Seventeenth International Conference on
Geometry, Integrability and Quantization
June 5–10, 2015, Varna, Bulgaria
Ivaïlo M. Mladenov, Guowu Meng
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2016, pp 296–308
doi: 10.7546/giq-17-2016-296-308

SOME RESULTS ON FRENET RULED SURFACES ALONG THE EVOLUTE-INVOLUTE CURVES, BASED ON NORMAL VECTOR FIELDS IN E^3

ŞEYDA KILIÇOĞLU

Department of Mathematics Education, Başkent University, Ankara, Turkey

Abstract. In this paper we consider eight special Frenet ruled surfaces along to the involute-evolute curves, α^* and α respectively, with curvature $k_1 \neq 0$. First we find the excplit equation of Frenet ruled surfaces along the involute curves in terms of the Frenet apparatus of evolute curve α. Also normal vector fields of these Frenet ruled surfaces have been calculated too.

Further we give all results for sixteen positions of Normal vector fields of four Frenet ruled surfaces in terms of Frenet apparatus of evolute curve α. These results also give us the positions of eight special Frenet ruled surfaces along to the involute-evolute curves, based on their normal vectors, in terms of curvatures of evolute curve α. We can give the answer of the questions that in which condition we can produce orthogonal surfaces or surfaces with constant angle. For example Darboux ruled surface and *involutive tangent ruled surface* of an evolute α have the perpendicular normal vector fields.

MSC 2000: 53A04, 53A05
Keywords: Evolute curve, involute curve, ruled surfaces

1. Introduction and Preliminaries

Deriving curves based on the other curves is a subject in geometry. Involute-evolute curves, Bertrand curves are this kind of curves. By using the similiar method we produce a new ruled surface based on the other ruled surface. The Involutive B *scrolls* are defined in [11]. $\tilde{D}$ *scroll*, which is known as the rectifying developable surface, of any curve α and the *involute* $\tilde{D}$ *scroll* of the curve α are already defined, in Euclidean three-space. Also the differential geometric elements of the *involute* $\tilde{D}$ scroll are examined in [16]. In this paper we consider the following four special ruled surfaces associated to a space curve α with $k_1 \neq 0$. They are called as Frenet ruled surface, cause of their generators are the Frenet vector fields of a curve.

It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let the first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, D, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves. Here Darboux vector D is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. The Darboux vector provides a concise way of interpreting curvature k_1 and torsion k_2 geometrically as the curvature is the measure of the rotation of the Frenet frame about the binormal unit vector, and the torsion is the measure of the rotation of the Frenet frame about the tangent unit vector. For any unit speed curve α, in terms of the Frenet-Serret apparatus, the Darboux vector can be expressed as $D(s) = k_2(s)V_1(s) + k_1(s)V_3(s)$. The Darboux vector field of α and it has the following symmetrical properties [5, p.205]

$$D \times V_1 = \dot{V}_1, \qquad D \times V_2 = \dot{V}_2, \qquad D \times V_3 = \dot{V}_3.$$

Let a vector field be $\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s)$ along $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α [13]. Also it is trivial that $\tilde{D}(s)' = \left(\frac{k_2}{k_1}\right)' V_1$.The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where curvature functions are defined by $k_1(s) = \|V_1(s)\|$ and $k_2(s) = -\langle V_2, \dot{V}_3 \rangle$.

1.1. The Involute Curves and Frenet Apparatus

The involute of a given curve is a well-known concept in Euclidean three-space. We can say that evolute and involute is a method of deriving a new curve based on a given curve. The involute of the curve is called sometimes *the evolvent*. Involvents play a part in the construction of gears. The evolute is the locus of the centers of tangent circles of the given planar curve [15]. Let α and α^* be the curves in Euclidean three-space. The tangent lines to a curve α generate a surface called the tangent surface of α. If the curve α^* which lies on the tangent surface intersect the tangent lines orthogonally is called an involute of α. If a curve α^*

is an involute of α, then by definition α is an evolute of α^*. Hence given α, its evolutes are the curves whose tangent lines intersect α orthogonally. The quantities $\left\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\right\}$ and $\left\{V_1^*, V_2^*, V_3^*, \tilde{D}^*, k_1^*, k_2^*\right\}$ are collectively Frenet-Serret apparatus of the curve α and the *involute* α^*, respectively.

