ is in $\mathcal{C}^{\infty}(P)$. Restricting the prequantization representation to the Poisson algebra spanned by J_ξ, for $\xi \in \mathfrak{g}$, we get a representation $\xi \mapsto (i\hbar)^{-1}\boldsymbol{Q}_{J_\xi}$ of $\mathfrak{g}$ on $\mathcal{S}^{\infty}_F(L)$. If the action Φ of G on P lifts to an action of G on L by connection preserving automorphisms, then this representation of $\mathfrak{g}$ integrates to a linear representation

$$\boldsymbol{R} : G \times \mathcal{S}^{\infty}_F(L) \to \mathcal{S}^{\infty}_F(L) : (g, \sigma) \mapsto \boldsymbol{R}_g \sigma \tag{20}$$

of G on $\mathcal{S}^{\infty}_F(L)$. For each $g \in G$, $f \in \mathcal{C}^{\infty}(P)^0_F$ and $\sigma \in \mathcal{S}^{\infty}_F(L)$

$$\boldsymbol{R}_g(f\sigma) = (\Phi^*_{g^{-1}} f)\boldsymbol{R}_g \sigma.$$

We refer to $\boldsymbol{R} : G \times \mathcal{S}^{\infty}_F(L) \to \mathcal{S}^{\infty}_F(L)$ as the quantization representation of G.

4.1. Kähler Polarization

A Kähler polarization of (P, ω) is a strongly admissible polarization F such that $F \oplus \bar{F} = T^{\mathbb{C}}P$ and $\mathrm{i}\omega(w, \bar{w}) > 0$ for all non-zero $w \in F$. These assumptions imply that there is a complex structure $\boldsymbol{J}$ on P such that F is the space of antiholomorphic directions. Moreover, P is a Kähler manifold such that $-\omega$ is the Kähler form on P.

For a Kähler polarization F on (P, ω), the prequantization line bundle L over P is holomorphic and the space $\mathcal{S}^{\infty}_F(L)$ of polarized sections coincides with the space of holomorphic sections. Moreover, holomorphic sections of L, which are normalizable with respect to the scalar product (17), form a Hilbert space $\mathfrak{H}_F$. In other words,

$$\mathfrak{H}_F = \mathfrak{H} \cap S^{\infty}_F(L).$$

Hence, the linear representation $\boldsymbol{R}$ of G on $\mathcal{S}^{\infty}_F(L)$ gives rise to a unitary representation $\boldsymbol{U}$ of G on $\mathfrak{H}_F$.

Proposition 8. *A co-adjoint orbit (O, Ω) of a compact connected Lie group G admits a Kähler polarization.*

Proof: Since G is compact its Lie algebra $\mathfrak{g}$ admits a positive definite Ad_G-invariant metric $\boldsymbol{k}$, which allows for an identification of $\mathfrak{g}$ with $\mathfrak{g}^*$. Under this identification, co-adjoint orbits go to adjoint orbits. Hence, we can treat O as an adjoint orbit. For each $\xi \in O$, the tangent space $T_\xi O$ is the quotient of $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}_\xi$ of the isotropy group $H_\xi = \{g \in G \,;\, \mathrm{Ad}_g \xi = \xi\}$. The map $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g} : \zeta \mapsto [\xi, \zeta]$ preserves $\mathfrak{h}_\xi$ and it induces a map A_ξ of $T_\xi O$ onto itself. The map A_ξ is skew symmetric with respect to $\boldsymbol{k}$. Hence, eigenvalues of A_ξ are purely imaginary and half of them lie on the positive imaginary axis. Let $F_\xi \subset T_\xi O \otimes \mathbb{C}$ be the space spanned by these positive eigenvalues. One can show that the set $F = \cup_{\xi \in O} F_\xi \subset T^{\mathbb{C}} O$ is a Kähler polarization of the symplectic manifold (O, Ω). ■

Theorem 9. *Let O be a quantizable co-adjoint orbit. The unitary representation $\boldsymbol{U}$ of G on the Hilbert space $\mathcal{H}_F$, obtained by the quantization of (O, Ω) with respect to the Kähler polarization F, described in Proposition above, is irreducible. Moreover the map $O \mapsto \boldsymbol{U}$ is a bijection of the space of quantizable co-adjoint orbits of G onto the space of irreducible representations of G.*

This above result is the Borel-Weil Theorem in the formulation due to Kostant [23].

4.1.1. Quantization representation of $\mathrm{SO}(3)$

In Section 3.2.3, we showed that the representation space $\mathfrak{H}$ of the prequantization representation corresponding to integer spin s consists of square integrable sections σ of the prequantization line bundle L over S^2_r, where $r = s\hbar$. In $V_+ \cap V_- \subset S^2_r$

$$\sigma_{|V_\pm} = \psi_\pm \sigma_\pm$$

where $\sigma_\pm$ are trivializing sections of $L_{|V_\pm}$ such that

$$\langle \sigma_\pm, \sigma_\pm \rangle = \frac{1}{(1 + z_\pm \bar{z}_\pm)^{2s}}$$

and $\psi_\pm$ are complex valued functions on $V_\pm$

$$\psi_+(x) = z_-^{-2s}(x) \psi_-(x).$$

We observed that the functions z_+ and z_- define a complex structure on P. The distribution $F = \mathbb{C} \frac{\partial}{\partial \bar{z}_\pm}$ is a Kähler polarization of (S^2_{2s}, ω_{2s}). In this case $D = 0$ and $E = TS^2_{2s}$ so that F is strongly admissible. Moreover, F is positive semidefinite because

$$\begin{aligned} \mathrm{i}\omega_{|V_\pm} \left(\frac{\partial}{\partial \bar{z}_\pm}, \frac{\partial}{\partial z_\pm} \right) &= \mathrm{i} \left(-2\mathrm{i} r (1 + z_\pm \bar{z}_\pm)^{-2} \mathrm{d}\bar{z}_\pm \wedge \mathrm{d}z_\pm \right) \left(\frac{\partial}{\partial \bar{z}_\pm}, \frac{\partial}{\partial z_\pm} \right) \\ &= 2r (1 + z_\pm \bar{z}_\pm)^{-2} > 0. \end{aligned}$$

A section σ of L is holomorphic with respect to this complex structure if $\sigma_{|V_\pm} = \psi_\pm \sigma_\pm$, where ψ_+ and ψ_- depend only on the variables z_+ and z_-, respectively.

