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ANALYSIS OF THE ERROR FOR HARMONIC TRACKING USING AN ITERATIVE SCHEME IN GEOMETRIC CONTROL

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Abstract. Geometric Control Theory was initiated in the beginning of the 1970's and has now become a well established design methodology for problems of tracking prescribed reference signals while rejecting unwanted disturbance signals. In this paper we describe the error analysis for time-dependent harmonic signal tracking for general distributed parameter control systems with bounded input and output operators using an iterative numerical scheme based on the geometric design methodology.

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CONTENTS

1. Introduction

Regulator theory has a long history in both the finite- and infinite-dimensional cases for both liner and nonlinear systems and there is a vast literature on these subjects. In the last four decades geometric control theory has seen a significant development and has attracted a great deal of interest in addressing problems of tracking and disturbance rejection. This theory basically deals with geometric elements such as invariant and error zeroing subspaces (or manifolds) for the dynamics of the system in an infinite dimensional state space. In fact geometric properties can be easily understood by observing state and output trajectories of a system.

We first present some basic notation and the assumptions. Next we give a very brief overview of the classical geometric method for linear systems based on the regulator equations as developed in [4]. The main difficulty with the regulator equation approach is that it may be quite difficult to solve the regulator equations. This is the primary motivation for this study which is concerned with analyzing the error associated with a numerical scheme used to obtain approximate control laws for tracking harmonic signals using geometric control methods. The numerical scheme is based on a generalization of the regulator equations which we refer to as the dynamic regulator equations.

The dynamic regulator equations require solving for three unknowns which again are difficult to find explicitly. But it is possible to formulate an iterative algorithm in the context of geometric control. We explain the methodology behind the dynamic regulator equations in Section 2. It turns out that the dynamic regulator equations are ill-posed which leads us to introduce a regularization of the method in Section 3. In Section 4 we introduce the iterative scheme which we refer to as the β-iteration method. In Section 5 we introduce some useful identities. Then in Section 6 we present an analysis of the error in the iterative method and finally in Section 7 we present the main theorem of our work which bounds the norm of the error $e_n(t)$ at each step when time tends to infinity for tracking harmonic signals. The aforementioned theorem was established as the consequence of two lemmas whose proofs we do not include here but refer to the Master's Thesis [7] which is readily available online. Finally we illustrate this error estimate for harmonic tracking in one and two dimensional systems by numerically solving a pair of control problems using the finite element package "COMSOL".

2. Notation and Definitions

Let $\mathcal{Z}$ denote an infinite dimensional Hilbert space and A be an unbounded, closed, densely defined operator, with $\mathcal{D}(A) \subset \mathcal{Z}$. Assume that A generates an exponentially stable C_0 semigroup e^{At} in $\mathcal{Z}$. These assumptions guarantee that there are numbers $\omega > 0$ and $M \geq 1$ so that

$$\|e^{At}\| \leq M e^{-\omega t}. \tag{1}$$

Here the norm on the left is the operator norm defined for a bounded operator $L \in \mathcal{L}(\mathcal{Z})$ as[1]

$$\|L\| = \sup_{\substack{\phi \in \mathcal{Z} \\ \|\phi\| \neq 0}} \frac{\|L\phi\|}{\|\phi\|}.$$

Notice that we use the same notation $\|\cdot\|$ for both the operator norm and the norm in $\mathcal{Z}$. Furthermore, our assumption on A implies that the spectrum of A (denoted by $\sigma(A)$) lies entirely in the closed left half plane $\mathbb{C}_\omega^- = \{\zeta \in \mathbb{C} \ ; \text{ with } \Re(\zeta) < \omega\}$. We also assume that A satisfies the "spectrum determined growth condition" so that $\omega = \sup\{\Re(\lambda) \ ; \ \lambda \in \sigma(A)\}$ in (1).

These assumptions on A are common in applications governed by parabolic control systems on bounded domains.

