Seventeenth International Conference on
Geometry, Integrability and Quantization
June 5–10, 2015, Varna, Bulgaria
Ivaïlo M. Mladenov, Guowu Meng
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2016, pp 72–94
doi: 10.7546/giq-17-2016-72-94

GENERALIZED KEPLER PROBLEMS AND EUCLIDEAN JORDAN ALGEBRAS

GUOWU MENG

Department of Mathematics, Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Abstract. This article is a written version of author's lecture on generalized Kepler problems at the XVII-th International Conference on Geometry, Integrability and Quantization, June 5-10, 2015 Varna, Bulgaria. It begins with a review of the Kepler problem for planetary motion and its magnetized cousins, from which a surprising relationship with Lorentz transformation emerges. Next, we give a review for euclidean Jordan algebra and the associated universal Kepler problem. Finally, we demonstrate that, via the universal Kepler problem, a suitable Poisson realization of the conformal algebra for a simple euclidean Jordan algebra gives rise to a super integrable model that resembles the Kepler problem. In particular we demonstrate how the Kepler problem and its magnetized cousins are obtained this way.

MSC: 70Hxx, 17Cxx, 17Bxx, 37J35
Keywords: Jordan algebra, Kepler problem, Lorentz transformation, Poisson bracket, principal bundle, super integrable models, symmetric cone

CONTENTS

1. Introduction

The Kepler problem, introduced by Isaac Newton in the 17th century, is the mathematical model for the simplest solar system. As a classical dynamic problem, its configuration space is $\mathbb{R}^3_*$ (i.e., $\mathbb{R}^3$ with the origin removed) and its equation of motion is

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3}\cdot$$

Here, $\mathbf{r}$ is a $\mathbb{R}^3_*$-valued function of time t, $\mathbf{r}''$ is the second time derivative of $\mathbf{r}$ and r is length of $\mathbf{r}$. So, its hamiltonian is

$$\mathrm{H} = \frac{1}{2}|\mathbf{p}|^2 - \frac{1}{r}\cdot \tag{1}$$

Here, $\mathbf{p}$ is the (linear) momentum. Since the hamiltonian H is invariant under rotations of $\mathbb{R}^3$, thanks to Noether's theorem, the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

is conserved.

In 1911 E. Rutherford established the Kepler problem as the mathematical model for the simplest atom (i.e., hydrogen atom) via his famous alpha-particle scattering experiment [13], albeit it makes sense only at the quantum level.

The simplicity and beauty of the Kepler problem (or rather the inverse square law) comes from the conserved Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}\cdot$$

For example, via $\mathbf{A}$, we can obtain the trajectories quickly without actually solving the equation of motion.

The primary goal in this lecture is to convince the readers that the mathematical secret for the Kepler problem lies in the euclidean Jordan algebra [3]. (Historically Jordan introduced this algebra for a better formulation of the quantum mechanics, but that effort failed.) Once this secret is known, we have a general theory, under the name of *generalized Kepler problems* [8]. In this general theory, an isotropic 3D oscillator problem is just the bounded sector of a generalized Kepler problem, and its Fradkin tensor [2] is just the generalized Laplace-Runge-Lenz vector of the generalized Kepler problem. It is in this sense that we say that the Kepler problem and the oscillator problem are unified in the theory of generalized Kepler problems.

2. Kepler Problem and its Magnetized Cousins

Towards the end of 1960s, two groups of researchers [7, 16] independently discovered a family of super-integrable models which all resemble the Kepler problem. A model in this family, specified by a real parameter μ, describes a hypothetical hydrogen atom where the nucleus, besides carrying the unit electric charge e, carries magnetic charge μ as well. Here is its equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \mu^2\frac{\mathbf{r}}{r^4} - \mathbf{r}' \times \mu\frac{\mathbf{r}}{r^3}\cdot \tag{2}$$

These magnetized cousins of the Kepler problem, being usually referred to as the MICZ-Kepler problems in honor of their discoverers H. McIntosh, A. Cisneros and D. Zwanziger, share the characteristic beauty and simplicity of the Kepler problem. For example, we have the conserved angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{r}' + \mu\frac{\mathbf{r}}{r}$$

as well as the conserved Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}\cdot$$

Indeed, using equation (2) we have

$$\begin{aligned}\mathbf{L}' &= \mathbf{r} \times \mathbf{r}'' + \mu\left(\frac{\mathbf{r}}{r}\right)' = -\mathbf{r} \times \left(\mathbf{r}' \times \mu\frac{\mathbf{r}}{r^3}\right) + \mu\left(\frac{\mathbf{r}}{r}\right)' \\ &= \mu\left((\mathbf{r} \times \mathbf{r}') \times \left(-\frac{\mathbf{r}}{r^3}\right) + \left(\frac{\mathbf{r}}{r}\right)'\right) = \mu\left((\mathbf{r} \cdot \mathbf{r}')\frac{\mathbf{r}}{r^3} - \frac{\mathbf{r}'}{r} + \left(\frac{\mathbf{r}}{r}\right)'\right) = \mathbf{0}\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}' &= \mathbf{L}\times\mathbf{r}'' + \left(\frac{\mathbf{r}}{r}\right)' = (\mathbf{r}\times\mathbf{r}')\times\mathbf{r}'' + \mu\frac{\mathbf{r}}{r}\times\mathbf{r}'' + \left(\frac{\mathbf{r}}{r}\right)'\\
&= (\mathbf{r}\times\mathbf{r}')\times\left(-\frac{\mathbf{r}}{r^3} + \mu^2\frac{\mathbf{r}}{r^4}\right) + \mu\frac{\mathbf{r}}{r}\times\left(-\mathbf{r}'\times\mu\frac{\mathbf{r}}{r^3}\right) + \left(\frac{\mathbf{r}}{r}\right)'\\
&= (\mathbf{r}\times\mathbf{r}')\times\left(-\frac{\mathbf{r}}{r^3}\right) + \left(\frac{\mathbf{r}}{r}\right)' = \mathbf{0}.
\end{aligned}$$

It is easy to see that $\mathbf{L}\cdot\mathbf{A} = \mu$, and

$$\mathbf{L}\cdot\mathbf{r} = \mu r, \qquad r - \mathbf{A}\cdot\mathbf{r} = |\mathbf{L}|^2 - \mu^2. \tag{3}$$

Equation (3) gives an algebraic description for the orbits, from which, we deduce that a non-colliding orbit is elliptic, parabolic, and hyperbolic according as $|\mathbf{A}|$ is less than, equal to, and bigger than one, though $|\mathbf{A}|$ may not be equal to the eccentricity of the conic.

Note that, for a motion with non-colliding orbit, $|\mathbf{L}|^2 - \mu^2 = |\mathbf{r}\times\mathbf{r}'|^2 > 0$ and the total energy $E = \frac{1}{2}|\mathbf{r}'|^2 + \frac{\mu^2}{2r^2} - \frac{1}{r}$ is completely determined by $\mathbf{L}$ and $\mathbf{A}$

$$E = -\frac{1-|\mathbf{A}|^2}{2(|\mathbf{L}|^2-\mu^2)}. \tag{4}$$

Indeed, since

$$\begin{aligned}
|\mathbf{A}|^2 &= |\mathbf{L}\times\mathbf{r}'|^2 + 2\frac{\mathbf{r}\cdot(\mathbf{L}\times\mathbf{r}')}{r} + 1\\
&= |\mathbf{L}|^2|\mathbf{r}'|^2 - (\mathbf{L}\cdot\mathbf{r}')^2 - 2\frac{\mathbf{L}}{r}\cdot\left(\mathbf{L} - \mu\frac{\mathbf{r}}{r}\right) + 1\\
&= |\mathbf{L}|^2|\mathbf{r}'|^2 - \mu^2 r'^2 - 2\frac{|\mathbf{L}|^2}{r} + 2\frac{\mu^2}{r} + 1 \quad \text{(using the identity } \mathbf{L}\cdot\mathbf{A} = \mu r\text{)}\\
&= 2|\mathbf{L}|^2\left(\frac{1}{2}|\mathbf{r}'|^2 - \frac{1}{r}\right) - \mu^2\frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^2} + 2\frac{\mu^2}{r} + 1\\
&= 2|\mathbf{L}|^2 E - \mu^2\frac{|\mathbf{L}|^2}{r^2} - \mu^2\frac{|\mathbf{r}\times\mathbf{r}'|^2 - |\mathbf{r}|^2|\mathbf{r}'|^2}{r^2} + 2\mu^2\frac{1}{r} + 1\\
&= 2(|\mathbf{L}|^2-\mu^2)E + 1 \qquad \text{(using the identity } |\mathbf{L}|^2 = \mu^2 + |\mathbf{r}\times\mathbf{r}'|^2\text{)}
\end{aligned}$$

we have $E = -\frac{1-|\mathbf{A}|^2}{2(|\mathbf{L}|^2-\mu^2)}$.