Theorem 10. *The space $\mathfrak{H}_F$ consisting of square integrable sections σ of L such that $\sigma_{|V_\pm} = \psi_\pm \sigma_\pm$, where ψ_+ and ψ_- are holomorphic functions of z_+ and z_-, respectively, is an invariant subspace of the prequantization representation of* $\mathrm{SO}(3)$.

Proof: If ψ_+ and ψ_- are holomorphic functions of z_+ and z_-, respectively, then $\frac{\partial \psi_\pm}{\partial z_\pm}$ are holomorphic functions of z_+ and z_-, respectively, and $\frac{\partial \psi_\pm}{\partial \bar{z}_\pm} = 0$. In this case, expressions for $\boldsymbol{P}_{s^1}$, $\boldsymbol{P}_{s^2}$ and $\boldsymbol{P}_{s^3}$ in Section 3.2.3 yield

$$\begin{aligned} \boldsymbol{P}_{s^1}(\psi_\pm \sigma_\pm) &= -\frac{\hbar}{2}\Big((z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\Big)\sigma_\pm + s\hbar z_\pm \psi_\pm \sigma_\pm \\ &= -\frac{\hbar}{2}(z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm}\sigma_\pm + s\hbar z_\pm \psi_\pm \sigma_\pm \end{aligned}$$

$$\begin{aligned} \boldsymbol{P}_{s^2}(\psi_\pm \sigma_\pm) &= \mp\frac{\mathrm{i}\hbar}{2}\Big((z_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial z_\pm} + (\bar{z}_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\Big)\sigma_\pm \pm \mathrm{i}s\hbar z_\pm \psi_\pm \sigma_\pm \\ &= \mp\frac{\mathrm{i}\hbar}{2}(z_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial z_\pm}\sigma_\pm \pm \mathrm{i}s\hbar z_\pm \psi_\pm \sigma_\pm \end{aligned}$$

$$\begin{aligned} \boldsymbol{P}_{s^3}(\psi_\pm \sigma_\pm) &= \pm\hbar\Big(\bar{z}_\pm \frac{\partial \psi_\pm}{\partial \bar{z}_\pm} - z_\pm \frac{\partial \psi_\pm}{\partial z_\pm}\Big)\sigma_\pm \pm \frac{s\hbar}{1 + z_\pm \bar{z}_\pm}\psi_\pm \sigma_\pm \\ &= \mp\hbar z_\pm \frac{\partial \psi_\pm}{\partial z_\pm}\sigma_\pm \pm s\hbar \psi_\pm \sigma_\pm . \end{aligned}$$

Thus, $\mathfrak{H}_0$ is invariant under the action of the operators $\boldsymbol{P}_{s^1}$,$\boldsymbol{P}_{s^2}$ and $\boldsymbol{P}_{s^3}$ on $\mathfrak{H}$. Since the representation of $\mathrm{SO}(3)$ on $\mathfrak{H}$ is obtained by integration of the operators $(\mathrm{i}/\hbar)\boldsymbol{P}_{s^1}$, $(\mathrm{i}/\hbar)\boldsymbol{P}_{s^2}$ and $(\mathrm{i}/\hbar)\boldsymbol{P}_{s^3}$, it follows that $\mathfrak{H}_0$ is invariant under the prequantization representation of $\mathrm{SO}(3)$. ■

The restriction of the prequantization representation $(\mathrm{i}/\hbar)\boldsymbol{P}$ to $\mathfrak{H}_F$ is the quantization representation $(\mathrm{i}/\hbar)\boldsymbol{Q}$ corresponding to spin s. It follows that

$$\begin{aligned} \boldsymbol{Q}_{s^1}(\psi_\pm \sigma_\pm) &= -\frac{\hbar}{2}(z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm}\sigma_\pm + s\hbar z_\pm \psi_\pm \sigma_\pm \\ \boldsymbol{Q}_{s^2}(\psi_\pm \sigma_\pm) &= \mp\frac{\mathrm{i}\hbar}{2}(z_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial z_\pm}\sigma_\pm \pm \mathrm{i}s\hbar z_\pm \psi_\pm \sigma_\pm \\ \boldsymbol{Q}_{s^3}(\psi_\pm \sigma_\pm) &= \mp\hbar z_\pm \frac{\partial \psi_\pm}{\partial z_\pm}\sigma_\pm \pm s\hbar \psi_\pm \sigma_\pm . \end{aligned}$$

4.2. Cotangent Polarization

Suppose that $P = T^*Q$ is the cotangent bundle of a manifold Q, ω is the canonical symplectic form of T^*Q, and the polarization F is the complexification of $\ker T\pi$, where $\pi : T^*Q \to Q$ is the cotangent bundle projection.

The canonical symplectic form of T^*Q is $\omega = \mathrm{d}\theta$, where θ is the Liouville form; see equation (3). Since ω is exact, the prequantization line bundle is trivial; that is $L = \mathbb{C} \times P$. We denote by $\sigma_0 : P \to L : p \mapsto (1, p)$ the trivializing section of L. We choose the covariant derivative operator ∇ such that

$$\nabla \sigma_0 = -\mathrm{i}\hbar^{-1}\theta \otimes \sigma_0.$$

Moreover, we normalize the Hermitian form $\langle \sigma_1, \sigma_2 \rangle$ appearing in equation (17) so that $\langle \sigma_0, \sigma_0 \rangle = 1$.