Theorem 1. *In the Euclidean three-space $\mathbf{E}^3$, α, $\alpha^* \subset \mathbf{E}^3$, α and α^* are the arclengthed curves with the arcparametres s and s^*, respectively*

$$\alpha^*(s) = \alpha(s) + \lambda V_1(s)$$

is the equation of involute of the curve α. Then we have the equalities

$$\langle V_1^*, V_1 \rangle = 0, \qquad V_2 = V_1^*.$$

(*For more detail see* [6], [14]).

Theorem 2. *The Frenet vectors of the* involute α^*, *based on the its evolute curve* α [6] *are*

$$\begin{aligned} V_1^* &= V_2, \qquad V_2^* = \frac{-k_1 V_1 + k_2 V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}}, \qquad V_3^* = \frac{k_2 V_1 + k_1 V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}} \\ \tilde{D}^* &= \frac{k_2}{\sqrt{k_1^2 + k_2^2}} V_1 - \frac{k_1' k_2 - k_1 k_2'}{\left(k_1^2 + k_2^2\right)^{\frac{3}{2}}} V_2 + \frac{k_1 V_3}{\sqrt{k_1^2 + k_2^2}}. \end{aligned} \tag{1}$$

The first curvature of involute α^* is

$$k_1^* = \frac{\sqrt{k_1^2 + k_2^2}}{(\sigma - s) k_1}, \qquad (\sigma - s) k_1 > 0, \qquad k_1 \neq 0.$$

The second curvature of involute α^* is

$$k_2^* = \frac{k_2' k_1 - k_1' k_2}{(\sigma - s) k_1 \left(k_1^2 + k_2^2\right)} = \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)'}{\lambda k_1 \left(k_1^2 + k_2^2\right)}, \quad \lambda = \sigma - s \ is\ constant.$$

Theorem 3. *The product of Frenet vector fields of the* involute α^*, *and its evolute curve α has the following matrix form*

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{bmatrix} V_1^* & V_2^* & V_3^* \end{bmatrix} = \frac{1}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}} \begin{bmatrix} 0 & -k_1 & k_2 \\ \left(k_1^2 + k_2^2\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & k_2 & k_1 \end{bmatrix}.$$

1.2. Ruled Surface

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean three-space [2, 3]. Choosing a directrix on the surface, i.e., a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\left\langle \alpha', v \right\rangle = 0$. To illustrate the current situation, we bring here the famous example of Graves (see [4]), so called the B-*scroll*. The special ruled surfaces B-*scroll* over null curves with null rulings in three-dimensional Lorentzian space form has been introduced by Graves. The Gauss map of B-scrolls has been examined in [12].

Definition 1. *In the Euclidean three-space, let* α *and* β *be the arclengthed curves. Let* $\beta(s)$ *be the involute of the curve* $\alpha(s)$ *. The Frenet vector fields* $V_1(s), V_2(s)$, $V_3(s)$ *and* $V_1^*(s), V_2^*(s), V_3^*(s)$ *of* α *and* involute β*, respectively. The equation* $\varphi^*(s,v) = \beta(s) + vV_3^*(s)$ *is the parametrization of the ruled surface which is called* involutive B-scroll (*binormal scroll*) *of the curve* α [11]. *The directrix of this surface is the involute curve* $\beta(s) = \alpha(s) + (\sigma - s)V_1(s)$ *of the curve* $\alpha(s)$*. The generating space of* B-scroll *is spaned by binormal subvector* V_3^**. Here* $\mathrm{Sp}\{V_1^*, V_2^*\}$ *is the osculator plane of the curve* β*.*

Definition 2. *The ruled surface* B-scroll *is a surface which can be parametrized as* $X(s,t) = \alpha(s) + tB(s)$*, a "ruled surface" in Lorentzian three-space* $\mathbf{L}^3$ *with null directrix curve and null rulings, i.e.,* $\alpha(s)$ *being a null curve and* $B(s)$ *a null normal vector field along* $\alpha(s)$*, satisfying* $\langle \dot{\alpha}, B \rangle = -1$ [4].

The fundamental forms of the B-*scroll* with null directrix and Cartan frame in the Minkowskian three-space is examined in [10]. The properties of the $B-scroll$ are also examined in Euclidean three-space and n-*space* and in Lorentzian three-space and n-*space* with time-like directrix curve and null rulings (see [7–9]).