2.1. A New Description for the Non-Colliding Orbits

Equation (3) can be rewritten as

$$\mu r - \mathbf{L}\cdot\mathbf{r} = 0, \qquad r - \mathbf{A}\cdot\mathbf{r} = |\mathbf{L}|^2 - \mu^2. \tag{5}$$

Assume that the orbit is non-colliding, i.e., $|\mathbf{L}|^2 - \mu^2 = |\mathbf{r} \times \mathbf{r}'|^2 > 0$. Then, we can introduce Lorentz four-vectors

$$l = \frac{1}{\sqrt{|\mathbf{L}|^2 - \mu^2}}(\mu, \mathbf{L}), \qquad a = \frac{1}{|\mathbf{L}|^2 - \mu^2}(1, \mathbf{A}), \qquad x = (r, \mathbf{r}) \tag{6}$$

so that equation (5) can be put in this form

$$l \cdot x = 0, \qquad a \cdot x = 1. \tag{7}$$

Here, $\cdot$ between Lorentz four-vectors is the Lorentz dot product

$$(x_0, \mathbf{x}) \cdot (y_0, \mathbf{y}) = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}.$$

It is easy to see that $l^2 := l \cdot l = -1$, $l \cdot a = 0$, $a_0 > 0$, and formula (4) for the total energy becomes

$$E = -\frac{a^2}{2a_0}\cdot \tag{8}$$

Note that equation (7) is for $\mathbf{r} \in \mathbb{R}^3_*$, but it is also for $x \in \mathbb{R}^{1,3}$ – the 4D Lorentz space, provided that x is on the future light cone

$$\Lambda_+ := \{x \in \mathbb{R}^{1,3};\, x^2 = 0, x_0 > 0\}.$$

Therefore, by lifting to the future light cone Λ_+, a non-colliding orbit becomes the intersection curve of the future light cone with an affine plane of the form given by equation (7), hence it must be a conic by the original ancient Greek's definition of conics. One can verify that this conic is an ellipse, a parabola, and a branch of hyperbola according as the total energy E is negative, zero, and positive.

2.2. MICZ-Kepler Problems and Lorentz Transformations

The new description for the non-colliding orbits has an advantage: it enables us to see a connection between the MICZ-Kepler problems and Lorentz transformations. The intuitive reason is that a Lorentz transformation leaves the light cone invariant and turns an affine plane into another affine plane, so it turns a non-colliding orbit into another non-colliding orbit.

To state this connection precisely, let us first introduce a few terminologies. A non-colliding orbit in a MICZ-Kepler problem shall be referred to as a **MICZ-Kepler orbit**. There are three types of such orbit: elliptic, parabolic, and hyperbolic. A Lorentz transformation is called **small** if it can be continuously deformed to the identity map on $\mathbb{R}^{1,3}$. The scalar multiplication of vectors in $\mathbb{R}^{1,3}$ by a positive real number shall be referred to as a **scaling transformation**. Note that each MICZ-Kepler orbit can be oriented by the velocity of the motion.

Theorem 1 (Meng, 2011)**.** *For the* MICZ-Kepler *orbits, the following statements are true.*

a) *Any two oriented parabolic* MICZ-Kepler *orbits can be transformed from one to the other via a little Lorentz transformation.*
b) *Any two oriented elliptic* MICZ-Kepler *orbits can be transformed from one to the other via a little Lorentz transformation together with a scaling transformation.*

The proof of this theorem can be found in [10].

3. Euclidean Jordan Algebra

We have learned that the MICZ-Kepler orbits have a very attractive description on the future light cone Λ_+. Is this a coincidence? More precisely, one may ask this

Question: Can the Kepler problem and its magnetized cousins be naturally formulated on the future light cone Λ_+?

Answer: Yes, provided that we can employ the more refined Jordan algebra structure behind the Lorentz structure on the Lorentz space $\mathbb{R}^{1,3}$. Since

$$\mathbb{R}^3_* \to \Lambda_+, \qquad \mathbf{r} \mapsto (r, \mathbf{r})$$

is a diffeomorphism, in hindsight, this may not be a surprise at all.

To know the details, we have to take a detour.

3.1. The Formally Real Jordan Algebra Structure on $\mathbb{R}^{1,3}$

Write $\mathbf{x} = (x_1, x_2, x_3)$, then $x = (x_0, x_1, x_2, x_3)$. Write X for $x_0 I + \mathbf{x}\cdot\sigma$, i.e.,

$$X = \begin{bmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{bmatrix}.$$

Let $\mathrm{H}_2(\mathbb{C})$ is the set of all complex hermitian matrices of order two. Note that $\det X = x^2$ and the map $x \mapsto X$ is an isometry between the Lorentz space $\mathbb{R}^{1,3}$ and $(\mathrm{H}_2(\mathbb{C}), \det)$. Introducing the symmetrized matrix multiplication

$$X \circ Y := \frac{1}{2}(XY + YX)$$

we observe that the symmetrized matrix multiplication turns $\mathrm{H}_2(\mathbb{C})$ into a *real commutative algebra with unit*. Moreover this algebra is

a) *weakly associative*. This means that, for X, Y in $\mathrm{H}_2(\mathbb{C})$, we have $(X \circ Y) \circ X^2 = X \circ (Y \circ X^2)$. Here, $X^2 = X \circ X = XX$
b) *formally real*. This means that, for X, Y in $\mathrm{H}_2(\mathbb{C})$, $X^2 + Y^2 = 0 \implies X = Y = 0$.

Using the correspondence $x \leftrightarrow X$, one can see that the corresponding Jordan algebra structure behind $\mathbb{R}^{1,3}$ is the one given by the following multiplication rule

$$(x_0, \mathbf{x}) \circ (y_0, \mathbf{y}) = (x_0 y_0 + \mathbf{x} \cdot \mathbf{y}, x_0 \mathbf{y} + y_0 \mathbf{x}).$$

This Jordan algebra is customarily denoted by $\Gamma(3)$.

3.2. The Euclidean Structure on $\Gamma(3)$

Up to scaling, there is a unique inner product on $\mathrm{H}_2(\mathbb{C})$ with respect to which the multiplication by any $u \in \mathrm{H}_2(\mathbb{C})$ is a self-adjoint operator on $\mathrm{H}_2(\mathbb{C})$. One should view $\mathrm{H}_2(\mathbb{C})$ as the analogue of a real compact simple Lie algebra and the inner product mentioned here as the analogue of the negative-definite Killing form in a real compact simple Lie algebra.