The space $\mathcal{C}^\infty(T^*Q)^0_F$ consists of complex-valued functions on $P = T^*Q$ that are constant along the fibres of the cotangent bundle projection. In other words

$$\mathcal{C}^\infty(P)^0_F = \{\pi^* f \; ; \; f \in (\mathbb{C} \otimes \mathcal{C}^\infty(Q))\}.$$

The space $\mathcal{C}^\infty_F(T^*Q)$ of functions whose Hamiltonian vector fields preserve $F = (\ker T\pi)^{\mathbb{C}}$ consists of functions on T^*Q that restrict to linear functions on fibres of the cotangent bundle projection $\pi : T^*Q \to Q$. The space $\mathcal{S}^\infty_F(L)$ of polarized sections of L is given by

$$\mathcal{S}^\infty_F(L) = \{\pi^*(\psi)\sigma_0 \; ; \; \psi \in \mathbb{C} \otimes \mathcal{C}^\infty(Q)\}.$$

By Definition 7, for every $f \in \mathcal{C}^\infty_F(T^*Q)$ and $\sigma = \pi^*(\psi)\sigma_0 \in \mathcal{S}^\infty_F(L)$

$$\boldsymbol{Q}_f \sigma = \boldsymbol{P}_f \sigma = (-\mathrm{i}\hbar\nabla_{X_f} + f)\pi^*\psi\sigma_0 = \{-\mathrm{i}\hbar X_f(\pi^*\psi) + (f - \langle \theta \mid X_f \rangle)\pi^*\psi\}\sigma_0. \tag{21}$$

In order to simplify equation (21), we use local coordinates $(q^1, ..., q^n)$ on Q and the corresponding coordinates $(q^1, ..., q^n, p_1, ..., p_n)$ in T^*P. In these coordinates, $\theta = p_k \mathrm{d}q^k$, $\omega = \mathrm{d}p_k \wedge \mathrm{d}q^k$

$$X_{q^k} = -\frac{\partial}{\partial p_k} \qquad \text{and} \qquad X_{p_k} = \frac{\partial}{\partial q_k}\cdot$$

Therefore, if $\pi^*\psi = \Psi(q^1, ..., q^n)$,

$$\begin{aligned} Q_{q^k}\Psi\sigma_0 &= \left\{ \mathrm{i}\hbar\frac{\partial \Psi}{\partial p_j} + \left(q^k - \left\langle p_j \mathrm{d}q^j \mid \frac{-\partial}{\partial p_k} \right\rangle \right))\Psi \right\} \sigma_0 = q^k \Psi \sigma_0 \\ Q_{p_k}\Psi\sigma_0 &= \left\{ -\mathrm{i}\hbar\frac{\partial \Psi}{\partial q_k} + \left(p_k - \left\langle p_j \mathrm{d}q^j \mid \frac{\partial}{\partial q_k} \right\rangle \right) \Psi \right\} \sigma_0 = -\mathrm{i}\hbar \frac{\partial \Psi}{\partial q_k}\sigma_0. \end{aligned}$$

Further, if $f = a(q^1, ..., q^n) + a^k(q^1, ..., q^n)p_k$, then

$$X_f = a^k \frac{\partial}{\partial q^k} - \frac{\partial a}{\partial q^i}\frac{\partial}{\partial p_k} - \frac{\partial a^k}{\partial q^j} p_k \frac{\partial}{\partial p_j}$$

and

$$f - \langle \theta \mid X_f \rangle = a + a^k p_k - \left\langle p_l \mathrm{d}q^l \mid a^k \frac{\partial}{\partial q^k} - \frac{\partial a}{\partial q^i}\frac{\partial}{\partial p_k} - \frac{\partial a^k}{\partial q^j} p_k \frac{\partial}{\partial p_j} \right\rangle = a.$$

Hence, equation (21) can be written as

$$\boldsymbol{Q}_f\sigma = \left(-\mathrm{i}\hbar a^k \frac{\partial \Psi}{\partial q^k} + a\Psi\right)\sigma_0. \tag{22}$$

For each $\sigma = \pi^*(\psi)\sigma_0$, we have

$$\langle \pi^*(\psi)\sigma_0, \pi^*(\psi)\sigma_0\rangle = \pi^*(\bar{\psi}\psi) = (\bar{\psi}\psi)\circ\pi.$$

Since the fibres of the cotangent bundle projection π are not compact, it follows that

$$\int_{T^*Q} \langle \sigma, \sigma\rangle\, \omega^n = \int_{T^*Q} \vartheta^*(\bar{\psi}\psi)\,\omega^n = \infty$$

unless $\sigma = 0$. Thus, $\mathfrak{H}_F = \mathfrak{H} \cap S_F^\infty(L) = 0$. This implies that by passing to polarized sections we have lost the scalar product in the space of polarized states.

4.2.1. Examples

Quantization of $T^*\mathbb{R}^3$. Let $P = T^*\mathbb{R}^3$ be the phase space of a particle with coordinates $(p_1, p_2, p_3, q^1, q^2, q^3)$, Liouville form

$$\theta = \sum_i p_i \mathrm{d}q^i$$

and symplectic form

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i.$$

Since ω is exact, the prequantization line bundle is trivial; that is $L = \mathbb{C}\times P$. We denote by

$$\sigma_0 : P \to L : (p_1, p_2, p_3, q^1, q^2, q^3) \mapsto (1, (p_1, p_2, p_3, q^1, q^2, q^3))$$

the trivializing section of L, and the covariant derivative operator ∇ such that

$$\nabla\sigma_0 = -\mathrm{i}\hbar^{-1}\left(\sum_i p_i \mathrm{d}q^i\right)\otimes\sigma_0.$$

Moreover, we normalize the Hermitian form $\langle\sigma_1, \sigma_2\rangle$ appearing in equation (17) so that $\langle\sigma_0, \sigma_0\rangle = 1$. We take $F = (\ker T\pi)\otimes\mathbb{C}$.