The control input operator $B \in \mathcal{L}(\mathbb{R}, \mathcal{Z})$ is given as the multiplication operator $Bu = bu$ for some $b \in \mathcal{Z}$ and any $u \in \mathbb{R}$. Similarly the output operator $C \in \mathcal{L}(\mathcal{Z}, \mathbb{R})$ is given by $C\phi = \langle \phi, c \rangle$ for some vector $c \in \mathcal{Z}$. Clearly in this case B and C are bounded, rank one operators and compositions like BC and CB make sense. In particular, for $\phi \in \mathcal{Z}$ we have $BC\phi = b\langle \phi, c\rangle$ is a rank one operator and $CB\phi = \langle b\phi, c\rangle \in \mathbb{R}$ is a scalar.

Without loss of generality we will make the important simplifying assumption that the operator triple (A, B, C) satisfies

$$CA^{-1}B = -1. \tag{2}$$

This assumption will simplify many of the formulas that we encounter in this work. This can be achieved in general by simply modifying the definition of either B or C by multiplying by a constant value. Namely we can easily see that the norm of C is obtained as $\|C\| = \|c\|$ since

$$\|C\| = \sup_{\substack{\phi \in \mathcal{Z} \\ \|\phi\| \neq 0}} \frac{|\langle \phi, c\rangle|}{\|\phi\|} \leq \sup_{\substack{\phi \in \mathcal{Z} \\ \|\phi\| \neq 0}} \frac{\|\phi\|\|c\|}{\|\phi\|} \leq \|c\|.$$

Since $c \in \mathcal{Z}$ we see that the supremum is achieved, i.e.,

$$\|C\| = \sup_{\substack{\phi \in \mathcal{Z} \\ \|\phi\| \neq 0}} \frac{|\langle \phi, c\rangle|}{\|\phi\|} \geq \frac{|\langle c, c\rangle|}{\|c\|} = \|c\|.$$

Suppose now that we are given a B corresponding to multiplication by $b \in \mathcal{Z}$. Then we compute $G = CA^{-1}B \in \mathcal{Z}$ which we assume is nonzero. In order to

[1] The symbol $\mathcal{L}(W_1, W_2)$ denotes the set of all bounded linear operators from a Hilbert space W_1 to a Hilbert space W_2. When $W = W_1 = W_2$ we write $\mathcal{L}(W)$.

obtain (2), in a slight abuse of notation we redefine $b \in \mathcal{Z}$ by $b \equiv b/G$. So from here on we assume that (2) holds. We again see that the norm of B is then given by $\|B\| = \|b\|$ (where this is the new b as described above).

We are concerned with a linear control system, with state variable $z(t) \in \mathcal{Z}$ in the form

$$z_t(t) = Az(t) + Bu(t), \qquad z(0) = z_0, \qquad y(t) = Cz(t). \tag{3}$$

The function $u(t) \in \mathbb{R}$ for all time $t > 0$ is a time dependent control, $y(t) = Cz(t) \in \mathbb{R}$ denotes the measured output at time t and $z_0 \in \mathcal{Z}$ is the initial condition.

Problem 1 (Regulation Tracking Problem)**.** *Find the time dependent control $u(t)$ so that response $y(t)$ of the plant satisfies*

$$\lim_{t\to\infty} |y(t) - y_r(t)| = 0$$

for a given reference signal $y_r(t)$ and all initial conditions z_0.

In this work we are interested in reference signals $y_r(t)$ that are time dependent and harmonic. The simplest examples are obtained as solutions of a harmonic oscillator

$$w_t = Sw, \qquad w(0) = w_0 \tag{4}$$

where

$$S = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad w_0 = \begin{bmatrix} 0 \\ M \end{bmatrix}$$

for all $t > 0$ and for some fixed frequency α. The solution to this so-called exosystem provides the desired reference signal $y_r = Qw = M\sin(\alpha t)$ (with $Q = [1, 0]$), a sinusoidal signal with frequency α and amplitude M. Since much of our work holds for more general reference signals we carry out many calculations with only the assumption that $y_r(t)$ is a smooth function with each derivative uniformly bounded for $0 \le t < \infty$. These assumptions are clearly satisfied by sinusoidal signals.