To be precise, for any $u \in \mathrm{H}_2(\mathbb{C})$, we let L_u be the endomorphism on $\mathrm{H}_2(\mathbb{C})$ defined by $v \mapsto u \circ v$. Let $\langle \mid \rangle \colon \mathrm{H}_2(\mathbb{C}) \times \mathrm{H}_2(\mathbb{C}) \to \mathbb{R}$ be defined as follows

$$\langle u \mid v \rangle := \frac{1}{2} \operatorname{tr}(u \circ v) = \frac{1}{2} \operatorname{tr}(uv) = \frac{1}{4} \operatorname{tr} L_{u \circ v}.$$

One can verify that

i) $\langle \mid \rangle$ is an inner product on $\mathrm{H}_2(\mathbb{C})$ such that

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthonormal basis. Note that $\operatorname{tr} \sigma_0 = 2$ and $\operatorname{tr} \sigma_i = 0$.

ii) L_u is self-adjoint with respect to $\langle \mid \rangle$, i.e., $\langle v \mid u \circ w \rangle = \langle u \circ v \mid w \rangle$ for any $v, w \in \mathrm{H}_2(\mathbb{C})$.

iii) The multiplication law for $\mathrm{H}_2(\mathbb{C})$ is given by

$$\sigma_i \circ \sigma_j = \delta_{ij} \sigma_0, \qquad \sigma_0 \text{ is the unit } e.$$

Under the correspondence $x \leftrightarrow X$, we have the correspondence $\mathbf{e}_\mu \leftrightarrow \sigma_\mu$ where

$$\mathbf{e}_0 = (1,0,0,0), \quad \mathbf{e}_1 = (0,1,0,0), \quad \mathbf{e}_2 = (0,0,1,0), \quad \mathbf{e}_3 = (0,0,0,1).$$

Therefore, the corresponding inner product on $\Gamma(3)$ is the standard dot product on $\mathbb{R}^4$.

In summary, on the real vector space $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$, there is an algebra structure given by the multiplication law

$$\mathbf{e}_i \mathbf{e}_j = \delta_{ij} \mathbf{e}_0, \qquad \mathbf{e}_0 \text{ is the multiplicative unit } e$$

and a compatible euclidean structure given by the dot product. (Here, compatible means that the multiplication by any element u is self-adjoint with respect to this dot product) Moreover, this algebra (denoted by $\Gamma(3)$) is commutative, weakly associated, formally real, isomorphic to $\mathrm{H}_2(\mathbb{C})$, and yields the Lorentz structure on $\mathbb{R}^4$.

3.3. Relevance to the Kepler Problem

We are now ready to give a preliminary explanation for the relevance of the euclidean Jordan algebra $\Gamma(3)$ to the Kepler problem. There are three main points

i) The future light cone Λ_+ = the set of rank one, semi-positive elements in $\mathrm{H}_2(\mathbb{C})$. Indeed, if the rank of X is less than two, then $\det X = 0$, also, if $X \neq 0$ is semi-positive, then $\operatorname{tr} X > 0$. So $x^2 = 0$ and $x_0 > 0$.

ii) For the Kepler problem, the potential term is

$$-\frac{1}{\langle e \mid x\rangle}.$$

iii) The kinetic term for the Kepler problem (or rather the Riemannian metric on Λ_+), angular momentum, and Laplace-Runge-Lenz vector can all be naturally expressed in terms of Jordan algebra structure as well. (The detailed elaboration of this point will be reviewed in the later sections.)

To proceed, we need to review euclidean Jordan algebra now.

3.4. Definition of Euclidean Jordan algebra

Jordan algebras are the unfavored cousins of Lie algebras, and euclidean Jordan algebras are the unfavored cousins of compact real Lie algebras. Having $\mathrm{H}_2(\mathbb{C})$ or $\Gamma(3)$ in mind, we have from [3]

Definition 2. *A finite dimensional* **euclidean Jordan algebra** *is a finite dimensional real commutative algebra V with unit e such that, for any elements a, b in V, we have*

1) $a(ba^2) = (ab)a^2$ (*weakly associative*)
2) $a^2 + b^2 = 0 \implies a = b = 0$ (*formally real*).

The simplest example is $\mathbb{R}$, the other example is $\mathrm{H}_2(\mathbb{C})$. We use $L_a\colon V \to V$ to denote the multiplication by a. Then condition 1) amounts to saying that

1´) $[L_a, L_{a^2}] = 0$ (Jordan Identity)
and condition 2) can be replaced by
2´) The “Killing form" $\langle a \,|\, b\rangle = \frac{1}{\dim V} \operatorname{tr} L_{ab}$ is positive definite.

One can check that $\langle b \mid ac\rangle = \langle ab \mid c\rangle$. Note that, euclidean Jordan algebras are called formally real Jordan algebras in the old literature. Note also that, by dropping condition 2) in the above definition, we end up with the definition for *finite dimensional real Jordan algebra*. Finally, we note that any euclidean Jordan algebra is semi-simple in the sense that its "Killing form" is non-degenerate.

3.5. The Classification Theorem of Jordan, von Neumann and Wigner

A typical way of obtaining a real Lie algebra is to start with a real associative algebra and then anti-symmetrize its associative product. Similarly, a typical way of obtaining a real Jordan algebra is to start with a real associative algebra and then symmetrize its associative product. For example, starting with the real Clifford algebra $\mathrm{Cl}(\mathbb{R}^n$, dot product), we get a finite dimensional real Jordan algebra. This Jordan algebra contains a Jordan subalgebra $\Gamma(n)$ whose underlying real vector space is the linear subspace $\mathbb{R} \oplus \mathbb{R}^n$. It turns out that $\Gamma(n)$ is formally real, and it is simple unless $n = 1$. Similarly, if we start with the real algebra of $n \times n$ matrices over $\mathbb{R}$ ($\mathbb{C}$ and $\mathbb{H}$ respectively), we get a finite dimensional real Jordan algebra which contains a Jordan subalgebra $\mathrm{H}_n(\mathbb{R})$ ($\mathrm{H}_n(\mathbb{C})$ and $\mathrm{H}_n(\mathbb{H})$ respectively) consisting of real symmetric (complex-hermitian and quaternionic-hermitian respectively) matrices of order n. This Jordan subalgebra is also formally real and simple. Note that, since the multiplication of octonions is not associative, $\mathrm{H}_n(\mathbb{O})$ is not a Jordan algebra unless $n \le 3$.

The following theorem is due to Jordan, von Neumann and Wigner [4].

Theorem 3. *Euclidean Jordan algebras are semi-simple, and the simple ones consist of four infinite series and one exceptional*
$\mathbb{R}$, $\Gamma(n)$, $n \ge 2$, $\mathrm{H}_n(\mathbb{R})$, $n \ge 3$, $\mathrm{H}_n(\mathbb{C})$, $n \ge 3$, $\mathrm{H}_n(\mathbb{H})$, $n \ge 3$, $\mathrm{H}_3(\mathbb{O})$.

Note that $\Gamma(1)$ is not simple, $\Gamma(0) = \mathrm{H}_1(\mathbb{F}) = \mathbb{R}$ where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, and there are isomorphisms

$$\mathrm{H}_2(\mathbb{R}) \cong \Gamma(2), \quad \mathrm{H}_2(\mathbb{C}) \cong \Gamma(3), \quad \mathrm{H}_2(\mathbb{H}) \cong \Gamma(5), \quad \mathrm{H}_2(\mathbb{O}) \cong \Gamma(9).$$

In particular the isomorphism $\Gamma(3) \cong \mathrm{H}_2(\mathbb{C})$ is the bijection $x \leftrightarrow X$ appeared in Subsection 3.1.

Note also that, in the classification list of the above theorem, $\mathbb{R}$ is the only associative Jordan algebra, and $\mathrm{H}_3(\mathbb{O})$ is the only one [1] that is not associated with an associative algebra, a reason for $\mathrm{H}_3(\mathbb{O})$ to be called exceptional.

Elements of a Jordan algebra should be viewed as "generalized hermitian matrices" so that the notions of trace, rank, eigenvalue, and diagonalization all make sense for elements in a Jordan algebra. The rank of a Jordan algebra is defined to be the maximum of the rank of its elements. For the simple Jordan algebras in the classification list of the above theorem, we have following table for trace and rank

Jordan algebra	trace	rank
$\mathbb{R}$	$\mathrm{tr}\ x = x$	1
$\Gamma(n), n \geq 2$	$\mathrm{tr}\ x = 2x_0$ for $x = (x_0, \mathbf{x})$	2
$\mathrm{H}_n(\mathbb{F})$	$\mathrm{tr}\ x =$ the sum of diagonal entries of x	n

Here, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ if $n > 3$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ if $n = 3$. In terms of trace and rank of the Jordan algebra, one can also have the notion of determinant. For example, since $\Gamma(n)$ has rank 2, for $x \in \Gamma(n)$, we have

$$\det\ x = \frac{1}{2!}(\mathrm{tr}\ x)^2 - \frac{1}{2}\,\mathrm{tr}\,(x^2)$$

which is precisely the Lorentz dot product of x with x. Note that, the identity $\det(xy) = \det\ x\ \ \det\ y$ does not hold in general. Finally we would like to say that the notions of positive-definite and semi-positive definite are valid for elements in an euclidean Jordan algebra.

