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A HARMONIC ENDOMORPHISM IN A SEMI-RIEMANNIAN CONTEXT

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Abstract. On the total space of the cotangent bundle T^*M of a Riemannian manifold (M, h) we consider the natural Riemann extension $\bar{g}$ with respect to the Levi-Civita connection of h. In this setting, we construct on T^*M a new para-complex structure P, whose harmonicity with respect to $\bar{g}$ is characterized here by using the reduction of $\bar{g}$ to the (classical) Riemann extension.

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1. Introduction

Let M be a connected smooth n-dimensional manifold and let T^*M be its cotangent bundle. We suppose that the manifold M is endowed with a symmetric linear connection ∇. In [12], Patterson and Walker introduced the (classical) Riemann extension that was generalized by Sekizawa and Kowalski to natural Riemann extension, which is a semi-Riemannian metric of signature (n, n), on the total space of T^*M, (see [14] and [11]). Later, Bejan and Kowalski [5] characterized harmonic functions with respect to the natural Riemann extension $\bar{g}$ on T^*M. Also, the natural Riemann extension is a special class of modified Riemann extensions which is studied in [7] and [10].

Harmonicity is a very interesting topic in some mathematical fields, such as differential geometry, analysis, partial differential equations, theoretical physics and so on. We recall that a C^2- map $\varphi : (N, h) \to (\bar{N}, \bar{h})$ between (semi-)Riemannian manifold is harmonic if its tension field $\tau(\varphi)$ vanishes identically. This means that φ satisfies the Euler-Lagrange equations.

Later on, Garciá-Río, Vanhecke and Vázquez-Abal introduced the harmonicity of a (1,1)-type tensor field T on a (semi-)Riemannian manifold N. In [9], they say that a (1,1)-type tensor field on a (semi-)Riemannian manifold (N, h) is called harmonic if it is a harmonic map when it is viewed as a map $T : (TN, h^c) \to (TN, h^c)$ between (semi-)Riemannian manifolds, where c denotes the complete lift. Also, they characterized the harmonicity of a (1,1)-type tensor field as being divergence-free, i.e., $\delta T = 0$.

If (M, ∇) is a manifold endowed with a symmetric linear connection, we have constructed on the total space of T^*M a canonical almost product structure P (i.e., $P^2 = \mathrm{Id}$ and $P \neq \pm\mathrm{Id}$) which preserves the vertical and the complete lift [6]. We proved there that P was almost para-complex, since its eigen values $+1$ and -1 have the same multiplicity. (For the notion of almost para-complex structure, we refer the reader to [8] and [1–3]). Moreover, in [6] we characterized the harmonicity of P, (viewed as an endomorphism field, or as a (1,1)-tensor field) in the sense of [9], with respect to the natural Riemann extension $\bar{g}$ on T^*M.

In the present paper, we construct a new structure P on the total space T^*M of the Riemannian manifold (M, h), which inverts the vertical and the complete lifts and we prove that P is para-complex. Then, we give a necessary and sufficient condition such that the endomorphism field P is harmonic in the sense of [9] with respect to the natural Riemann extension $\bar{g}$ on T^*M, constructed with the Levi-Civita connection ∇ of h.

2. Preliminaries

Let M be a connected smooth n-dimensional manifolds and T^*M denotes its cotangent bundle. Let $p : T^*M \to M$ be a natural projection from the cotangent bundle T^*M to a manifold M. At any arbitrary point $x \in M$, any local chart $\left(U; x^1, ..., x^n\right)$ correspond to $(p^{-1}(U); x^1, ..., x^n, x^{1*}, ..., x^{n*})$ at $(x, w) \in T^*M$. We define the function $x^i \circ p$ on $p^{-1}(U)$ with x^i on U, where $x^{i*} = w_i = w\left(\left(\frac{\partial}{\partial x^i}\right)_x\right)$ at each point $(x, w) \in T^*M$, $i = 1, \ldots, n$. Then, at any point (x, w), we get a basis for the tangent space of $(T^*M)_{(x,w)}$

$$\left\{(\partial_1)_{(x,w)}, ..., (\partial_n)_{(x,w)}, (\partial_{1*})_{(x,w)}, ..., (\partial_{n*})_{(x,w)}\right\}.$$

We denote $\partial_i = \dfrac{\partial}{\partial x^i}$ and $\partial_{i*} = \dfrac{\partial}{\partial w_i}$, $i = 1, \ldots, n$.