The representation space $\mathcal{S}_F^\infty(L)$ consists of sections of L of the form $\Psi(\boldsymbol{q})\sigma_0$, where $\Psi \in C^\infty(\mathbb{R}^3)$. The space $\mathcal{C}^\infty(P)_F$ of quantizable functions consists of linear functions in $\boldsymbol{p}$ with coefficients that are smooth functions of $\boldsymbol{q}$. Dynamical variables directly quantizable in this representation are linear functions of momenta with coefficients given by smooth functions of $\boldsymbol{q}$. In particular, we can quantize smooth functions $V(q)$ of the position variables $\boldsymbol{q} = (q^1, q^2, q^3)$, linear momentum $\boldsymbol{p} = (p_1, p_2, p_3)$, and angular momentum $\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{p}$. Equation (22) gives

$$\begin{aligned} (\boldsymbol{Q}_{V(\boldsymbol{q})}\Psi\sigma_0)(\boldsymbol{q}) &= V(\boldsymbol{q})\Psi(\boldsymbol{q})\sigma_0(\boldsymbol{q}) \\ (\boldsymbol{Q}_{\boldsymbol{p}}\Psi\sigma_0)(\boldsymbol{q}) &= -\mathrm{i}\hbar\,(\mathrm{grad}\Psi(\boldsymbol{q}))\,\sigma_0(\boldsymbol{q}) \\ (\boldsymbol{Q}_{\boldsymbol{J}}\Psi\sigma_0)(\boldsymbol{q}) &= -\mathrm{i}\hbar\boldsymbol{q} \times (\mathrm{grad}\Psi(\boldsymbol{q}))\,\sigma_0(\boldsymbol{q}). \end{aligned} \tag{23}$$

Since the kinetic energy function $K = \frac{1}{2}\boldsymbol{p}^2$ is a quadratic function of $\boldsymbol{p}$, the Hamiltonian vector field X_K of K does not preserve the polarization. Therefore, the approach presented here is insufficient to give quantization of energy. We shall discuss this problem below.

Schrödinger wave mechanics. In the Schrödinger formulation of wave mechanics of a single particle, the representation space is the space $L^2(\mathbb{R}^3)$ of square integrable complex valued functions Ψ of $\boldsymbol{q}$, and the quantization equations (23) are satisfied, with the factor σ_0 omitted. Moreover, the quantization of the kinetic energy is postulated to be given by the Laplace operator Δ. In other words,

$$\boldsymbol{Q}_K\Psi = -\frac{\hbar^2}{2}\Delta\Psi.$$

The scalar product in the Schrödinger theory is

$$(\Psi_1 \mid \Psi_2) = \int_{\mathbb{R}^3} \overline{\Psi}_1\Psi_2 \mathrm{d}q$$

where $\mathrm{d}q$ is the Lebesgue measure on $\mathbb{R}^3$. With this scalar product, the operators $\boldsymbol{Q}_{V(\boldsymbol{q})}$, $\boldsymbol{Q}_{\boldsymbol{p}}$, $\boldsymbol{Q}_{\boldsymbol{J}}$ and $\boldsymbol{Q}_K$ are self-adjoint.

The function $f = \boldsymbol{q} \cdot \boldsymbol{p}$, where $\cdot$ denotes the scalar product, is also quantizable and equation (22) adapted to the notation of this example gives

$$\boldsymbol{Q}_{\boldsymbol{q}\cdot\boldsymbol{p}}\Psi(\boldsymbol{q}) = -\mathrm{i}\hbar\boldsymbol{q} \cdot \mathrm{grad}\Psi(\boldsymbol{q}).$$

However, the operator $-\mathrm{i}\hbar\boldsymbol{q} \cdot \mathrm{grad}$ is not self-adjoint on $L^2(\mathbb{R}^3)$. The usual explanation in texts of quantum mechanics is that we should symmetrize this operator to obtain

$$\begin{aligned} \boldsymbol{Q}^{\mathrm{sym}}_{\boldsymbol{q}\cdot\boldsymbol{p}}\Psi(\boldsymbol{q}) &= -\frac{\mathrm{i}\hbar}{2}\left(\boldsymbol{q} \cdot \mathrm{grad}\Psi(\boldsymbol{q}) + \mathrm{div}(\boldsymbol{q}V(\boldsymbol{q}))\right) \\ &= -\mathrm{i}\hbar\boldsymbol{q} \cdot \mathrm{grad}\Psi(\boldsymbol{q}) - \frac{3}{2}\mathrm{i}\hbar\Psi(\boldsymbol{q}) \end{aligned} \tag{24}$$

which is self-adjoint.

4.2.2. Scalar product

We have seen in examples above that the approach presented above is insufficient to derive the Schrödinger quantization of the kinetic energy of a single particle, and that there are quantizable functions that do not give self-adjoint operators. Suppose that in the Schrödinger theory, we replace the Lebesgue measure $\mathrm{d}q$ by an absolutely continuous measure $\mu \mathrm{d}q$, where μ is a positive density on $\mathbb{R}^3$. Then

$$\begin{aligned}(\Psi_1 \mid (-\mathrm{i}\hbar \boldsymbol{q} \cdot \mathrm{grad}\Psi_2))_\mu &= \int_{\mathbb{R}^3} \overline{\Psi}_1(-\mathrm{i}\hbar \boldsymbol{q} \cdot \mathrm{grad}\Psi_2)\mu \mathrm{d}q \\ &= \int_{\mathbb{R}^3} \Big(\overline{-\mathrm{i}\hbar \boldsymbol{q} \cdot \mathrm{grad}\Psi_1(\boldsymbol{q})} \\ &\qquad + \mathrm{i}\hbar[3 + (\boldsymbol{q}^k \cdot \mathrm{grad}\mu)\mu^{-1}\overline{\Psi}_1]\Psi_2\Big)\,\mathrm{d}q.\end{aligned}$$

We can rewrite this equation in the form

$$\begin{aligned}&\left(\Psi_1 \mid -\mathrm{i}\hbar \boldsymbol{q} \cdot \mathrm{grad}\Psi_2 - \frac{\mathrm{i}\hbar}{2}[3 + \left(3 + (\boldsymbol{q}^k \cdot \mathrm{grad}\mu\right)\mu^{-1}]\Psi_2\right)_\mu \\ &\qquad = \left(-\mathrm{i}\hbar \boldsymbol{q} \cdot \mathrm{grad}\Psi_1 - \frac{\mathrm{i}\hbar}{2}[3 + (\boldsymbol{q}^k \cdot \mathrm{grad}\mu)\mu^{-1}]\Psi_1 \mid \Psi_2\right)_\mu\end{aligned}$$

which shows explicitly the dependence on μ of the correction term in equation (24).