3.6. The Structure Algebra

The Jordan algebra $\Gamma(3)$ is more refined than the 4D Lorentz space $\mathbb{R}^{1,3}$, because the Lorentz dot product can be derived from the Jordan algebra structure. For $\mathbb{R}^{1,3}$, the following nested sequence of real Lie algebras

$$\mathfrak{so}(3) \subset \mathfrak{so}(1,3) \subset \mathfrak{so}(2,4) \tag{9}$$

is well known to theoretical physicists and their group versions are called rotational, Lorentz and conformal group respectively.

It turns out that the nested sequence (9) has an analogue for a general Jordan algebra V

$$\mathfrak{Der}(V) \subset \mathfrak{str}'(V) \subset \mathfrak{co}(V) \tag{10}$$

where $\mathfrak{Der}(V)$ is the Lie algebra of derivations on V, and $\mathfrak{str}'(V)$ is called the restricted structure algebra of V and $\mathfrak{co}(V)$ is called the conformal algebra of V.

To introduce the structure algebra, we need to do some preparations. For a, b in V, we let

$$S_{ab} := [L_a, L_b] + L_{ab}, \qquad \{abc\} := S_{ab}(c)$$

and $\mathfrak{str}(V)$ (or simply $\mathfrak{str}$) be the span of $\{S_{ab}\ ;\ a, b \in V\}$ over $\mathbb{R}$. One can verify that

$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}}$$

so $\mathfrak{str}(V)$ becomes a real Lie algebra – the **structure algebra** of V. For example

1) $\mathfrak{str}(V) \cong \mathbb{R}$ for $V = \mathbb{R}$
2) $\mathfrak{str}(V) \cong \mathfrak{so}(1,3) \oplus \mathbb{R}$ for $V = \Gamma(3)$.

This Lie algebra is not simple, actually not even semi-simple, because it has a non-trivial central element: $L_e = S_{ee}$. (In general $L_u = S_{ue}$) Note that the set $\{L_u \mid u \in V\}$ is a generating set for $\mathfrak{str}(V)$, and its subset

$$\{L_u \; ; \; u \in V, \; \mathrm{tr} \; u = 0\}$$

generates a Lie subalgebra of $\mathfrak{str}(V)$, called the restricted structure algebra of V, and is denoted by $\mathfrak{str}'(V)$.

Finally, we would like to remark that $\mathfrak{str}(V)$ is the Lie algebra of the Lie subgroup of $\mathrm{GL}(V)$ which leaves the (symmetric) cone

$$V_+ := \{x \in V \; ; \; x > 0\}$$

invariant. In the case $V = \Gamma(3)$, this Lie subgroup is $\mathrm{O}^+(1,3) \times \mathbb{R}_+$ where $\mathrm{O}^+(1,3)$ is the group of Lorentz transformations that preserve the direction of time, and $\mathbb{R}_+$ is the group of dilations of $\mathbb{R}^{1,3}$.

3.7. The Conformal Algebra of Tits, Köcher and Kantor

The structure algebra is not simple, but it can be extended to a simple real Lie algebra provided that V is a simple euclidean Jordan algebra.

The following theorem was obtained by J. Tits [15], M. Köcher [6] and I. Kantor [5] independently in the 1960s.

Theorem 4. *Let V be a simple euclidean Jordan algebra, and write $z \in V$ as X_z and $\langle w \mid \;\rangle \in V^*$ as Y_w. Then, the structure algebra of V can be extended to a simple real Lie algebra whose underlying real vector space is $V \oplus \mathfrak{str}(V) \oplus V^*$, and commutation relations are*

$$[X_u, X_v] = 0, \qquad [Y_u, Y_v] = 0, \qquad [X_u, Y_v] = -2S_{uv}$$

$$[S_{uv}, X_z] = X_{\{uvz\}}, \qquad [S_{uv}, Y_z] = -Y_{\{vuz\}} \tag{11}$$

$$[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}$$

where u, v, z and w are arbitrary elements of V.

This real simple Lie algebra is called the TKK algebra in the literature, but shall be called the **conformal algebra** here and is denoted by $\mathfrak{co}(V)$ or simply $\mathfrak{co}$. For example

1) $\mathfrak{str} = \mathfrak{so}(3,1) \oplus \mathbb{R}, \qquad \mathfrak{str}' = \mathfrak{so}(3,1), \qquad \mathfrak{co} = \mathfrak{so}(4,2)$ for $V = \Gamma(3)$
2) $\mathfrak{str} = \mathbb{R}, \qquad \mathfrak{str}' = \{0\}, \qquad \mathfrak{co} = \mathfrak{sl}(2,\mathbb{R})$ for $V = \mathbb{R}$.

In general, $\mathfrak{co}$ is the Lie algebra of the bi-holomorphic automorphism group of the complex domain $V \oplus \mathrm{i}V_+ \subset V \otimes_{\mathbb{R}} \mathbb{C}$.

After spending so much effort on the basic facts on euclidean Jordan algebras, an impatient reader may ask this

Question: How could the euclidean Jordan algebra $\mathrm{H}_2(\mathbb{C})$ (or $\Gamma(3)$) be relevant to the Kepler problem?

Well, the answer will become clear after we review the Lenz algebra for the Kepler problem.

4. Lenz Algebra for the Kepler Problem

The phase space for the Kepler problem, i.e., $T^*\mathbb{R}^3_*$, is a Poisson manifold. In terms of the standard canonical coordinates $x^1, x^2, x^3, p_1, p_2, p_3$, its Poisson structure can be described by the following basic Poisson bracket relations:

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0.$$

Recall that the hamiltonian, angular momentum, and Laplace-Runge-Lenz vector are

$$\mathrm{H} = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r}, \qquad \mathbf{L} = \mathbf{r}\times\mathbf{p}, \qquad \mathbf{A} = \mathbf{L}\times\mathbf{p} + \frac{\mathbf{r}}{r}$$

respectively. In terms of Poisson bracket, the fact that $\mathbf{L}$ and $\mathbf{A}$ are constants of motion can be restated as

$$\{\mathbf{L}, \mathrm{H}\} = 0, \qquad \{\mathbf{A}, \mathrm{H}\} = 0.$$

To show that, we first note that L_i (the i-th component of $\mathbf{L}$) is the infinitesimal generator of the rotation about the i-th axis. Indeed, since $L_3 = x^1p_2 - x^2p_1$, we have

$$\{L_3, x^1\} = -x^2\{p_1, x^1\} = x^2, \qquad \{L_3, x^2\} = -x^1, \qquad \{L_3, x^3\} = 0.$$

Similarly, we have

$$\{L_3, p_1\} = p_2, \qquad \{L_3, p_2\} = -p_1, \qquad \{L_3, p_3\} = 0.$$

Then, it is clear that $\{\mathbf{L}, \mathrm{H}\} = 0$, moreover

$$\begin{aligned}
\{\mathbf{A}, \mathrm{H}\} &= \mathbf{L}\times\{\mathbf{p}, \mathrm{H}\} + \{\frac{\mathbf{r}}{r}, \mathrm{H}\} = \mathbf{L}\times\{\mathbf{p}, -\frac{1}{r}\} + \{\frac{\mathbf{r}}{r}, \frac{1}{2}\mathbf{p}^2\}\\
&= \mathbf{L}\times\nabla\frac{1}{r} + \sum_i\{\frac{\mathbf{r}}{r}, p_i\}p_i = -\mathbf{L}\times\frac{\mathbf{r}}{r^3} + \sum_i p_i\partial_{x^i}\frac{\mathbf{r}}{r}\\
&= -(\mathbf{r}\times\mathbf{p})\times\frac{\mathbf{r}}{r^3} + \frac{\mathbf{p}}{r} - \frac{\mathbf{r}\cdot\mathbf{p}}{r^3}\mathbf{r} = \mathbf{0}.
\end{aligned}$$

In fact, it is fairly routine to verify the following theorem.