Let $W \in \chi(T^*M)$ denote the canonical vertical vector field on T^*M which is a global vector field defined in local coordinate systems, by

$$W = \sum_{i=1}^{n} w_i \partial_{i*}. \tag{1}$$

For any $\alpha \in \Omega^1(M)$ (which is a differential form on M), its *vertical lift* α^v is a vector field which is tangent to T^*M and defined by

$$\alpha^v(Z^v) = (\alpha(Z))^v, \qquad Z \in \chi(M). \tag{2}$$

In local coordinates one has

$$\alpha^v = \sum_{i=1}^{n} \alpha_i \partial_{i*} \tag{3}$$

where $\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{d}x^i$. Also, for any $f^v = f \circ p \in \mathcal{F}(T^*M)$ with $f \in \mathcal{F}(M)$, we note that $\alpha^v(f^v) = 0$, $f \in \mathcal{F}(M)$.

For any vector field $X \in \chi(M)$, *the complete lift* is defined as a vector field $X^c \in \chi(T^*M)$ such that

$$X^c(Z^v) = [X, Z]^v, \qquad Z \in \chi(M). \tag{4}$$

In local coordinates, one has

$$X^c_{(x,w)} = \sum_{i=1}^{n} \xi^i(x)(\partial_i)_{(x,w)} - \sum_{h,i=1}^{n} w_h \left(\partial_i \xi^h\right)(x)(\partial_{i*})_{(x,w)}$$

where $X = \xi^i \partial_i$. We also have $X^c(f^v) = (Xf)^v$, $f \in \mathcal{F}(M)$.

Now, we recall the following definition which was given in [14], as a generalization of the (classical) Riemann extension defined in [12]

Definition 1. *Let M be a manifold endowed with symmetric linear connection ∇. Then, the natural Riemann extension $\bar{g}$ is defined at each point $(x, w) \in T^*M$ such that*

$$\begin{aligned} &\bar{g}_{(x,w)}(X^c, Y^c) = -aw(\nabla_{X_x} Y + \nabla_{Y_x} X) + bw(X_x)w(Y_x) \\ &\bar{g}_{(x,w)}(X^c, \alpha^v) = a\alpha_x(X_x), \qquad \bar{g}_{x,w}(\alpha^v, \beta^v) = 0 \end{aligned} \tag{5}$$

for all vector fields X, Y and all differential one-forms α, β on M, where a, b are arbitrary constants. We may assume $a > 0$ without loss of generality.

We note that if $a = 1$ and $b = 0$, then $(T^*M, \bar{g})$ is the classical Riemann extension of M endowed with ∇ (see [12] and [15]).

Let (x, w) be an arbitrary fixed point of T^*M, where $w \neq 0$. We take $\{\alpha_1, ..., \alpha_n\}$ to be a basis of covectors on T_x^*M such that

$$\alpha_1 = w \tag{6}$$

and let $\{e_1, ..., e_n\}$ be its dual basis on T_xM. We denote by the same letter e_i the parallel extension of each e_i (along geodesic starting at x) to a normal neighborhood of x in M, for $i = 1, \ldots, n$, (see [11]). We obtain a local frame $\{e_1, ..., e_n\}$ defined around x in M, such that

$$(\nabla_{e_i} e_j)_x = 0, \qquad i, j = 1, \ldots, n. \tag{7}$$

We note that

$$\bar{g}_{(x,w)} \left(e_i^c, e_j^c\right) = bw\left(e_{i,x}\right) w\left(e_j, x\right), \qquad i, j = 1, \ldots, n.$$

Next, we denote by the same letter $\{\alpha_1, ..., \alpha_n\}$ the local coframe defined around x on M, which is dual to the local frame $\{e_1, ..., e_n\}$, i.e., $\alpha_i(e_j) = \delta_{ij},\ i, j = 1, \ldots, n$, and we have automatically $\alpha_{1,x} = w$.