2. Normal Vector Fields of Frenet and Involutive Frenet Ruled Surfaces

2.1. Frenet Ruled Surfaces

Frenet ruled surface is one which can be generated by the motion of a Frenet vector of any curve in Euclidean three-space. In this subsection tangent, normal, binormal, Darboux ruled surfaces of any curve are collectively named Frenet ruled surfaces. They have the following equations.

Definition 3 (Tangent ruled surface). *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^1(s,u_1) = \alpha(s) + u_1 V_1(s)$ is the parametrization of the ruled surface which is called V_1-*scroll *(tangent ruled surface). The directrix of this V_1-*scroll *is the curve $\alpha(s)$. The generating space of this $V_1 - scroll$ is spaned by tangent subvector V_1. Here* $\mathrm{Sp}\{V_2, V_3\}$ *is the normal plane of the curve α.*

Definition 4 (Normal ruled surface). *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^2(s,u_2) = \alpha(s) + u_2 V_2(s)$ is the parametrization of the ruled surface which is called $V_2 - scroll$ (normal ruled surface). The directrix of this $V_2 - scroll$ is the curve $\alpha(s)$. The generating space of this $V_2 - scroll$ is spaned by normal subvector V_2. Here* $\mathrm{Sp}\{V_1, V_3\}$ *is the rectifying plane of the curve α.*

Definition 5 (Binormal ruled surface). *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^3(s,u_3) = \alpha(s) + u_3 V_3(s)$ is the parametrization of the ruled surface which is called $V_3 - scroll$ (binormal ruled surface). The directrix of this $V_3 - scroll$ is the curve $\alpha(s)$. The generating space of this $V_3 - scroll$ is spaned by binormal subvector V_3. Here* $\mathrm{Sp}\{V_1, V_2\}$ *is the osculator plane of the curve α.*

Definition 6 (Darboux ruled surface). *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^4(s,u_4) = \alpha(s) + u_4 \tilde{D}(s)$ is the parametrization of the ruled surface which is called rectifying developable surface of the curve α in* [13]. *Here, it is referred to as the* ***Darboux ruled surface*** *because the generator vector is modified Darboux vector field $\tilde{D}$.*

2.2. Normal Vector Fields of the Frenet Ruled Surfaces

The following theorem gives us four normal vector fields of Frenet ruled surfaces with a simple matrix product.

Theorem 4. *In the Euclidean three-space, let η_1, η_2, η_3, and η_4 be the normal vector fields of ruled surfaces $\varphi^1, \varphi^2, \varphi^3$, and φ^4, respectively, along the curve α. They can be expressed by the following matrix*

$$[\eta] = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} [A]\,[V] = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where

$$a = \frac{-u_2 k_2}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}, \qquad c = \frac{-u_3 k_2}{\sqrt{(u_3 k_2)^2 + 1}}$$
$$b = \frac{(1 - u_2 k_1)}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}, \qquad d = \frac{-1}{\sqrt{(u_3 k_2)^2 + 1}}.$$

Proof: The normal vector fields η_1, η_2, η_3, and η_4 of ruled surfaces $\varphi^1, \varphi^2, \varphi^3$, and φ^4 can be expressed as in the following four equalities

$$\eta_1 = -V_3, \;\; \eta_2 = \frac{-u_2 k_2 V_1 + (1 - u_2 k_1) V_3}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}, \;\; \eta_3 = \frac{-u_3 k_2 V_1 - V_2}{\sqrt{(u_3 k_2)^2 + 1}}, \;\; \eta_4 = -V_2.$$

■

2.3. Involutive Frenet Ruled Surfaces

In this subsection, the tangent, normal, binormal, Darboux Frenet ruled surfaces of the *involute* α^* of the evolute α have been given as in the following definitions. First we find the equations of Frenet ruled surfaces along the involute curves. Then we write their parametric equations in terms of the Frenet apparatus of the evolute curve α. Hence they are called "*the involutive tangent, normal, binormal, Darboux Frenet ruled surfaces of evolute* α" as in the following way.