Theorem 5. *Let L_i (respectively A_i) be the i-th component of* $\mathbf{L}$ (*respectively* $\mathbf{A}$). *Then*

$$\begin{array}{c}\{L_i, \mathrm{H}\} = 0, \qquad \{A_i, \mathrm{H}\} = 0, \qquad \{L_i, L_j\} = \epsilon_{ijk} L_k \\ \{L_i, A_j\} = \epsilon_{ijk} A_k, \qquad \{A_i, A_j\} = -2\mathrm{H}\epsilon_{ijk} L_k.\end{array} \tag{12}$$

Here $\epsilon_{ijk} = 1$ (respectively -1) if ijk is an even (respectively odd) permutation of 123, and equals to 0 otherwise. A summation over the repeated index k is assumed.

The real associated algebra with generators H, $L_1, L_2, L_3, A_1, A_2, A_3$ and relations in equation (12) is called the **Lenz algebra**.

With this in mind, we are now ready to introduce the notion of **universal Kepler problem**.

5. Universal Kepler Problem

Let us fix a simple euclidean Jordan algebra V. Let $\mathcal{TKK}$ be the complexified universal enveloping algebra for the conformal algebra $\mathfrak{co}(V)$, but with Y_e being formally inverted. The following definition was first introduced in [12].

Definition 6. *The universal angular momentum is*

$$L : V \times V \to \mathcal{TKK}, \qquad (u, v) \mapsto L_{u,v} := [L_u, L_v] \tag{13}$$

The universal hamiltonian is

$$H := \frac{1}{2} Y_e^{-1} X_e - (\mathrm{i} Y_e)^{-1} \tag{14}$$

The universal Laplace-Runge-Lenz vector is

$$A : V \to \mathcal{TKK}, \qquad u \mapsto A_u := (\mathrm{i}Y_e)^{-1} [L_u, (\mathrm{i}Y_e)^2 H] \tag{15}$$

5.1. Universal Lenz Algebra

Via the commutation relations (11) for the conformal algebra $\mathfrak{co}(V)$, one can verify

Theorem 7. *For u, v, z and w in V*

$$\begin{array}{c}[L_{u,v}, H] = 0, \qquad [A_u, H] = 0 \\ [L_{u,v}, L_{z,w}] = L_{L_{u,v}z,w} + L_{z,L_{u,v}w} \\ [L_{u,v}, A_z] = A_{L_{u,v}z}, \qquad [A_u, A_v] = -2HL_{u,v}.\end{array} \tag{16}$$

Proof: Since $S_{uv} = L_{u,v} + L_{uv}$, part of the commutation relations (11) for the conformal algebra can be rewritten as

$$\begin{array}{c}[L_{u,v}, X_z] = X_{L_{u,v}z}, \qquad [L_{u,v}, Y_z] = Y_{L_{u,v}z} \\ [L_{u,v}, L_z] = L_{L_{u,v}z}, \qquad [L_{u,v}, L_{z,w}] = L_{L_{u,v}z,w} + L_{z,L_{u,v}w}.\end{array}$$

Then we have $[L_{u,v}, X_e] = [L_{u,v}, Y_e] = 0$, so $[L_{u,v}, H] = 0$, so

$$\begin{aligned}[L_{u,v}, A_z] &= \mathrm{i}Y_e^{-1}[L_{u,v}, [L_z, Y_e^2 H]] \\ &= \mathrm{i}Y_e^{-1}\left([[L_{u,v}, L_z], Y_e^2 H] + [L_z, [L_{u,v}, Y_e^2 H]]\right) \\ &= \mathrm{i}Y_e^{-1}[L_{L_{u,v}z}, Y_e^2 H] = A_{L_{u,v}z}.\end{aligned}$$

The rest of the proof is left to the reader and anyway it can be found in [12]. ■

We are now ready to outline a scheme to obtain a super integrable model that shares Kepler problem's characteristic features such as existence of Laplace-Rungs-Lenz vector and hidden symmetry: Let V be a simple euclidean Jordan algebra, then

$$\begin{array}{c}\text{a concrete realization of the conformal algebra } \mathfrak{co}(V) \\ \Downarrow \\ \text{a concrete model of the Kepler type.}\end{array}$$

To be more precise, we have

$$\begin{array}{c}\text{a suitable operator realization of } \mathfrak{co}(V) \\ \Downarrow \\ \text{a quantum generalized Kepler problem}\end{array}$$

and

$$\begin{array}{c}\text{a suitable Poisson realization of } \mathfrak{co}(V) \\ \Downarrow \\ \text{a classical generalized Kepler problem.}\end{array} \tag{17}$$

For simplicity, hereafter we shall stick to Poisson realizations of the conformal algebra $\mathfrak{co}(V)$ and the associated classical generalized Kepler problems.

5.2. Poisson Realizations and Classical Generalized Kepler Problems

In a Poisson realization of the conformal algebra, S_{uv}, X_z, Y_w are respectively represented as real functions $\mathcal{S}_{uv}$, $\mathcal{X}_z$, $\mathcal{Y}_w$ on a Poisson manifold so that the commutation relations are represented by the Poisson bracket relations: for u, v, z, w in V, we have

$$\begin{gathered}\{\mathcal{X}_u, \mathcal{X}_v\} = 0, \qquad \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, \qquad \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv} \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, \qquad \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}} \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}.\end{gathered} \tag{18}$$

Then, H, A_u and $L_{u,v}$ can be realized as real functions

$$\mathcal{H} = \frac{\frac{1}{2}\mathcal{X}_e - 1}{\mathcal{Y}_e}, \qquad \mathcal{A}_u := \frac{\{\mathcal{L}_u, \mathcal{Y}_e^2\mathcal{H}\}}{\mathcal{Y}_e}, \qquad \mathcal{L}_{u,v} := \{\mathcal{L}_u, \mathcal{L}_v\} \tag{19}$$

respectively. Note that

$$\mathcal{A}_u = \frac{1}{2}\left(\mathcal{X}_u - \mathcal{Y}_u \frac{\mathcal{X}_e}{\mathcal{Y}_e}\right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}. \tag{20}$$

Note also that, in a suitable Poisson realization, $\mathcal{Y}_e$ is a positive real function, so the expressions in the preceding two equations all make sense.

5.3. A Poisson Realization on TV

Via the (unique up to scaling) canonical inner product on V, $TV \cong T^*V$. So TV becomes a symplectic manifold. Denote an element of $TV = V \times V$ by (x, π) and fix an orthonormal basis $\{e_\alpha\}$ for V so that we can write $x = x^\alpha e_\alpha$ and $\pi = \pi^\alpha e_\alpha$. Then we have the basic Poisson bracket relations

$$\{x^\alpha, \pi^\beta\} = \delta^{\alpha\beta}, \qquad \{x^\alpha, x^\beta\} = 0, \qquad \{\pi^\alpha, \pi^\beta\} = 0$$

on TV.

In coordinate free form, we have

$$\{\langle x \mid u\rangle, \langle \pi \mid v\rangle\} = \langle u \mid v\rangle, \qquad \{\langle x \mid u\rangle, \langle x \mid v\rangle\} = \{\langle \pi \mid u\rangle, \langle \pi \mid v\rangle\} = 0.$$

Claim: The real functions

$$\mathcal{S}_{uv} := \langle S_{uv}(x) \mid \pi\rangle, \qquad \mathcal{X}_u := \langle x \mid \{\pi u \pi\}\rangle, \qquad \mathcal{Y}_v := \langle x \mid v\rangle \tag{21}$$

yield a Poisson realization on TV for S_{uv}, X_z, Y_w respectively.