We want to modify quantization rules so that the correction terms of this type are automatically included. This can be obtained by representing quantum states as tensor products $\Psi \otimes \sqrt{\mu \mathrm{d}q}$ and defining the scalar product of the states as

$$\left(\Psi_1 \otimes \sqrt{\mu \mathrm{d}q} \mid \Psi_2 \otimes \sqrt{\mu \mathrm{d}q}\right) = \int \int_{\mathbb{R}^3} \overline{\Psi}_1 \Psi_2 \mu \mathrm{d}q. \tag{25}$$

Equation (25) is purely symbolic. We have to define what we mean by $\sqrt{\mu \mathrm{d}q}$ and how we extract the density μ on the right hand side, which is beyond the scope of these lectures. For details see [4], [5] and [34].

4.3. Completely Integrable Systems

4.3.1. Action-angle coordinates

A completely integrable system on a symplectic manifold (P, ω) is given by a $n = \frac{1}{2}\dim P$ Poisson commuting functions $f_1, ..., f_n$ that are independent on a dense open subset P_0 of P. The span of the Hamiltonian vector fields of

$$f_1, ..., f_n$$

is a generalized involutive distribution

$$D = \mathrm{span}\{X_{f_1}, ..., X_{f_n}\}$$

D on P, which restricts to an involutive Lagrangian distribution $D_0 = D \cap TP_0$ on P_0. Thus, we may think of the complexification $F = D^{\mathbb{C}}$ as a singular polarization of (P, ω).

We assume that the integral manifolds of D are orbits of a Hamiltonian action of the torus group $\mathbb{T}^n$ on (P, ω) with momentum map $J : P \to (\mathfrak{t}^n)^*$, where $\mathfrak{t}^n$ is the Lie algebra of $\mathbb{T}^n$. Let $(\xi_1, ..., \xi_n)$ be a basis in $\mathfrak{t}^n$, such that all orbits of the action of $\exp t\xi_i$ on P_0 are periodic with (minimal) period 2π. The corresponding momenta

$$A_i = \langle J, \xi_i \rangle$$

are called actions and the parameter along the orbits of $\exp t\xi_i$ is called an angle corresponding to A_i and is denoted by φ_i, for details see [8]. For simplicity of presentation, we assume that

$$\omega = \mathrm{d}\theta$$

where the restriction of θ to P_0 is

$$\theta_0 = \sum_{i=1}^{n} (A_i \mathrm{d}\varphi_i).$$

Then, the restriction of ω to P_0 is

$$\omega_0 = \mathrm{d}\theta_0 = \mathrm{d} \sum_{i=1}^{n} (A_i \mathrm{d}\varphi_i) = \sum_{i=1}^{n} (\mathrm{d}A_i \wedge \mathrm{d}\varphi_i).$$

If the boundary $\partial P = P \backslash P_0$ is not empty, integral manifolds of D through points of ∂P are not Lagrangian; they are isotropic tori in (P, ω) of dimension smaller than $n = \frac{1}{2} \dim P$. Thus, completely integrable systems lead to quantization of system with respect to singular polarization $F = D^{\mathbb{C}}$.

4.3.2. Bohr-Sommerfeld quantization

In this section, we reformulate the approach of Bohr and Sommerfeld, discussed before, in the framework of geometric quantization of completely integrable systems that satisfy assumptions made above.

Since $\omega = d\theta$ is exact, the prequantization line bundle L is trivial, and we may use the trivialization section $\sigma_0 : P \to L : p \mapsto (1, p)$ as in the case of cotangent bundle. A section $\sigma = \psi\sigma_0$ is covariantly constant along F if $\nabla_{\frac{\partial}{\partial \varphi_j}} (\psi\sigma_0) = 0$ for all $j = 1, ..., n$. But

$$\nabla_{\frac{\partial}{\partial \varphi_j}} (\psi\sigma_0) = \frac{\partial \psi}{\partial \varphi_j} \sigma_0 + \psi \mathrm{i}\hbar^{-1} \left\langle \theta \mid \frac{\partial}{\partial \varphi_j} \right\rangle \sigma_0 = \left(\frac{\partial \psi}{\partial \varphi_j} + \mathrm{i}\hbar^{-1} A_j \psi \right) \sigma_0.$$

Hence, $\psi\sigma_0$ is covariantly constant along F if, for all $j = 1, ..., n$

$$\frac{\partial \psi}{\partial \varphi_j} + \mathrm{i}\hbar^{-1} A_j \psi = 0.$$

If, for $\psi(p) \neq 0$, then ψ does not vanish on the orbit O_p of $\mathbb{T}^n$ through p, and

$$\frac{\mathrm{d}\psi}{\psi} = -\mathrm{i}\hbar^{-1} A_j \mathrm{d}\varphi_j$$

along the orbit. Integrating this equation from 0 to 2π, along orbits $O_{j,p}$ of $\exp t\xi_j$ we get

$$\ln\psi \mid_{\varphi_j=0}^{\varphi_j=2\pi} = -\mathrm{i}\hbar^{-1} \int_0^{2\pi} A_j \mathrm{d}\varphi_j.$$

Since $\psi = \mathrm{e}^{\ln\psi}$ is single valued, it follows that $\ln\psi \mid_{\varphi_j=0} = \ln\psi \mid_{\varphi_j=2\pi} + 2\pi m\mathrm{i}$ for some integer m. Therefore

$$-\mathrm{i}\hbar^{-1} \int_0^{2\pi} A_j \mathrm{d}\varphi_j = \ln\psi \mid_{\varphi_j=0}^{\varphi_j=2\pi} = -2\pi m\mathrm{i}$$

and

$$\int_0^{2\pi} A_j \mathrm{d}\varphi_j = 2\pi m\hbar = mh. \tag{26}$$

Observe that we have recovered the Bohr-Sommerfeld condition of the old quantum theory. The collection of orbits of the action of $\mathbb{T}^n$ on P that satisfy the Bohr-Sommerfeld conditions (26) is called the Bohr-Sommerfeld set; individual orbits in the Bohr-Sommerfeld set are called Bohr-Sommerfeld tori.