We construct as in [5], an orthonormal basis $\{E_i, E_{i*}\}_{i=1,\ldots,n}$ with respect to $\bar{g}$ in $T_{(x,w)}\left(T^*M\right)$ which is defined at any point $(x, w) \in T^*M$ by the formulas

$$\begin{aligned} E_1 &= e_1^c + \frac{1-b}{2a}\alpha_1^v, & E_{1*} &= e_1^c - \frac{1+b}{2a}\alpha_1^v \\ E_k &= \frac{1}{\sqrt{2a}}\left(e_k^c + \alpha_k^v\right), & E_{k*} &= \frac{1}{\sqrt{2a}}\left(e_k^c - \alpha_k^v\right). \end{aligned} \tag{8}$$

It follows that $\bar{g}\left(E_i, E_i\right) = 1$ and respectively $\bar{g}\left(E_{i*}, E_{i*}\right) = -1,\ i = 1, \ldots, n$, from which we can see that $\bar{g}$ is of signature (n, n).

Now, we recall the following statement which is given in [16]

Proposition 2. *Let X and Y be two vector fields on T^*M. If $X\left(Z^v\right) = Y\left(Z^v\right)$ holds for all $Z \in \chi\left(M\right)$, then $X = Y$.*

Later, we use the following

Notation 1. If $\mathcal{T}$ is a $(1,1)$-tensor field on a manifold M, then *the contracted vector field $\mathcal{C}\left(\mathcal{T}\right) \in \chi\left(T^*M\right)$* is defined at any point $(x, w) \in T^*M$, by its value on any vertical lift as follows

$$\mathcal{C}\left(\mathcal{T}\right)\left(X^v\right)_{(x,w)} = \left(\mathcal{T}X\right)^v_{(x,w)} = w\left(\left(\mathcal{T}X\right)_x\right), \qquad X \in \chi\left(M\right).$$

For the Levi-Civita connection $\bar{\nabla}$ of the Riemann extension $\bar{g}$, we get the following formulas (see e.g. [11])

$$\begin{aligned}
\left(\bar{\nabla}_{X^c}Y^c\right)_{(x,w)} &= (\nabla_X Y)^c_{(x,w)} + C_w\left((\nabla X)(\nabla Y) + (\nabla Y)(\nabla X)\right)_{(x,w)} \\
&\quad + C_w\left(R_x(\cdot, X)Y + R_x(\cdot, Y)X\right)_{(x,w)} - \frac{c}{2}\{w(Y)X^c \\
&\quad + w(X)Y^c + 2w(Y)C_w(\nabla X) + 2w(X)C_w(\nabla Y) \\
&\quad + w(\nabla_X Y + \nabla_Y X)W\}_{(x,w)} + c^2 w(X)w(Y)W_{(x,w)} \qquad (9) \\
\left(\bar{\nabla}_{X^c}\beta^v\right)_{(x,w)} &= (\nabla_X\beta)^v_{(x,w)} + \frac{c}{2}\{w(X)\beta^v + \beta(X)W\}_{(x,w)} \\
\left(\bar{\nabla}_{\alpha^v}Y^c\right)_{(x,w)} &= -\left(i_\alpha(\nabla Y)\right)^v_{(x,w)} + \frac{c}{2}\{w(Y)\alpha^v + \alpha(Y)W\}_{(x,w)} \\
\left(\bar{\nabla}_{\alpha^v}\beta^v\right)_{(x,w)} &= 0, \qquad X, Y \in \chi(M), \qquad \alpha, \beta \in \Omega^1(M)
\end{aligned}$$