Our approach to solve for the control $u(t)$ is based on the geometric theory which in turn is based on the existence of an attractive invariant dynamical system. The existence of such a system is guaranteed by our assumption that A satisfies the spectrum decomposition condition at $-\alpha_0$, for some $\alpha_0 > 0$, as defined in [6] (see also [5, p 71 and p 232]). But this, by itself, is not enough to guarantee the existence of a control solving the regulation tracking problem since not all problems of regulation are solvable. On the other hand there is a large literature concerned with the geometric method and characterizations of solvability in terms of the so called “Regulator Equations” (see [3, 4]). The regulator equations are a pair of operator equations with unknowns $\Pi \in \mathcal{L}(\mathbb{R}^2, \mathcal{Z})$ and $\Gamma \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ satisfying

$$\Pi S w = A\Pi w + B\Gamma w, \qquad C\Pi w = Qw \tag{5}$$

for all $w \in \mathbb{R}^2$. As proved in [4] the regulator problem is solvable if and only if the regulator equations are solvable. And, in case the regulator equations are solvable the the desired control is given by $u(t) = \Gamma w(t)$.

Unfortunately for general reference signals, i.e., for much more general $S \in \mathcal{L}(\mathbb{R}^k)$ solving the regulator equations for the desired control gain Γ can be very computationally challenging.

Rather than consider the static regulator equations as given in (5) we will consider what we refer to as the Dynamic Regulator Equations.

The Dynamic Regulator Equations [3] (DRE) for state variable $\Pi(t) \in \mathcal{Z}$ and control gain $\gamma(t) \in \mathbb{R}$ are given by the following system (which we also refer to as the controller)

$$\Pi_t(t) = A\Pi(t) + B\Gamma(t), \qquad \Pi(0) = \Pi_0, \qquad C\Pi(t) = y_r(t), \quad t > 0. \tag{6}$$

Notice that there are three unknown expressions in the controller: $\Pi(t)$, $\Gamma(t)$, and Π_0. Here Π_0 is the unique initial condition that, together with the time dependent control $\Gamma(t)$, forces the error zeroing condition expressed in equation (6). In fact $\Pi_0 = \Pi w_0$ where Π is the solution of the classical regulator equations in (5) and w_0 is the initial condition for the reference signal obtained from the exo-system in (4). But since we don't know Π the initial condition Π_0 is also unknown.

On the other hand, if we can solve equations (6) then the desired control for solving Problem 1 is given by $u(t) = \Gamma(t)$.

Since knowing the precise initial condition Π_0 needed in (6) is equivalent to solving the static regulator equations (5) it is not easy to solve the DRE. The main goal of this work to introduce an iterative scheme that provides a sequence of increasingly more accurate approximate values $\Gamma_n(t)$ that converge to $\Gamma(t)$ as n goes to infinity. We will show that by introducing a regularization of the problem we can obtain the desired iterative sequence. We also show that we can obtain precise values for the error at the nth step for the case of harmonic tracking.

These approximations suggest that we consider replacing Problem 1 with a more practical design objective. Rather than ask for exact asymptotic tracking as expressed in Problem 1 we will seek controls providing solutions that eventually remain within a small neighborhood of the reference signal.

Problem 2 (Practical Regulation Problem). *Given $\varepsilon > 0$ find a time dependent control $u(t)$ so that response $y(t)$ of the plant satisfies*

$$\lim_{t\to\infty} |y(t) - y_r(t)| \le \varepsilon$$

for a given reference signal $y_r(t)$ *and all initial conditions* z_0.