Since the actions A_j are independent of φ_j, we conclude that if, for some $p \in P_0$, $\psi(p) \neq 0$, then

$$A_j(p) = m_j h/2\pi = m_j\hbar$$

for all $j = 1, ..., n$ and some integers $m_1, ..., m_n$. In this case

$$\frac{\mathrm{d}\psi}{\psi} = -\mathrm{i}m_j \mathrm{d}\varphi_j$$

and

$$\psi = \psi_0 \exp\left(-\mathrm{i}(m_1\varphi_1 + ... + m_n\varphi_n)\right)$$

where ψ_0 is a constant. Thus, there are no smooth sections of L that are covariantly constant along F, but there exist distribution sections covariantly constant along F that are supported on Bohr-Sommerfeld tori.

If $p \in \partial P = P \backslash P_0$, then the orbit of $\mathbb{T}^n$ through p is a torus of dimension smaller than n. In this case, not all actions are independent at p, and only some angle functions are well defined. Restricting the Bohr-Sommerfeld conditions to case, we get lower dimensional Bohr-Sommerfeld tori.

The Bohr-Sommerfeld set $\mathcal{S}$ in P is the union of $\mathbb{T}^n$-orbits that satisfy the Bohr-Sommerfeld conditions. For each $\boldsymbol{m} = (m_1, ..., m_n) \in \mathbb{Z}^n$, Bohr-Sommerfeld orbits $\mathcal{O}_{\boldsymbol{m}}$ with quantum numbers $\boldsymbol{m} = (m_1, ..., m_n)$ are connected components of the set

$$\mathcal{S}_{\boldsymbol{m}} = \{p \in P\ ;\ A_j(p) = m_j\hbar \ \text{ for } \ j = 1, ..., n\}.$$

The Bohr-Sommerfeld orbits in ∂P are connected components of level sets of $(f_1, ..., f_n)$ determined by a smaller number of Bohr-Sommerfeld conditions. To each Bohr-Sommerfeld orbit $\mathcal{O} \subseteq \mathcal{S}_{\boldsymbol{m}}$, we may associate a non-zero distribution section $\sigma_{\mathcal{O}}$ given by

$$\sigma_{\mathcal{O}}(p) = \begin{cases} \exp\left(-\mathrm{i}(m_1\varphi_1 + ... + m_n\varphi_n)\right)(p) & \text{if } p \in \mathcal{O} \\ 0 & \text{if } p \notin \mathcal{O}. \end{cases}$$

The collection $\{\sigma_{\mathcal{O}}\}$ of distribution sections of L forms a basis of an infinite dimensional vector space $\mathfrak{E}$, in which we may define a scalar product $(.\mid.)$ such that the basis $\{\sigma_{\mathcal{O}}\}$ is orthonormal. The Hilbert space $\mathfrak{H}$ of distribution sections in $\mathfrak{E}$ with of finite norm is the space of quantum states of Bohr-Sommerfeld quantization.

The space $C^\infty(P_0)_F$ of functions $f \in C^\infty(P_0)$ such that X_f preserves the polarization $F_0 = D_0^{\mathbb{C}}$ coincides with the space $C^\infty(P_0)_F^0$ of functions that are constant along D_0. Thus, quantizable functions in the Bohr-Sommerfeld theory are smooth functions of the action variables $A_1, ..., A_n$. For each $j = 1, ..., n$, the quantum operator $\boldsymbol{Q}_{A_j}$ corresponding to A_j is diagonal in the basis $\{\sigma_O\}$. For $\mathcal{O} \subseteq \mathcal{S}_{(m_1,...,m_n)}$

$$\boldsymbol{Q}_{A_j}\sigma_{\mathcal{O}} = m_j\hbar\sigma_{\mathcal{O}} \qquad \text{for} \qquad j = 1, ..., n.$$

Thus, the Bohr-Sommerfeld approach gives quantization only of functions on P that only depend on the actions. The corresponding operators are diagonal in the basis $\{\sigma_O\}$.

4.3.3. Bohr-Sommerfeld-Heisenberg quantization

In his 1925 paper [19], Heisenberg stressed the importance of operators of transitions between different states. However, our recipe of geometric quantization does not provide any such operators. On the other hand the infinite dimensional vector space $\mathfrak{E}$ with basis $\{\sigma_O\}$ has a natural structure of a local lattice defined by the Bohr-Sommerfeld conditions. For each $i = 1, ..., n$, a Bohr-Sommerfeld orbit $\mathcal{O}$ with quantum numbers $(m_1, ..., m_i, ..., m_n)$ has a predecessor $\mathcal{O}_i^-$ with quantum numbers $(m_1, ..., m_i - 1, ..., m_n)$ and a successor $\mathcal{O}_i^+$ with quantum numbers $(m_1, ..., m_i + 1, ..., m_n)$. If $\mathcal{O} \subset \partial P$, then some predecessors or successors of $\mathcal{O}$ may be empty sets. This *local lattice* structure of the Bohr-Sommerfeld set was discovered by Cushman and Duistermaat [10]. The Bohr-Sommerfeld-Heisenberg quantization was introduced in 2012 by Cushman and Śniatycki, see [12] and [13].

Global lattice

Suppose first that the local lattice structure described here is global, and that for every $\boldsymbol{m} \in \mathbb{Z}^n$, the set $\mathcal{S}_{\boldsymbol{m}}$ consists of a single Bohr-Sommerfeld orbit $\mathcal{O}$. In this case, we can label basic vectors not by Bohr-Sommerfeld orbits O but the corresponding quantum numbers $\boldsymbol{m}$. In other words, if $\mathcal{O} = \mathcal{S}_{\boldsymbol{m}}$, we write $\sigma_{\mathcal{O}} = \sigma_{\boldsymbol{m}}$.