where the coefficient c denotes the fraction $\frac{b}{a}\cdot$ For any $(1,1)$-tensor field $\mathcal{T}$ and any one-form α on M, we denote by $i_\alpha(\mathcal{T})$ the one-form of M defined by

$$(i_\alpha(\mathcal{T}))(X) = \alpha(\mathcal{T}X), \qquad X \in \chi(M).$$

3. Harmonicity of an Almost Para-Complex Structure

In this section, we assume (M, h) to be a Riemannian n-dimensional manifold and let ∇ be the Levi-Civita connection of h. Here we construct an almost product structure P on T^*M and we show that P is para-complex. We provide a necessary and sufficient condition for which P is harmonic on T^*M, with respect to the natural Riemann extension $\bar{g}$. Then, as a consequences, we characterize the classical Riemann extension in terms of the harmonicity of P.

Definition 3. *We define a linear transformation by*

$$P : \chi(T^*M) \to \chi(T^*M), \quad \textit{where} \quad PX^c = \alpha^v, \qquad P\alpha^v = X^c \qquad (10)$$

where X^c and α^v are respectively the complete lift of any vector field $X \in \chi(M)$ and the vertical lift of a differential one-form α which is dual to X, with respect to h on M.

Remark 4. *Different from the endomorphism P constructed in* [6]*, which preserves both the vertical and complete lifts, here the $(1,1)$-tensor field P given by* (10) *inverts the vertical and complete lifts.*

In what follows, we use the standard notation for the musical isomorphism $\alpha \in T^*M \to \alpha^\sharp \in TM$ defined by h, such that $h(\alpha^\sharp, Y) = \alpha(Y), \ Y \in \chi(M)$.

Proposition 5. *Let (M, h) be a Riemannian manifold. Then, the endomorphism P constructed by* (10) *is an almost para-complex structure on the total space of T^*M.*

Proof: First, we note that P is an almost product structure, since $P^2 = \mathrm{Id}$ and $P \neq \pm\mathrm{Id}$. We remark that if $X \in \chi(M)$ is a vector field and $\alpha \in \Omega^1(M)$ is its dual one-form with respect to h, then $X^c + \alpha^v$ and $X^c - \alpha^v$ are eigen vector fields of P corresponding to the eigen values $(+1)$ and (-1), respectively. Now, we note that the rank of the eigen distributions corresponding to the eigen values $(+1)$ and (-1) coincide (being equal to n), and therefore P is para-complex, which complete the proof. ■

We recall the following notion from [9]

Definition 6. *Any $(1, 1)$-tensor field T on a (semi-) Riemannian manifold (N, h) is called harmonic if T viewed as an endomorphism field*

$$T : (TN, h^c) \to (TN, h^c) \tag{11}$$

is a harmonic map, where h^c denotes the complete lift of the semi-Riemannian metric h.

Using [9], we have the following characterization

Proposition 7. *Let (N, h) be a (semi-)Riemannian manifold and let ∇ be the Levi-Civita connection of h. Then any $(1, 1)$-tensor field $\mathcal{T}$ on (N, h) is harmonic if and only if $\delta\mathcal{T} = 0$, where*

$$\delta\mathcal{T} = \mathrm{trace}_h\,(\nabla\mathcal{T}) = \mathrm{trace}_h\,\{(X, Y) \to (\nabla_X\mathcal{T})\,Y\}\,.$$

We have the following characterization

Theorem 8. *Let (M, h) be a Riemannian n-dimensional manifold with the total space of its cotangent bundle T^*M endowed with the natural Riemann extension $\bar{g}$. Then, the almost product structure P defined by* (10) *is harmonic with respect to $\bar{g}$ if and only if*

$$\sum_{i=1}^{n}[(\nabla_{e_i}\alpha_i)^\sharp]^c + \sum_{k=2}^{n}\bar{\nabla}_{e_k^c}e_k^c + c((n+1)e_1^c - c\alpha_1^v) - \bar{\nabla}_{e_1^c}e_1^c \;=\; 0 \tag{12}$$

where the basis $\{e_1, ..., e_n\}$ and its dual basis $\{\alpha_1, ..., \alpha_n\}$ were constructed in Section 2 *on T^*M at an arbitrary fixed point (x, w) of T^*M, such that $w \neq 0$.*