Definition 7 (The involutive tangent ruled surface)**.** *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^{*1}(s, v_1) = \alpha^*(s) + v_1 V_1^*(s)$ is the parametrization of the ruled surface which is called V_1^**-scroll (*tangent ruled surface*)*. The directrix of this V_1^**-scroll *is the curve $\alpha^*(s)$. The generating space of this V_1^**-scroll *is spaned by tangent subvector V_1^*. Here* $\mathrm{Sp}\{V_2^*, V_3^*\}$ *is the normal plane of the curve α. Also we can write*

$$\varphi^{*1}(s, v_1) = \alpha(s) + (\sigma - s) V_1(s) + v_1 V_2(s).$$

Here we renamed this surface as ***the involutive tangent ruled surface of the curve*** *α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.*

Definition 8 (The involutive normal ruled surface)**.** *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^{*2}(s, v_2) = \alpha^*(s) + v_2 V_2^*(s)$ is the parametrization of the ruled surface which is called V_2^**-scroll (*normal ruled surface*)*. The directrix of this V_2^**-scroll *is the curve $\alpha^*(s)$. The generating space of this V_2^**-scroll *is spaned by normal subvector V_2^*. Here* $\mathrm{Sp}\{V_1^*, V_3^*\}$ *is the* *rectifying plane of the curve α^*. Also we can write*

$$\varphi^{*2}(s, v_2) = \alpha(s) + (\sigma - s)V_1(s) + v_2\Big(\frac{-k_1V_1 + k_2V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}}\Big).$$

Here we renamed this surface as ***the involutive normal ruled surface of the curve*** *α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.*

Definition 9 (The involutive binormal ruled surface)**.** *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^{*3}(s, v_3) = \alpha^*(s) + v_3V_3^*(s)$ is the parametrization of the ruled surface which is called V_3^*-*scroll *(binormal ruled surface). The directrix of this V_3^*-*scroll *is the curve α^*. The generating space of this V_3^*-*scroll *is spaned by binormal subvector V_3^*. Here* $\mathrm{Sp}\{V_1^*, V_2^*\}$ *is the osculator plane of the curve α^*. Also we can write*

$$\varphi^{*3}(s, v_3) = \alpha(s) + (\sigma - s)V_1(s) + v_3\Big(\frac{k_2V_1 + k_1V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}}\Big).$$

Here we renamed this surface as ***the involutive binormal ruled surface of the curve*** *α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.*

Definition 10 (The involutive Darboux ruled surface)**.** *In the Euclidean three-space, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^{*4}(s, v_4) = \alpha^*(s) + v_4\tilde{D}^*(s)$ is the parametrization of the ruled surface which is called rectifying developable surface of the curve α in* [13]. *Also we can write*

$$\begin{aligned}\varphi^{*4}(s, v_4) \;=\; & \alpha(s) + \Big((\sigma - s) + v_4\frac{k_2}{\sqrt{k_1^2 + k_2^2}}\Big)V_1 \\ & -v_4\frac{k_1'k_2 - k_1k_2'}{\left(k_1^2 + k_2^2\right)^{\frac{3}{2}}}V_2 + v_4\frac{k_1}{\sqrt{k_1^2 + k_2^2}}V_3.\end{aligned}$$

Here we renamed this surface as ***the involutive Darboux ruled surface of the curve*** *α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.*

2.4. Normal Vector Fields of the Involutive Frenet Ruled Surfaces

Theorem 5. *In the Euclidean three-space, the normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of ruled surfaces $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$, respectively, along the curve $involute\ \alpha^*$, can* *be expressed by the following matrix*

$$[\eta^*] = \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \\ \eta_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

where

$$a^* = \frac{-v_2 k_2^*}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}}, \qquad c^* = \frac{-v_3 k_2^*}{\sqrt{(v_3 k_2^*)^2 + 1}}$$

$$b^* = \frac{(1 - v_2 k_1^*)}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}}, \qquad d^* = \frac{-1}{\sqrt{(v_3 k_2^*)^2 + 1}}.$$