3. Regularization of the DRE

In this section we explain why the most direct approach to solving the system (6) for $\Gamma(t)$ does not work which explains why we employ a regularization method which, in turn, produces a sequence of iterations which converge to the desired result. Let us formally solve (6) for $\Pi(t)$ to obtain

$$\Pi(t) = A^{-1}\Pi_t(t) - A^{-1}B\Gamma(t)$$

and then apply C to this equation and use (6) to obtain

$$y_r(t) = CA^{-1}\Pi_t(t) - CA^{-1}B\Gamma(t) = CA^{-1}\Pi_t(t) + \Gamma(t).$$

So we can write

$$\Gamma(t) = y_r(t) - CA^{-1}\Pi_t(t). \tag{7}$$

Plugging this back into (6) we can write

$$\Pi_t(t) = A\Pi(t) + B(y_r(t) - CA^{-1}\Pi_t(t))$$

which can then be written as

$$(I + BCA^{-1})\Pi_t(t) = A\Pi(t) + By_r(t). \tag{8}$$

The coefficient $(I+BCA^{-1})$ of the time derivative term on the left is not generally invertible since it is singular. Indeed, notice that, due to our assumption in (2), (BCA^{-1}) satisfies

$$(BCA^{-1})^2 = B(CA^{-1}B)CA^{-1} = -(BCA^{-1})$$

which implies that $(I + BCA^{-1})$ has a nontrivial null space containing the vector (BCA^{-1}). Namely we have

$$(I + BCA^{-1})(BCA^{-1}) = \big((BCA^{-1}) - (BCA^{-1})\big) = 0.$$

Since equation (8) is singular we proceed by considering a regularization in which we replace $\Gamma(t)$ given in (7) by the regularized approximation (for a $0 < \beta < 1$)

$$\overline{\Gamma}(t) = y_r(t) - (1-\beta)CA^{-1}\Pi_t(t).$$

Then the system in (6) becomes

$$\overline{\Pi}_t(t) = A\overline{\Pi}(t) + B\overline{\Gamma}(t), \qquad \overline{\Gamma}(t) = y_r(t) - (1-\beta)CA^{-1}\overline{\Pi}_t(t) \tag{9}$$

and we can replace (8) by

$$(I + (1-\beta)BCA^{-1})\overline{\Pi}_t(t) = A\overline{\Pi}(t) + By_r(t). \tag{10}$$

We expect that for β close to 1 we will have $\overline{\Pi}(t)$ close to $\Pi(t)$. In this way we also hope that the resulting control $\overline{\Gamma}(t)$ (which depends on β) obtained from this procedure will be close to the desired control $\Gamma(t)$.

For β close enough to 1 we can see that the operator $(I + (1-\beta)BCA^{-1})$ is close to the the identity since $\|(1-\beta)BCA^{-1})\|$ can be made as small as we like. Therefore $(I + (1-\beta)BCA^{-1})$ can be inverted using the formula for the sum of a geometric series. As we have seen above for any $j \geq 1$, we have $(BCA^{-1})^j = (-1)^{j-1}(BCA^{-1})$. With this the operator $(I + (1-\beta)BCA^{-1})^{-1}$ can be written explicitly as

$$
\begin{aligned}
(I + (1-\beta)BCA^{-1})^{-1} &= \sum_{j=1}^{\infty}[-(1-\beta)]^j(BCA^{-1})^j \\
&= I - \left[\sum_{j=1}^{\mathrm{i}}(1-\beta)^j\right](BCA^{-1}) = I - \frac{(1-\beta)}{\beta}(BCA^{-1}). \quad (11)
\end{aligned}
$$

So we can write (10) as

$$
\overline{\Pi}_t(t) = \left(I - \frac{(1-\beta)}{\beta}(BCA^{-1})\right) A\overline{\Pi}(t) + \left(I - \frac{(1-\beta)}{\beta}(BCA^{-1})\right) By_r(t).
$$

Notice that we can use (2) to simplify the second term as follows

$$
\left(I - \frac{(1-\beta)}{\beta}(BCA^{-1})\right) B = \left(B + \frac{(1-\beta)}{\beta}B\right) = \frac{1}{\beta}B.
$$

We then define

$$
A_\beta = \left(A - \frac{(1-\beta)}{\beta}BC\right). \quad (12)
$$

Notice from (11) that

$$
A_\beta^{-1} = (I + (1-\beta)BCA^{-1}). \quad (13)
$$

For β close to 1 we expect that the operator A_β will posses a negative growth bound $-\omega_\beta$ and generate an asymptotically stable semigroup. We make this formal with the following lemma.