Proof: Let $\bar{\nabla}$ be the Levi-Civita connection of the natural Riemann extension $\bar{g}$ which is given by (9). Also, we note that any relation written here will be calculated at each point $(x, w) \in T^*M$. Using Proposition 7, we have the following equivalences: The almost para-complex structure P on $(T^*M, \bar{g})$ is harmonic $\Leftrightarrow$

$$\delta P = \operatorname{trace}_{\bar{g}} \bar{\nabla} P \ = \ 0.$$

Hence

$$\delta P = \operatorname{trace}_{\bar{g}} \bar{\nabla} P = \sum_{i,j=0}^{2n} \bar{g}^{ij} (\bar{\nabla}_{H_i} P) H_j \ = \ 0 \tag{13}$$

where $\{H_i\}_{i=1,\dots,2n}$ is a local basis of vector fields on T^*M and $\bar{g}^{ij}$ is the inverse matrix of the matrix $\bar{g}(H_i, H_j)_{i,j=1,\dots,2n}$. Then

$$(13) \Leftrightarrow \sum_{i=1}^{2n} \varepsilon_i (\bar{\nabla}_{F_i} P) F_i \ = \ 0 \tag{14}$$

where $\{F_i\}_{i=1,\dots,2n}$ is a local orthonormal basis on $(T^*M, \bar{g})$ and $\varepsilon_i = \bar{g}(F_i, F_i)$, $i = 1, \dots, 2n$. From (8), the equivalences can be derived

$$(14) \ \Leftrightarrow \ \sum_{s=1}^{n} \{(\bar{\nabla}_{E_s} P) E_s - (\bar{\nabla}_{E_{s*}} P) E_{s*}\} = 0 \tag{15}$$

$$\Leftrightarrow \ \bar{\nabla}_{E_1} P E_1 - P \bar{\nabla}_{E_1} E_1 - \bar{\nabla}_{E_{1*}} P E_{1*} + P \bar{\nabla}_{E_{1*}} E_{1*} \tag{16}$$

$$= \sum_{k=2}^{n} \bar{\nabla}_{E_{k*}} P E_{k*} - P \bar{\nabla}_{E_{k*}} E_{k*} - \bar{\nabla}_{E_k} P E_k + P \bar{\nabla}_{E_k} E_k.$$

We recall also the expression from [5, equation (4.6)]

$$\left(\bar{\nabla}_{E_{1*}} E_{1*} - \bar{\nabla}_{E_1} E_1\right)_{(x,w)} = -\frac{1}{a} \left\{(\nabla_{e_1} \alpha_1)^v + c\alpha_1^v + cW - (i_{\alpha_1} \nabla e_1)^v\right\}_{(x,w)}.$$

By applying P defined by (10), we get

$$P\left(\bar{\nabla}_{E_{1*}} E_{1*} - \bar{\nabla}_{E_1} E_1\right)_{(x,w)} = -\frac{1}{a} \left\{[(\nabla_{e_1} \alpha_1)^\sharp]^c + 2ce_1^c\right\}. \tag{17}$$

From (9), we get

$$\left(\bar{\nabla}_{E_1} P E_1 - \bar{\nabla}_{E_{1*}} P E_{1*}\right)_{(x,w)} = \frac{1}{a} \left\{\bar{\nabla}_{e_1^c} e_1^c - c^2 \alpha_1^v\right\} \tag{18}$$

By using (17) and (18), the left hand side of (16) becomes

$$\left(\bar{\nabla}_{E_1} P E_1 - P \bar{\nabla}_{E_1} E_1 - \bar{\nabla}_{E_{1*}} P E_{1*} + P \bar{\nabla}_{E_{1*}} E_{1*}\right)_{(x,w)}$$

$$= -\frac{1}{a} \left\{[(\nabla_{e_1} \alpha_1)^\sharp]^c + 2ce_1^c - \bar{\nabla}_{e_1^c} e_1^c + c^2 \alpha_1^v\right\} \tag{19}$$

where we have made use of (1), (3) and (7).