Proof: It is trivial. ■

Theorem 6. *In the Euclidean three-space, the normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of ruled surfaces $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$, respectively, can be expressed in terms of the Frenet apparatus of evolute curve α by the following matrix*

$$a^* = \frac{v_2 k_2^2 \left(\frac{k_1}{k_2}\right)'}{\sqrt{v^2 \left(-k_2^2 \left(\frac{k_1}{k_2}\right)'\right)^2 + \left(k_1^2 + k_2^2\right)^2 \left(\lambda k_1 - v\sqrt{k_1^2 + k_2^2}\right)^2}}$$

$$b^* = \frac{\left(\lambda k_1 \left(k_1^2 + k_2^2\right) - v_2 \left(k_1^2 + k_2^2\right)^{\frac{3}{2}}\right)}{\sqrt{v^2 \left(-k_2^2 \left(\frac{k_1}{k_2}\right)'\right)^2 + \left(k_1^2 + k_2^2\right)^2 \left(\lambda k_1 - v\sqrt{k_1^2 + k_2^2}\right)^2}}$$

$$c^* = \frac{v_3 k_2^2 \left(\frac{k_1}{k_2}\right)'}{\sqrt{v_3^2 \left(-k_2^2 \left(\frac{k_1}{k_2}\right)'\right)^2 + \left(\lambda k_1 \left(k_1^2 + k_2^2\right)\right)^2}}$$

and

$$d^* = \frac{-\lambda k_1 \left(k_1^2 + k_2^2\right)}{\sqrt{v_3^2 \left(k_2^2 \left(\frac{k_1}{k_2}\right)'\right)^2 + \left(\lambda k_1 \left(k_1^2 + k_2^2\right)\right)^2}}.$$

3. Some Results on Normal Vector Fields of Frenet Ruled Surfaces

In this section the unit normal vector fields of Frenet ruled surfaces and involutive Frenet ruled surface are examined with the equality of the following matrices.

Theorem 7. *In the Euclidean three-space, the position of the unit normal vector field η_1^*, η_2^*, η_3^*, and η_4^* of ruled surfaces φ^{*1}, φ^{*2}, φ^{*3}, and φ^{*4}, respectively, along the curve $involute$ α^*, can be expressed by the following matrix*

$$[\eta]\,[\eta^*]^T = \begin{bmatrix} \frac{k_1}{\sqrt{k_1^2+k_2^2}} & \frac{-k_1 b^*}{\sqrt{k_1^2+k_2^2}} & \frac{-k_2 d^*}{\sqrt{k_1^2+k_2^2}} & \frac{k_2}{\sqrt{k_1^2+k_2^2}} \\ \frac{-(ak_2+bk_1)}{\sqrt{k_1^2+k_2^2}} & \frac{b^*(ak_2+bk_1)}{\sqrt{k_1^2+k_2^2}} & \frac{(-ak_1+bk_2)d^*}{\sqrt{k_1^2+k_2^2}} & \frac{-(-ak_1+bk_2)}{\sqrt{k_1^2+k_2^2}} \\ \frac{-ck_2}{\sqrt{k_1^2+k_2^2}} & \frac{a^*d\sqrt{k_1^2+k_2^2}+ck_2b^*}{\sqrt{k_1^2+k_2^2}} & \frac{\sqrt{k_1^2+k_2^2}dc^*-k_1cd^*}{\sqrt{k_1^2+k_2^2}} & \frac{ck_1}{\sqrt{k_1^2+k_2^2}} \\ 0 & -a^* & -c^* & 0 \end{bmatrix}. \tag{2}$$

Proof: Let $[\eta] = [A]\,[V]$ and $[\eta^*] = [A^*]\,[V^*]$, hence

$$\begin{aligned} [\eta]\,[\eta^*]^T &= [A]\,[V]\,([A^*]\,[V^*])^T = [A]\left([V]\,[V^*]^T\right)[A^*]^T \\ &= \frac{1}{\sqrt{k_1^2+k_2^2}}[A]\begin{bmatrix} 0 & -k_1 & k_2 \\ \sqrt{k_1^2+k_2^2} & 0 & 0 \\ 0 & k_2 & k_1 \end{bmatrix}[A^*]^T \end{aligned}$$

we get the proof. ■

Theorem 8. *In the Euclidean three-space, the position of the unit normal vector field $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of ruled surfaces $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$, respectively, along the curve $involute$ α^*, can be expressed by the following equations*