For each $i = 1, ..., n$, let

$$\mathbf{m}_i = \{m_1, ..., m_{i-1}, m_i - 1, m_{i+1}, ..., m_n\}$$

and

$$\mathbf{m}^i = \{m_1, ..., m_{i-1}, m_i + 1, m_{i+1}, ..., m_n\}.$$

We define shifting operators $\mathbf{a}_i$ on $\mathfrak{H}$ by

$$\mathbf{a}_i \sigma_{\mathbf{m}} = \sigma_{\mathbf{m}_i}.$$

The adjoint operators $\mathbf{a}_i^\dagger$ are given by

$$\mathbf{a}_i^\dagger \sigma_{\mathbf{m}} = \sigma_{\mathbf{m}^i}.$$

The Poisson bracket relations between actions and angles are

$$\{\mathrm{e}^{-\mathrm{i}\varphi_k}, A_j\} = -\mathrm{i}\delta_{kj}\, \mathrm{e}^{-\mathrm{i}\varphi_k}.$$

Hence, Dirac's quantization conditions

$$[\mathbf{Q}_{f_1}, \mathbf{Q}_{f_2}] = \mathrm{i}\hbar\, \mathbf{Q}_{\{f_1, f_2\}}$$

suggest the identification $\mathbf{a}_k = \mathbf{Q}_{\mathrm{e}^{-\mathrm{i}\varphi_k}}$ and $\mathbf{a}_k^\dagger = \mathbf{Q}_{\mathrm{e}^{\mathrm{i}\varphi_k}}$, where φ_k is the angle coordinate corresponding to the action A_k, provided that the exponential functions $\mathrm{e}^{-\mathrm{i}\varphi_k}$ are globally defined.

Globalization. In reality, the exponential functions $\mathrm{e}^{-\mathrm{i}\varphi_k}$ are not defined on all of P, but they are defined on the open dense subset P_0 of P. We can try to replace $\mathrm{e}^{-\mathrm{i}\varphi_k}$ by a globally defined smooth function $\chi_k = r_k \mathrm{e}^{-\mathrm{i}\varphi_k}$, where the coefficient r_k depends only on the actions and vanishes at the points at which $\mathrm{e}^{-\mathrm{i}\varphi_k}$ is not defined.

We have the following Poisson bracket relations

$$\{\chi_k, A_j\} = -\mathrm{i}\delta_{kj}\, \chi_k \qquad \text{and} \qquad \{\bar{\chi}_k, A_j\} = \mathrm{i}\delta_{kj}\, \bar{\chi}_k.$$

By Dirac's quantization conditions, we get

$$[\mathbf{Q}_{\chi_k}, \mathbf{Q}_{A_j}] = \delta_{kj}\hbar\, \mathbf{Q}_{\chi_k}, \qquad [\mathbf{Q}_{\bar{\chi}_k}, \mathbf{Q}_{A_j}] = -\delta_{kj}\hbar\, \mathbf{Q}_{\bar{\chi}_k}.$$

For each basic vector $\sigma_{\mathbf{m}}$ of $\mathfrak{H}$,

$$\begin{aligned} \mathbf{Q}_{A_j}(\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}) &= \mathbf{Q}_{\chi_j}(\mathbf{Q}_{A_j}\sigma_{\mathbf{m}}) - [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{A_j}]\sigma_{\mathbf{m}} \\ &= \mathbf{Q}_{\chi_j}(\hbar m_j \sigma_{\mathbf{m}}) - \hbar \mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} = \hbar(m_j - 1)\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}. \end{aligned}$$

Thus, $\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}$ is proportional to $\sigma_{\mathbf{m}_j}$. A similar argument shows that $\mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}}$ is proportional to $\sigma_{\mathbf{m}^j}$. Hence, $\mathbf{Q}_{\chi_j}$ and $\mathbf{Q}_{\bar{\chi}_j}$ act as shifting operators, namely

$$\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} = b_{\mathbf{m},j}\sigma_{\mathbf{m}_j} \qquad \text{and} \qquad \mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}} = c_{\mathbf{m},j}\sigma_{\mathbf{m}^j}$$

for some coefficients $b_{\mathbf{m},j}$ and $c_{\mathbf{m},j}$.

We can use Dirac's quantization conditions

$$[\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\chi_k}] = \mathrm{i}\hbar\, \mathbf{Q}_{\{\chi_j,\chi_k\}} \qquad \text{and} \qquad [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\bar{\chi}_k}] = \mathrm{i}\hbar\, \mathbf{Q}_{\{\chi_j,\bar{\chi}_k\}}$$

and the identification

$$\mathbf{Q}^{\dagger}_{\chi_j} = \mathbf{Q}_{\bar{\chi}_j}$$

to determine the coefficients $b_{\mathbf{m},j}$ and $c_{\mathbf{m},j}$, which must satisfy the consistency conditions

$$b_{\mathbf{m},j} = 0 \quad \text{if} \quad \mathcal{S}_{\mathbf{m}_j} = \emptyset \qquad \text{and} \qquad c_{\mathbf{m},j} = 0 \quad \text{if} \quad \mathcal{S}_{\mathbf{m}^j} = \emptyset.$$

The choice of functions χ_j depends on the completely integrable system under consideration.

The procedure described here assumes i) the local lattice structure of the Bohr-Sommerfeld set can has no monodromy and it extends to a global lattice structure with singularities, ii) the local lattice structure of the set eigenstates of the action operators are simple, so that there is a basis in $\mathfrak{H}$ parametrized by quantum numbers. Our approach does not apply to a completely integrable system with monodromy. However, in presence of monodromy, we may apply this procedure to the double covering of the classical system.