Relying on ([5], equation (4.8)), we recall the relation

$$\sum_{k=2}^{n}\left(\bar{\nabla}_{E_{k*}}E_{k*}-\bar{\nabla}_{E_k}E_k\right)_{(x,w)}=-\frac{1}{a}\sum_{k=2}^{n}\{(\nabla_{e_k}\alpha_k)^v+cW-(i_{\alpha_k}\nabla e_k)^v\}\,.$$

By applying P defined in (10), we get

$$\sum_{k=2}^{n}P\left(\bar{\nabla}_{E_{k*}}E_{k*}-\bar{\nabla}_{E_k}E_k\right)_{(x,w)}=-\frac{1}{a}\sum_{k=2}^{n}\left\{[(\nabla_{e_k}\alpha_k)^\sharp]^c+ce_1^c\right\}. \tag{20}$$

From (9), we obtain

$$\sum_{k=2}^{n}\left(\bar{\nabla}_{E_k}PE_k-\bar{\nabla}_{E_{k*}}PE_{k*}\right)_{(x,w)}=-\frac{1}{a}\sum_{k=2}^{n}\bar{\nabla}_{e_k^c}e_k^c. \tag{21}$$

By using (20) and (21), the right hand side of (16) becomes

$$\sum_{k=2}^{n}\left(\bar{\nabla}_{E_{k*}}PE_{k*}-P\bar{\nabla}_{E_{k*}}E_{k*}-\bar{\nabla}_{E_k}PE_k+P\bar{\nabla}_{E_k}E_k\right)_{(x,w)}$$
$$=\frac{1}{a}\sum_{k=2}^{n}\left\{\bar{\nabla}_{e_k^c}e_k^c+[(\nabla_{e_k}\alpha_k)^\sharp]^c+ce_1^c\right\}. \tag{22}$$

From (19), (22) and (7), we obtain (12). ■

Using the above theorem, where we take in particular M to be a flat manifold, then from (9), we obtain the following:

Corollary 9. *Let (M,h) be a Riemannian n-dimensional flat manifold with the total space of its cotangent bundle T^*M endowed with the natural Riemann extension $\bar{g}$ constructed with respect to the Levi-Civita connection ∇ of h. Then, the almost para-complex structure P given by* (10) *is harmonic if and only if at any point (x,w) of T^*M, the following condition is satisfied, under the notations made in* Section 2, *for $w\neq 0$*

$$\sum_{i=1}^{n}[(\nabla_{e_i}\alpha_i)^\sharp]^c+c((n+1)e_1^c-c\alpha_1^v)-\bar{\nabla}_{e_1^c}e_1^c\;=\;0. \tag{23}$$

Corollary 10. *Let (M,h) be a Riemannian n-dimensional flat manifold with the total space of its cotangent bundle T^*M endowed with the natural Riemann extension $\bar{g}$ constructed with respect to the Levi-Civita connection ∇ of h. Then any two of the following conditions imply the third one*

i) *The almost para-complex structure P is given by* (10) *is harmonic with respect to $\bar{g}$*

ii) *$\bar{g}$ reduces to the* (*classical*) *Riemann extension*

iii) *At any point* (x, w) *of* T^*M*, the following condition is satisfied, under the notations made in* Section 2*, for* $w \neq 0$

$$\sum_{i=1}^{n}[(\nabla_{e_i}\alpha_i)^\sharp]^c \;=\; \bar{\nabla}_{e_1^c}e_1^c. \tag{24}$$

Proof: If we assume that i) and ii) are satisfied, then we obtain $c((n+1)e_1^c - c\alpha_1^v) = 0$. Since a vertical lift and a complete one coincide if and only if they both vanish identically, it follows that $c = 0$, which implies ii). The rest of implications follows directly from Theorem 9. ■

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