$$[\eta]\,[\eta^*]^T = \begin{bmatrix} \langle\eta_1,\eta_1^*\rangle & \langle\eta_1,\eta_2^*\rangle & \langle\eta_1,\eta_3^*\rangle & \langle\eta_1,\eta_4^*\rangle \\ \langle\eta_2,\eta_1^*\rangle & \langle\eta_2,\eta_2^*\rangle & \langle\eta_2,\eta_3^*\rangle & \langle\eta_2,\eta_4^*\rangle \\ \langle\eta_3,\eta_1^*\rangle & \langle\eta_3,\eta_2^*\rangle & \langle\eta_3,\eta_3^*\rangle & \langle\eta_3,\eta_4^*\rangle \\ \langle\eta_4,\eta_1^*\rangle & \langle\eta_4,\eta_2^*\rangle & \langle\eta_4,\eta_3^*\rangle & \langle\eta_4,\eta_4^*\rangle \end{bmatrix} \tag{3}$$

here $[\eta^*]^T$ is the tranpose matrix of $[\eta^*]$.

In the Euclidean three-space, the position of two surface, basicly, can be examined by the position of their unit normal vector fields. According the equalities of the last matrice we have the simle results as in the following theorems. In this section using the equality of the matrices (2) and (3), we give sixteen interesting results according to the normal vector fields with the following theorems.

Theorem 9. *Tangent ruled surface and* involutive tangent ruled surface *of the evolute α have normal vector fields with angle ϕ between $\eta_1$and η_1^* which is a nonzero function of the curvatures k_1 and k_2 of the evolute α*

$$\cos\phi = \frac{k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}}.$$

Proof: Since $\langle \eta_1, \eta_1^* \rangle = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$, it is trivial. ■

Theorem 10. *Tangent ruled surface and* involutive normal ruled surface *of the evolute α have perpendicular along the curve*

$$\varphi^{*2}(s, v_2) = \alpha(s) + (\sigma - s)V_1(s) + v_2 \left(\frac{-k_1 V_1 + k_2 V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}} \right)$$

$$\varphi^{*2}(s) = \alpha(s) + \left(\lambda - \frac{\lambda k_1^2}{k_1^2 + k_2^2} \right) V_1(s) + \frac{\lambda k_1 k_2}{k_1^2 + k_2^2} V_3(s).$$

Proof: Since $\langle \eta_1, \eta_2^* \rangle = \frac{-k_1\left(1-v_2 k_1^*\right)}{\sqrt{k_1^2+k_2^2}\sqrt{\left(v_2 k_2^*\right)^2+\left(1-v_2 k_1^*\right)^2}}$, and for $\langle \eta_1, \eta_2^* \rangle = 0$ we have perpendicular normal vector fields under the condition $v_2 = \frac{\lambda k_1}{\sqrt{k_1^2+k_2^2}}$ we have the proof. ■

Theorem 11. *Tangent ruled surface and* involutive binormal ruled surface *of the evolute α have not perpendicular normal vector fields, except $\lambda = 0$.*

Proof: Since $\langle \eta_1, \eta_3^* \rangle = \frac{-k_2 d^*}{\sqrt{k_1^2+k_2^2}}$, and $\langle \eta_1, \eta_3^* \rangle = 0$ we have $\lambda k_1 k_2 \left(k_1^2 + k_2^2\right) = 0$. ■

Theorem 12. *Tangent ruled surface and* involutive Darboux ruled surface *of the evolute α have normal vector fields with angle μ which is the nonzero function of the curvatures k_1 and k_2 $\cos\mu = \frac{k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}}$.*

Proof: Since $\langle \eta_1, \eta_4^* \rangle = \frac{k_2}{\sqrt{k_1^2+k_2^2}}$, it is trivial. ■

Theorem 13. *Normal ruled surface and* involutive tangent ruled surface *of the evolute α have the perpendicular normal vector fields along the curve $\varphi^2(s) = \alpha(s) + \frac{k_1}{k_1^2+k_2^2} V_2(s)$.*

Proof: Since $\langle \eta_2, \eta_1^* \rangle = \frac{-ak_2-bk_1}{\sqrt{k_1^2+k_2^2}}$, and for $\langle \eta_2, \eta_1^* \rangle = 0$ we have perpendicular normal vector fields under the condition $u_2 = \frac{k_1}{k_1^2+k_2^2}$, we have the proof. ■