Proof: It is clear that $\{\mathcal{Y}_u, \mathcal{Y}_v\} = 0$.

$$\begin{aligned}
\{\mathcal{X}_u, \mathcal{Y}_v\} &= \{\langle x \mid \{\pi u \pi\}\rangle, \langle x \mid v\rangle\} \\
&= -2\langle x \mid \{v u \pi\}\rangle = -2\langle S_{uv}(x) \mid \pi\rangle = -2\mathcal{S}_{uv}.
\end{aligned}$$

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{Y}_z\} &= \{\langle S_{uv}(x) \mid \pi\rangle, \langle x \mid z\rangle\} \\
&= -\langle S_{uv}(x) \mid z\rangle = -\langle x \mid \{v u z\}\rangle = -\mathcal{Y}_{\{vuz\}}.
\end{aligned}$$

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} &= \{\langle S_{uv}(x) \mid \pi\rangle, \langle S_{zw}(x) \mid \pi\rangle\} \\
&= \langle S_{uv} S_{zw}(x) \mid \pi\rangle - \langle S_{zw} S_{uv}(x) \mid \pi\rangle \\
&= \langle [S_{uv}, S_{zw}](x) \mid \pi\rangle = \langle (S_{\{uvz\}w} - S_{z\{vuw\}})(x) \mid \pi\rangle \\
&= \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}.
\end{aligned}$$

The rest of the proof is left to the readers and anyway it can be found in [12]. ■

Let us try to implement the scheme outlined in equation (17) for this Poisson realization. Since

$$\mathcal{X}_u = \langle x \mid \{\pi u \pi\}\rangle \qquad \text{and} \qquad \mathcal{Y}_v = \langle x \mid v\rangle$$

and $\mathcal{H} = \frac{1}{2}\frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e}$, we have

$$\mathcal{H} = \frac{1}{2}\frac{\langle x \mid \pi^2\rangle}{r} - \frac{1}{r}$$

where $r = \langle x \mid e\rangle = \frac{1}{\mathrm{rank} V}\,\mathrm{tr}\, x$. However, since $\mathrm{tr}\, x = 0$ for some $x \in V$, $\mathcal{H}$ is NOT even a real-valued function on the entire TV. In other words, this Poisson realization is not suitable for the implementation of scheme in equation (17).

To salvage this Poisson realization, we restrict it to certain symplectic sub-manifolds of TV, for example $T\mathcal{C}_k$, where $\mathcal{C}_k$ is the set of rank k semi-positive elements of V, with k being a positive integer less than or equal to the rank of V. Indeed, restricting $\mathcal{H}$ to $T\mathcal{C}_k$ yields an integrable model of Kepler type. Moreover, this model is the Kepler problem when $V = \Gamma(3)$ and $k = 1$, as we shall demonstrate in Section 7.

6. Kepler Cones and Generalized Kepler Problems

In reference [8] the notion of Kepler cone was first introduced. Let us begin with a theorem there.

Theorem 8. *Let V be a simple euclidean Jordan algebra, k be a positive integer which is at most* $\mathrm{rank} V$*, and $\mathcal{C}_k$ be the set of rank-k semi-positive elements of V. Then $\mathcal{C}_k$ is a manifold. Moreover, for any $x \in \mathcal{C}_k$*

1) $T_x\mathcal{C}_k = \{x\} \times \mathrm{Im} L_x$
2) *The map*

$$\langle \pi | \ \rangle \mapsto \frac{\langle \pi \mid x\pi\rangle}{r} \tag{22}$$

is a positive-definite quadratic form on $T^*_x\mathcal{C}_k$.

These quadratic forms in the theorem define a Riemannian metric on $\mathcal{C}_k$ (called the **Kepler metric**), denoted by $(\ ,\ )_K$. The Riemannian manifold $(\mathcal{C}_k, (\ ,\ )_K)$ is refereed to as the rank-k **Kepler cone** of V.

A Kepler cone shall serve as the configuration space for a generalized Kepler problem. Indeed, if $V = \Gamma(3)$, then $(\mathcal{C}_1, (\ ,\ )_K)$ is isometric to the Riemannian submanifold $\mathbb{R}^3_*$ of the euclidean space $\mathbb{R}^3$, i.e., the configuration space for the Kepler problem.

For more details, please consult [8].

6.1. The Poisson Realization on $T\mathcal{C}_k$

Before presenting this Poisson realization of $\mathfrak{co}(V)$ on $T\mathcal{C}_k$, we need to do some preparations. First of all, $T\mathcal{C}_k$ shall be identified with $T^*\mathcal{C}_k$ via the Riemannian metric (22). With this identification understood, $T\mathcal{C}_k$ becomes a Poisson manifold. Next, we write the inclusion map

$$T\mathcal{C}_k \hookrightarrow TV = V \times V$$

as (x, π), and view both x and π as vector-valued smooth functions on $T\mathcal{C}_k$. Note that, at any point Q of $T\mathcal{C}_k$, $x(Q) \in \mathcal{C}_k$ and $\pi(Q) \in \mathrm{Im}L_{x(Q)}$.

We use q^i to denote a system of local coordinates on $\mathcal{C}_k$, ∂_{q^i} to denote the resulting local tangent frame, and let

$$g_{ij} := \langle \partial_{q^i} | \partial_{q^i} \rangle, \qquad g := [g_{ij}], \qquad g^{ij} := (g^{-1})_{ij}, \quad E^i = g^{ij}\partial_{q^i}.$$

Under the identification of $T^*\mathcal{C}_k$ with $T\mathcal{C}_k$ mentioned early, one can see that the local cotangent frame $(\mathrm{d}q^1, \mathrm{d}q^2, \ldots)$ becomes the local tangent frame $(E^1, E^2, \ldots)$, in terms of which we can write

$$\pi = p_i E^i. \tag{23}$$

Here, each p_i is a local function on $T\mathcal{C}_k$. For notational sanity, we use the same notation for both a local function on $\mathcal{C}_k$ and its pull-back under the tangent bundle map τ: $T\mathcal{C}_k \to \mathcal{C}_k$. For example, q^i denotes both a local function on $\mathcal{C}_k$ and its pullback to $T\mathcal{C}_k$. In terms of q^i and p_j, we have the following local canonical Poisson bracket relations

$$\{q^i, q^j\} = 0, \qquad \{q^i, p_j\} = \delta^i_j, \qquad \{p_i, p_j\} = 0. \tag{24}$$

We are now ready to state the Poisson realization of the conformal algebra $\mathfrak{co}$ on $T\mathcal{C}_k$.

Theorem 9 (Meng 2011)**.** *Let V be a simple euclidean Jordan algebra. For any vectors u, v in V, define functions*

$$\mathcal{X}_u := \langle x | \{\pi u \pi\} \rangle, \qquad \mathcal{S}_{uv} := \langle S_{uv}(x) | \pi \rangle, \qquad \mathcal{Y}_v := \langle v | x \rangle \tag{25}$$

on $T\mathcal{C}_k$. Then, for any vectors u, v, z, w in V, the following Poisson bracket relations hold

$$\{\mathcal{X}_u, \mathcal{X}_v\} = 0, \qquad \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, \qquad \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}$$

$$\{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, \qquad \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}$$

$$\{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}.$$

This theorem is the classical limit of a quantum analogue, i.e., part i) of Proposition 4.6 in [8]. A direct proof has been given in [17].

Remark 10. *The generalized Kepler problem corresponding to the Poisson realization in* Theorem 9 *is the hamiltonian system with phase space $T\mathcal{C}_k$, hamiltonian*

$$\mathcal{H} = \frac{1}{2}\frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e}$$

and Laplace-Runge-Lenz vector

$$\mathcal{A}_u = \frac{1}{2}\left(\mathcal{X}_u - \mathcal{Y}_u\frac{\mathcal{X}_e}{\mathcal{Y}_e}\right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}.$$

The following section is a detailed demonstration of this remark for the Kepler problem.

7. Example: Kepler Problem and Future Light-Cone

The purpose here is to demonstrate explicitly that if $V = \Gamma(3)$ and $k = 1$, the generalized Kepler problem mentioned in Remark 10 is exactly the Kepler problem.