Lemma 1. *There is a $\beta_0 > 0$ so that for all $\beta_0 < \beta \leq 1$ the operator A_β generates an exponentially stable C_0 semigroup.*

Indeed from [5, p 110, Theorem 3.2.1] we see that a conservative estimate for the growth bound for A_β is $-\omega_\beta = -(\omega - \gamma M\|BC\|)$ with $\gamma = \frac{(1-\beta)}{\beta}$. Thus for β close enough to one, together with our assumption that the growth bound $-\omega$ of A is negative implies that A_β generates an exponentially stable semigroup, i.e.,

$$
\|e^{A_\beta t}\| < M_\beta e^{-\omega_\beta t}. \quad (14)
$$

With the above simplifications the desired regularized equation can be written in the simple form

$$\overline{\Pi}_t(t) = A_\beta \overline{\Pi}(t) + \frac{1}{\beta} B y_r(t). \tag{15}$$

4. The Iterative Scheme

The main objective of this section is to find approximate values for $\overline{\Pi}(t)$ and $\overline{\Gamma}(t)$. We develop an iterative algorithm to find a sequence of $\overline{\Pi}^n(t)$and $\overline{\Gamma}^n(t)$ in the form

$$\overline{\Pi}^n(t) = \sum_{j=1}^{n} \overline{\Pi}_j(t), \qquad \overline{\Gamma}^n(t) = \sum_{j=1}^{n} \overline{\Gamma}_j(t)$$

such that $\overline{\Pi}^n(t)$ converges to $\overline{\Pi}(t)$ and $\overline{\Gamma}^n(t)$ converges to $\overline{\Gamma}(t)$.
We start with $n = 1$ and continue adding terms while calculating the values of $\overline{\Pi}_j(t)$ and $\overline{\Gamma}_j(t)$ for each step. Doing so, we also deduce by induction an exact formula for the error, $e_n(t)$, at the n-th iteration.

First Iteration: When $n = 1$ the system in (9) with unknowns

$$\overline{\Pi}^1 = \overline{\Pi}_1 \qquad \text{and} \qquad \overline{\Gamma}^1 = \overline{\Gamma}_1$$

can be written as

$$(\overline{\Pi}_1(t))_t = A\overline{\Pi}_1(t) + B\overline{\Gamma}_1(t), \quad \overline{\Gamma}_1(t) = y_r(t) - (1-\beta)CA^{-1}(\overline{\Pi}_1(t))_t. \tag{16}$$

As we have seen in (15) the above system can be written in the equivalent simpler form

$$(\overline{\Pi}_1(t))_t = A_\beta^{-1}\overline{\Pi}_1(t) + \frac{1}{\beta} B y_r(t).$$

In this case the error in the first iteration, $e_1(t)$, can be written as

$$e_1(t) = y_r(t) - C(\overline{\Pi}_1(t)).$$

We also take as initial data the solution of the boundary value problem

$$\overline{\Pi}_1(0) = -A^{-1} B y_r(0).$$

This choice yields the convenient relation

$$e_1(0) = y_r(0) - C(\overline{\Pi}_1(0)) = y_r(0) - C(-A^{-1}By_r(0)) = y_r(0) - y_r(0) = 0.$$

Second Iteration: Next we continue the same procedure considering $n = 2$ with unknowns

$$\overline{\Pi}^2 = (\overline{\Pi}_1 + \overline{\Pi}_2), \qquad \overline{\Gamma}^2 = (\overline{\Gamma}_1 + \overline{\Gamma}_2)$$

$$(\overline{\Pi}_1(t) + \overline{\Pi}_2(t))_t = A(\overline{\Pi}_1 + \overline{\Pi}_2)(t) + B(\overline{\Gamma}_1 + \overline{\Gamma}_2)(t).$$