Theorem 14. *Normal ruled surface and* involutive normal ruled surface *of the evolute α have the perpendicular normal vector fields along the curves $\varphi^2(s) = \alpha(s) + \frac{k_1}{k_1^2+k_2^2} V_2(s)$ or $\varphi^{*2}(s) = \alpha(s) + \lambda V_1(s) + \frac{-\lambda k_1^2 V_1 + \lambda k_1 k_2 V_3}{k_1^2+k_2^2}(s)$.*

Proof: Since $\langle \eta_2, \eta_2^* \rangle = \frac{-(ak_2+bk_1)b^*}{\sqrt{k_1^2+k_2^2}}$ and for $\langle \eta_2, \eta_2^* \rangle = 0$ under the condition $u_2 = \frac{k_1}{\left(k_1^2+k_2^2\right)}$ or $v_2 = \frac{\lambda k_1}{\sqrt{k_1^2+k_2^2}}$, we have the proof. ■

Theorem 15. *Normal ruled surface and* involutive binormal ruled surface *of the evolute α have not the perpendicular normal vector fields except $\lambda k_1 k_2 = 0$.*

Proof: Since $\langle \eta_2, \eta_3^* \rangle = \frac{(-ak_1+bk_2)d^*}{\sqrt{k_1^2+k_2^2}}$ and for $\langle \eta_2, \eta_3^* \rangle = 0$ it is trivial. ■

Theorem 16. *Normal ruled surface and* involutive Darboux ruled surface *of the evolute α have not the perpendicular normal vector fields unless $k_2 \neq 0$.*

Proof: Since $\langle \eta_2, \eta_4^* \rangle = \frac{ak_1-bk_2}{\sqrt{k_1^2+k_2^2}}$ and for $\langle \eta_2, \eta_4^* \rangle = 0$ it is trivial. ■

Theorem 17. *Binormal ruled surface and* involutive tangent ruled surface *of the evolute α have the perpendicular normal vector fields only along the evolute α.*

Proof: Since $\langle \eta_3, \eta_1^* \rangle = \frac{-k_2 c}{e}$, and for $\langle \eta_3, \eta_1^* \rangle = 0$, under the condition $u_3 = 0$, it completes the proof. ■

Theorem 18. *Binormal ruled surface and* involutive normal ruled surface *of the evolute α have the perpendicular normal vector fields under the condition*

$$v_2 = \frac{u_3 \lambda k_1 k_2^2 \sqrt{k_1^2 + k_2^2}}{k_2^{\prime} k_1 - k_1^{\prime} k_2 + u_3 k_2^2 \left(k_1^2 + k_2^2\right)}.$$

Proof: Since $\langle \eta_3, \eta_2^* \rangle = \frac{\sqrt{k_1^2+k_2^2} da^* + k_2 cb^*}{\sqrt{k_1^2+k_2^2}}$ and for $\langle \eta_3, \eta_2^* \rangle = 0$ it can be calculated easily. Where u_3 and v_2 are the parameters of binormal ruled surface and *involutive normal ruled surface* of the evolute α respectively. ■

Theorem 19. *Binormal ruled surface and* involutive binormal ruled surface *of the evolute α have the perpendicular normal vector fields under the condition*

$$v_3 = u_3 \frac{\lambda k_1^2 \sqrt{k_1^2 + k_2^2}}{-k_2 \left(\frac{k_1}{k_2}\right)^{\prime}}.$$

Proof: Since $\langle \eta_3, \eta_3^* \rangle = \frac{\sqrt{k_1^2+k_2^2} dc^* - k_1 cd^*}{\sqrt{k_1^2+k_2^2}}$ and for $\langle \eta_3, \eta_3^* \rangle = 0$ it can be calculated easily. ■

Theorem 20. *Binormal ruled surface and* involutive Darboux ruled surface *of the evolute α have not the perpendicular normal vector fields under the condition $u_3 = 0$, here u_3 is a parameter of binormal ruled surface of the evolute α.*

Proof: Since $\langle \eta_3, \eta_4^* \rangle = \frac{k_1 c}{\sqrt{k_1^2+k_2^2}}$ and $\langle \eta_3, \eta_4^* \rangle = 0$ it is trivial. ■

Theorem 21. *Darboux ruled surface and* involutive tangent ruled surface *of an evolute α have the perpendicular normal vector fields.*

Proof: Since $\langle \eta_4, \eta_1^* \rangle = 0$ it is trivial. ■

Theorem 22. *Darboux ruled surface and* involutive normal ruled surface *of an evolute α have perpendicular normal vector fields if α is a general helix or α is a planar curve.*