Recall that, in terms of the standard basis vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the Jordan multiplication can be determined by the following rules: $\mathbf{e}_0$ is the identity element, and

$$\mathbf{e}_i\mathbf{e}_j = \delta_{ij}\mathbf{e}_0$$

for $i, j > 0$. The trace tr: $V \to \mathbb{R}$ is given by the following rules

$$\mathrm{tr}\ \mathbf{e}_0 = 2, \qquad \mathrm{tr}\ \mathbf{e}_i = 0.$$

So the inner product on V is the one such that the standard basis is an orthonormal basis. Since V has rank two, the determinant of $x = x^\mu\mathbf{e}_\mu$ is

$$\det x = \frac{1}{2}((\mathrm{tr}\, x)^2 - \mathrm{tr}\, x^2) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

Therefore

$$\mathcal{C}_1 = \{x \in V\ ;\ \det\ x = 0, \mathrm{tr}\ x > 0\}$$

is precisely the future light-cone in the Lorentz space $\mathbb{R}^{1,3}$. It turns out that $\mathcal{C}_1$ has a global coordinate $q = (q^1, q^2, q^3)$ with $q^i(x) = x^i$. Since $x(q) = r\mathbf{e}_0 + \mathbf{r}$ where $\mathbf{r} = q^i\mathbf{e}_i$ and r is the length of $\mathbf{r}$, we have

$$\partial_{q^i} = \mathbf{e}_i + \frac{q^i}{r}\mathbf{e}_0, \quad g_{ij} = \delta_{ij} + \frac{q^iq^j}{r^2}, \quad g^{ij} = \delta_{ij} - \frac{q^iq^j}{2r^2}, \quad E^j = \mathbf{e}_j - \frac{q^j}{2r^2}\mathbf{r} + \frac{q^j}{2r}\mathbf{e}_0.$$

Here the first and last identities are understood with the natural identification of $T_x\mathcal{C}_1$ with $\mathrm{Im}L_x$ in mind.

Recall that $\pi = p_iE^i$. Let $\mathbf{p} = \sum_i p_i\mathbf{e}_i$ and $|\mathbf{p}|^2 = \mathbf{p}\cdot\mathbf{p}$. Then we have

$$\pi = \mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{r}}{2r^2}\mathbf{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{2r}\mathbf{e}_0, \qquad x\pi = (\mathbf{p}\cdot\mathbf{r})\mathbf{e}_0 + r\mathbf{p}$$

therefore

$$\begin{aligned}\mathcal{X}_e &= \langle x|\pi^2\rangle = \langle x\pi|\pi\rangle\\ &= \left\langle (\mathbf{p}\cdot\mathbf{r})\mathbf{e}_0 + r\mathbf{p}\,\middle|\, \mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{r}}{2r^2}\mathbf{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{2r}\mathbf{e}_0\right\rangle = r|\mathbf{p}|^2.\end{aligned}$$

Since $\mathcal{Y}_e = r$, we have the hamiltonian

$$\mathcal{H} = \frac{1}{2}\frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e} = \frac{1}{2}|\mathbf{p}|^2 - \frac{1}{r}$$

which is precisely the hamiltonian given by equation (1).
Since

$$\begin{aligned}\mathcal{L}_{\mathbf{e}_1,\mathbf{e}_2} &= \langle L_{\mathbf{e}_1,\mathbf{e}_2}x|\pi\rangle = \langle \mathbf{e}_1(\mathbf{e}_2x) - \mathbf{e}_2(\mathbf{e}_1x)|\pi\rangle\\ &= \langle q^2\mathbf{e}_1 - q^1\mathbf{e}_2|\mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{r}}{2r^2}\mathbf{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{2r}\mathbf{e}_0\rangle = q^2p_1 - q^1p_2\end{aligned}$$

and similarly

$$\mathcal{L}_{\mathbf{e}_2,\mathbf{e}_3} = q^3p_2 - q^2p_3, \qquad \mathcal{L}_{\mathbf{e}_3,\mathbf{e}_1} = q^1p_3 - q^3p_1$$

we have

$$\mathbf{L} := -(\mathcal{L}_{\mathbf{e}_1,\mathbf{e}_2}\mathbf{e}_3 + \mathcal{L}_{\mathbf{e}_2,\mathbf{e}_3}\mathbf{e}_1 + \mathcal{L}_{\mathbf{e}_3,\mathbf{e}_1}\mathbf{e}_2) = \mathbf{r}\times\mathbf{p}$$

which is precisely the angular momentum vector for the Kepler problem.
One can compute $\mathcal{X}_{\mathbf{e}_i} = \langle x|\{\pi\mathbf{e}_i\pi\}\rangle$ and $\mathcal{Y}_{\mathbf{e}_i} = \langle x|\mathbf{e}_i\rangle$ and arrive at

$$\sum_i \mathcal{X}_{\mathbf{e}_i}\mathbf{e}_i = 2(\mathbf{r}\cdot\mathbf{p})\mathbf{p} - |\mathbf{p}|^2\mathbf{r}, \qquad \sum_i \mathcal{Y}_{\mathbf{e}_i}\mathbf{e}_i = \mathbf{r}.$$

Then

$$\begin{aligned}\mathbf{A} &:= \sum_i A_{\mathbf{e}_i}\mathbf{e}_i = \sum_i\left(\frac{1}{2}\left(\mathcal{X}_{\mathbf{e}_i} - \mathcal{Y}_{\mathbf{e}_i}\frac{\mathcal{X}_e}{\mathcal{Y}_e}\right) + \frac{\mathcal{Y}_{\mathbf{e}_i}}{\mathcal{Y}_e}\right)\mathbf{e}_i\\ &= (\mathbf{r}\times\mathbf{p})\times\mathbf{p} + \frac{\mathbf{r}}{r}\end{aligned}$$

which is precisely the Laplace-Runge-Lenz vector for the Kepler problem.

Remark 11. *In the literature the Poisson realization of* $\mathfrak{so}(2,4)$ *on the total cotangent space of* $\mathbb{R}^3_*$*, i.e., the Poisson realization in* Theorem 9 *for* $k = 1$ *and* $V = \Gamma(3)$*, is called the classical dynamic symmetry for the Kepler problem. A far as we know, the fact that the Laplace-Runge-Lenz vector owes its existence to the dynamic symmetry was initially pointed out by this author in* [11, Subsection 7.1].

8. Generalized Kepler Problems with Magnetic Charges

We have learned that there are Kepler-type integrable models associated with a simple euclidean Jordan algebra V, one for each Kepler cone of V. We have also learned that the Kepler problem is one of these integrable models. Since the Kepler problem has magnetized cousins, one naturally wonders whether these Kepler-type integrable models also have magnetized cousins. The simple answer is "Yes". To know more, we need to introduce the notion of Sternberg phase space.

We begin with a basic technical setup.

i) G – a compact connected Lie group.

ii) $\mathfrak{g}$, $\mathfrak{g}^*$ – the Lie algebra of G and its dual.

iii) $P \to X$ – a smooth principal G-bundle over manifold X.

iv) Θ – a fixed principal connection form, i.e., Θ is a $\mathfrak{g}$-valued differential one-form on P which satisfies the following two conditions

$$1)\ {R_{a^{-1}}}^*\,\Theta = \mathrm{Ad}_a\Theta \text{ for any } a \in G, \qquad 2)\ \Theta(X_\xi) = \xi \text{ for any } \xi \in \mathfrak{g}.$$

Here, $R_{a^{-1}}$ denotes the right action of a^{-1} on P, Ad_a denotes the adjoint action of a on $\mathfrak{g}$, and vector field X_ξ denotes the infinitesimal right action of ξ on P.

v) ω_X – the canonical symplectic form on T^*X.

vi) F – a hamiltonian G-space.

vii) Ω – the symplectic form on F.

viii) Φ: $F \to \mathfrak{g}^*$ – the G-equivariant moment map.

ix) $\mathcal{F} \to X$ – the associated fiber bundle with fiber F.

x) $\mathcal{F}^\sharp$ – the pullback of diagram $T^*X \to X \leftarrow \mathcal{F}$, i.e., square

$$\begin{array}{ccc} \mathcal{F}^\sharp & \to & \mathcal{F} \\ \downarrow & & \downarrow \\ T^*X & \to & X \end{array}$$

is a pullback diagram in the category of smooth manifolds and smooth maps.