4.3.4. Examples

Co-adjoint orbits of $\mathrm{SO}(3)$ **revisited**

In Section 3.2.2, we constructed the prequantization line bundle L over a sphere

$$S^2_r = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \; ; \; (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2\}$$

of radius $r = \frac{n}{2}\hbar$, where n is an integer. The symplectic form on S^2_r is $\omega = \frac{1}{r}\mathrm{vol}_{S^2_r}$, where $\mathrm{vol}_{S^2_r}$ is the standard area form on S^2_r with $\int_{S^2_r} \mathrm{vol}_{S^2_r} = 4\pi r^2$.

For each $i = 1, 2, 3$, the restriction of x^i to the sphere S^2_r are components s^i of the spin vector $\boldsymbol{s}$.They satisfy the Poisson bracket relations $\{s^i, s^j\} = \sum_{k=1}^{3} \varepsilon_{ijk} s^k$. In spherical polar coordinates

$$s^1 = r \sin\theta \cos\varphi, \qquad s^2 = r\sin\theta\sin\varphi, \qquad s^3 = r\cos\theta$$

and

$$\omega = r\sin\theta \mathrm{d}\varphi \wedge \mathrm{d}\theta = -(r\cos\theta \mathrm{d}\varphi) = -s^3\mathrm{d}\varphi.$$

Thus, $(J^3, -\varphi)$ are action-angle coordinates for an integrable system (J^3, S^2_r, ω). In this case, we can choose $\chi = \sqrt{r^2 - (s^3)^2}\, \mathrm{e}^{\mathrm{i}\varphi}$. The resulting Bohr-Sommerfeld-Heisenberg quantization leads to the irreducible unitary representation of $\mathrm{SO}(3)$ corresponding to the co-adjoint orbit S^2_r.

The Bohr-Sommerfeld-Heisenberg quantization of co-adjoint orbits of $\mathrm{SO}(3)$ presented here looks very much like the construction of a highest weight representation of $\mathrm{SO}(3)$. In fact, it is a generalization of this construction, in which we get not only the highest weight representation but also identify functions on the orbit that correspond to shifting operators. It closely resembles the approach of Schwinger [31]. For more details, see [12].

Mathematical pendulum

The phase space of mathematical pendulum is the cotangent bundle space T^*S^1 of a circle S^1, with coordinates (p, α), symplectic form $\omega = \mathrm{d}\theta$, where $\theta = p\mathrm{d}\alpha$, and the Hamiltonian $H = \frac{1}{2}p^2 - \cos\alpha + 1$. Integral curves of the Hamiltonian vector field X_H of H give rise to a singular distribution D on P with singular leaves occurring at $H = 0$ and $H = 2$. The level set $H^{-1}(0)$ is a stable equilibrium at $(0, 0)$ and the level set $H^{-1}(2)$ is the union of an unstable equilibrium at $(0, \pi)$ and two homoclinic orbits. This section is based on [14].

The homoclinic orbits are the only non-compact orbits of X_H. However, the closures of homoclinic orbits are compact and we can apply Bohr-Sommerfeld conditions to all orbits of X_H. The integral of $\theta = p\mathrm{d}\alpha$ along each homoclinic orbit is 8. Hence, homoclinic orbits satisfy Bohr-Sommerfeld conditions if $\hbar = \frac{8}{n}$ for some integer n. Therefore, we may assume that $H^{-1}(2)$ is not a Bohr-Sommerfeld set. This implies that there is the largest quantum number N such that, $N\hbar < 8$. Hence

$$H_{|\mathcal{S}_n} \leq 2 \quad \text{for} \;\; n \leq N, \qquad H_{|\mathcal{S}_n} > 2 \quad \text{for} \quad n > N.$$

For $n \leq N$, Bohr-Sommerfeld sets $\mathcal{S}_n$ consist of a single orbit. On the other hand, $\mathcal{S}_n$ is the union of two disjoint orbits for $n > N$, which differ by the sign of p. Thus, we are lead to the following decomposition

$$T^*S = P_0 \cup P_+ \cup P_- \cup H^{-1}(2)$$

where

$$\begin{aligned} P_0 &= \{(p, \alpha) \in T^*S^1 \;;\; H(p, \alpha) < 2\} \\ P_\pm &= \{(p, \alpha) \in T^*S^1 \;;\; H(p, \alpha) > 2, \; \pm p > 0\}. \end{aligned}$$

The usual definition of an action gives a function I on $T^*S^1 \backslash H^{-1}(2)$ that can be continuously extended across $H^{-1}(2)$, but has no smooth extension. Using the assumption that $H^{-1}(2)$ is not a Bohr-Sommerfeld set, we show that there is $\varepsilon > 0$ such that $H^{-1}((2-\varepsilon, 2+\varepsilon))$ does not contain Bohr-Sommerfeld orbits. Moreover, we construct smooth functions $A_\pm$ on T^*S^1 such that

$$A_\pm(p, \alpha) = \begin{cases} I(p, \alpha) & \text{if} \quad 0 \leq H(p, \alpha) \leq 2 - \varepsilon \\ I(p, \alpha) & \text{if} \qquad\;\; H(p, \alpha) \geq 2 + \varepsilon \\ \;\;0 & \text{if} \qquad\;\; (p, \alpha) \in P_\mp \end{cases}$$

and $A_\pm$ vanish identically in a neighborhood of $H^{-1}(2)$. Thus, in neighborhoods of Bohr-Sommerfeld tori, functions $A_\pm$ coincide with the restrictions of I to $P_0 \cup P_\pm$.

In a similar way, we construct smooth functions $\Theta_\pm$ which, in neighborhoods of Bohr-Sommerfeld tori, coincide with the angle variables restricted to $P_0 \cup P_\pm$. Further, we construct non-negative functions $R_\pm$ of H, which vanish to infinite order at $H^{-1}(0)$ and $H^{-1}(2) \cup P_\mp$ and are equal to one on Bohr-Sommerfeld tori contained in $P_0 \cup P_\pm$. Finally, we show that quantizations of $\chi_\pm = R_\pm \mathrm{e}^{\mathrm{i}\Theta_\pm}$ yield operators of shifting along the lattice of Bohr-Sommerfeld tori in $P_0 \cup P_\pm$.

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