Proof: The angle between η_4 and η_2^* is a nonzero function of the curvatures of the curve α. Since

$$\langle \eta_4, \eta_2^* \rangle = -a^* = \frac{v_2 k_2^*}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}}$$

and for $\langle \eta_4, \eta_2^* \rangle = 0$, we have $v_2 \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)'}{\lambda k_1 \left(k_1^2 + k_2^2\right)} = 0$. We get the proof. ■

Theorem 23. *Darboux ruled surface and* involutive binormal ruled surface *of an evolute α have the perpendicular normal vector fields under the condition*

$$v_3 \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)'}{\lambda k_1 \left(k_1^2 + k_2^2\right)} = 0.$$

Proof: Since $\langle \eta_4, \eta_3^* \rangle = -c^* e$ and for $\langle \eta_4, \eta_3^* \rangle = 0$ we get the proof. The angle between η_4 and η_1^* is a nonzero function of the curvatures of the curve α

$$\langle \eta_4, \eta_3^* \rangle = -c^* = -\frac{-v_3 k_2^*}{\sqrt{(v_3 k_2^*)^2 + 1}} = \frac{v_3 k_2^*}{\sqrt{(v_3 k_2^*)^2 + 1}}.$$

■

Theorem 24. *Darboux ruled surface and* involutive Darboux ruled surface *of an evolute α have the perpendicular normal vector fields.*

Proof: Since $\langle \eta_4, \eta_4^* \rangle = 0$, it is trivial. ■

References

[1] Boyer C., *A History of Mathematics*, Wiley, New York 1968.

[2] Do Carmo M., *Differential Geometry of Curves and Surfaces,* Prentice-Hall, Englewood Cliffs 1976.

[3] Eisenhart P., *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover, New York 2004.

[4] Graves L., *Codimension One Isometric Immersions Between Lorentz Spaces*, Trans. Amer. Math. Soc. **252** (1979) 367–392.

[5] Gray A., *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2^{nd} Edn CRC Press, Boca Raton 1997.

[6] Hacisalihoğlu H., *Diferensiyel Geometri*, Cilt 1, Ínönü Üniversitesi Yayinlari, Malatya 1994.

[7] Kiliçoğlu Ş., *n-Boyutlu Lorentz uzayinda B-scrollar,* PhD Thesis, Ankara Üniversitesi Fen Bilimleri Enstitüsü, Ankara 2006.

[8] Kiliçoğlu Ş., *On the B-Scrolls with Time-Like Directrix in 3-Dimensional Minkowski Space,* Beykent University Journal of Science and Technology, **2** (2008) 206-.

[9] Kiliçoğlu Ş., *On the Generalized B-Scrolls with p th Degree in n-Dimensional Minkowski Space and Striction* (*Central Spaces*)*,* Sakarya Üniversitesi Fen Edebiyat Dergisi **10** (2008) 15-29.

[10] Kiliçoğlu Ş., Hacisalihoglu H. and Şenyurt S., *On the Fundamental Forms of the B-Scroll with Null Directrix and Cartan Frame in Minkowskian 3-Space,* Applied Mathematical Sciences **9** (2015) 3957 - 3965, doi.org/10.12988/ams.2015.53230.

[11] Kiliçoğlu Ş., *On the Involutive B-scrolls in the Euclidean Three-space E^3*, Geom. Integrability & Quantization **13** (2012) 205-214.

[12] L. Alias, Ferrandez A., Lucas P. and Merono M., *On the Gauss Map of B-Scrolls*, Tsukuba J. Math. **22** (1998) 371-377.

[13] Izumiya S. and Takeuchi N., *Special curves and Ruled surfaces,* Beitrage zur Algebra und Geometrie **44** (2003) 203-212.

[14] Lipschutz M., *Differential Geometry*, Schaum's Outlines.

[15] Springerlink, *Encyclopaedia of Mathematics*, Springer, Berlin 2002.

[16] Şenyurt, S. and Kiliçoğlu Ş., *On the Differential Geometric Elements of the* $Involute\ \tilde{D}\ Scroll$, Advances in Applied Clifford Algebras, Springer 2015, doi:10.1007/s00006-015-0535-z.