8.1. Sternberg Phase Space

For notational sanity here, we shall use the same notation for both a differential form (or a map) and its pullback under a fiber bundle projection map. The following theorem is obtained by Sternberg in [14].

Theorem 12. *With the technical setup given above, we have*

a) *There is a closed real differential two-form Ω_Θ on $\mathcal{F}$ which is of the form $\Omega - \mathrm{d}\langle A, \Phi\rangle$ under a local trivialization of $P \to X$ in which the connection form Θ is represented by the $\mathfrak{g}$-valued differential one-form A on X.*

b) *The differential two-form* $\omega_\Theta := \omega_X + \Omega_\Theta$ *is a symplectic form on* $\mathcal{F}^\sharp$*, where* ω_X *is the canonical symplectic form on* T^*X*, pulled back under* $\mathcal{F}^\sharp \to T^*X$*, and* Ω_Θ *is the pullback of* Ω_Θ *under* $\mathcal{F}^\sharp \to \mathcal{F}$*.*

We call the symplectic manifold $(\mathcal{F}^\sharp, \omega_\Theta)$ a **Sternberg phase space**.

Note that Ω_Θ is the right substitute for Ω when we go from a product bundle with the product connection to a generic bundle.

Note also that If $G = \mathrm{U}(1)$, then $(\mathcal{F}^\sharp, \omega_\Theta) = (T^*X, \omega_X - q_e\, \mathrm{d}A)$ where q_e is the electric charge of the particle. So, locally we have

$$\omega_\Theta = \mathrm{d}\left((p_i - q_e A_i)\mathrm{d}q^i\right).$$

Finally we note that, in the hamiltonian formalism, as shown by Sternberg, the Sternberg phase space $(\mathcal{F}^\sharp, \omega_\Theta)$ is the right substitute for (T^*X, ω_X) when particles move in a background gauge field.

8.2. Examples: MICZ-Kepler Problems in Dimension $2k+1$

If we specialize the technical setup to the case where

i) $X = \mathbb{R}^{2k+1}_*$ or the Kepler cone $\mathcal{C}_1$ of the Jordan algebra $\Gamma(2k+1)$.

ii) $G = \mathrm{SO}(2k)$.

iii) $P \to X$ is the pullback bundle of the principal $\mathrm{SO}(2k)$-bundle $\mathrm{SO}(2k{+}1) \to \mathrm{S}^{2k}$ under the map

$$X \to \mathrm{S}^{2k}, \qquad \mathbf{r} \mapsto \frac{\mathbf{r}}{r} \tag{26}$$

iv) Θ is the pullback of the canonical connection

$$\mathrm{Proj}_{\mathfrak{so}(2k)}(g^{-1}\,\mathrm{d}g)$$

on $\mathrm{SO}(2k+1) \to \mathrm{S}^{2k}$. Here $g^{-1}\,\mathrm{d}g$ is the Maurer-Cartan form for the Lie group $\mathrm{SO}(2k+1)$, and $\mathrm{Proj}_{\mathfrak{so}(2k)}\colon \mathfrak{so}(2k+1) \to \mathfrak{so}(2k)$ is the orthogonal projection.

v) F is a co-adjoint orbit of $\mathrm{SO}(2k)$ which is either $\{0\}$ or diffeomorphic to $\mathrm{SO}(2k)/\mathrm{U}(k)$.

vi) Ω is the Kostant-Kirillov-Souriau (KKS) symplectic form on F.

vii) $\Phi\colon F \to \mathfrak{g}^*$ is the inclusion map.

It has been proved already in [9] that the conformal algebra of the Jordan algebra $\Gamma(2k+1)$ has a suitable Poisson realization on Sternberg phase space determined by the above data. Moreover, this Poisson realization yields a magnetized cousin of the Kepler problem in dimension $2k+1$. When $k = 1$, these magnetized models are the MICZ Kepler problems. For more details, one may consult [9].

9. Conclusion

We have seen that, for each Kepler cone of a simple euclidean Jordan algebra, there associate a Kepler-type classical dynamical model and its magnetized cousins. Indeed, the MICZ-Kepler problems are associated with the rank-one Kepler cone of $\Gamma(3)$. Here are some further facts

- The quantum versions of these models are expected to give, among other things, a concrete geometric realization for all unitary highest weight modules of (the universal cover) of the following real non-compact Lie groups

$$\mathrm{SO}(2,n), \qquad \mathrm{Sp}(2n,\mathbb{R}), \qquad \mathrm{SU}(n,n), \qquad \mathrm{SO}^*(4n), \qquad \mathrm{E}_{7(-25)}.$$

  Indeed, this has been confirmed for all scalar-type unitary highest weight modules in [8].
- The three-dimensional isotropic harmonic oscillator is the bounded sector of the Kepler-type problem associated with the rank-one Kepler cone of $\mathrm{H}_3(\mathbb{R})$, with the Fradkin tensor [2] being the (generalized) Laplace-Runge-Lenz vector. It is in this sense that we say that the Kepler problem and the oscillator problem are unified in the theory of generalized Kepler problems.

Acknowledgements

The author was supported by the Hong Hong Research Grants Council under RGC Project # 16304014. He would like to thank I. Mladenov for inviting him to deliver this lecture at the XVII-th International Conference on Geometry, Integrability and Quantization, June 5-10, 2015 Varna, Bulgaria.

References

[1] Albert A., *On a Certain Algebra of Quantum Mechanics*, Ann. Math. **35** (1934) 65-73.

[2] Fradkin D.M., *Three-Dimensional Isotropic Harmonic Oscillator and* $\mathrm{SU}(3)$, Am. J. Phys. **33** (1965) 207-211.

[3] Jordan P., *Über die Multiplikation quantenmechanischer Größen,* Z. Phys. **80** (1933) 285-291.

[4] Jordan P., von Neumann J. and Wigner E.P., *On an Algebraic Generalization of the Quantum Mechanical Formalism*, Ann. Math. **35** (1934) 29-64.

[5] Kantor I., *Classification of Irreducible Transitively Differential Groups,* Sov. Math. Dokl. **5** (1964) 1404-1407.

[6] Köcher M., *Imbedding of Jordan Algebras into Lie Algebras,* Amer. J. Math. **89** (1967) 787-816.

[7] McIntosh H. and Cisneros A., *Degeneracy in the Presence of a Magnetic Monopole*, J. Math. Phys. **11** (1970) 896-916.

[8] Meng G., *Generalized Kepler Problems I: Without Magnetic Charges,* J. Math. Phys. **54** (2013) 012109.

[9] Meng G., *The Poisson Realization of* $\mathfrak{so}(2, 2k+2)$ *on Magnetic Leaves,* J. Math. Phys. **54** (2013) 052902.

[10] Meng G., *Lorentz Group and Oriented McIntosh-Cisneros-Zwanziger-Kepler Orbits,* J. Math. Phys. **53** (2012) 052901.

[11] Meng G., *Euclidean Jordan Algebras, Hidden Actions, and J-Kepler Problems,* J. Math. Phys. **52** (2011) 112104.

[12] Meng G., *The Universal Kepler Problems*, J. Geom. Symmetry Phys. **36** (2014) 47-57

[13] Rutherford E., *The Scattering of Alpha and Beta Particles by Matter and the Structure of the Atom*, Philosophical Magazine **21** (1911) 669-688.

[14] Sternberg S., *Minimal Coupling and the Symplectic Mechanics of a Classical Particle in the Presence of a Yang-Mills Field*, Proc Nat. Acad. Sci. **74** (1977) 5253-5254.

[15] Tits T., *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan,* Nederl. Akad. van Wetens. **65** (1962) 530-525.

[16] Zwanziger D., *Exactly Soluble Nonrelativistic Model of Particles with Both Electric and Magnetic Charges*, Phys. Rev. **176** (1968) 1480-1488.

[17] Bouarroudj S. and Meng G., *The Classical Dynamic Symmetry for the* $\mathrm{U}(1)$*-Kepler Problems*, arXiv:1509.08263.