By the assumption that $\overline{\Pi}_1(t)$ satisfies (16) we have by the above equation

$$(\overline{\Pi}_2(t))_t = A(\overline{\Pi}_2)(t) + B\overline{\Gamma}_2(t).$$

In the second iteration, we consider the reference signal that needs to be tracked as the error in the first iteration, $e_1(t)$. This follows from the observation that we want

$$y_r(t) = C(\overline{\Pi}_1(t) + \overline{\Pi}_2(t)) \qquad \Rightarrow \qquad C(\overline{\Pi}_2(t)) = y_r - C(\overline{\Pi}_1(t)) = e_1(t).$$

With the new reference signal $e_1(t)$, the regularized value for $\overline{\Gamma}_2(t)$ can be written as

$$\overline{\Gamma}_2(t) = e_1(t) - (1-\beta)CA^{-1}(\overline{\Pi}_2(t))_t$$

and the regularized system becomes

$$(\overline{\Pi}_2(t))_t = A_\beta \overline{\Pi}_2(t) + \frac{1}{\beta} B e_1(t).$$

Therefore the error of the second iteration, $e_2(t)$, can be written as

$$e_2(t) = y_r(t) - C(\overline{\Pi}_1(t) + \overline{\Pi}_2(t)) = e_1(t) - C(\overline{\Pi}_2(t)).$$

We also take as initial data the trivial solution

$$\overline{\Pi}_2(0) = -A^{-1} B e_1(0) = 0$$

and consequently

$$e_2(0) = e_1(0) - C(0) = 0.$$

***n*th Iteration:** Similarly, at the $(n-1)$th iteration we have a certain error denoted by $e_{n-1}(t)$ with $e_{n-1}(0) = 0$. At the nth iteration we consider the signal to be tracked to be exactly $e_{n-1}(t)$ and the corresponding regularized system to be solved given by

$$(\overline{\Pi}_n(t))_t = A_\beta \overline{\Pi}_n(t) + \frac{1}{\beta} B e_{n-1}(t), \quad \overline{\Gamma}_n(t) = e_{n-1}(t) - (1-\beta)CA^{-1}(\overline{\Pi}_n(t))_t.$$

Then the error in the nth iteration, e_n, can then be written as

$$e_n(t) = y_r(t) - C\left(\sum_{j=1}^{n} \overline{\Pi}_j(t)\right) = e_{n-1}(t) - C(\overline{\Pi}_n(t)).$$

Again we take as initial data the trivial solution

$$\overline{\Pi}_n(0) = -A^{-1} B e_{n-1}(0) = 0$$

and consequently

$$e_n(0) = e_{n-1}(0) - C(0) = 0.$$

5. Some Useful Formulas

In this section we present some useful relations between A_β, A, B and C. The proofs of all the relations are simple and based on the assumption given in equation (2). Therefore we will only include the proof of the first result. The complete proof can be found in the Master's Thesis by Pathiranage [7].

Lemma 2. 1. $CA_\beta^{-1} = \beta CA^{-1}$, 2. $CA_\beta^{-1}B = -\beta$, 3. $A_\beta^{-1}B = \beta\, A^{-1}B$, 4. $C\big(A_\beta^{-2}\big)B = \beta^2\, C\big(A^{-2}\big)B$, 5. $A_\beta A^{-1}B = \dfrac{1}{\beta}B$, 6. $AA_\beta^{-1}B = \beta B$, 7. $CA_\beta^{-1}A = \beta C$, 8. $CA^{-1}A_\beta = \dfrac{1}{\beta}C$.

As an example to show that $CA_\beta^{-1} = \beta CA^{-1}$ we use the formula for A_β^{-1} given in equation (13), from which it follows

$$\begin{aligned} CA_\beta^{-1} &= (C + (1-\beta)CA^{-1}BC)A^{-1} \\ &= (C + (1-\beta)(CA^{-1}B)C)A^{-1} \\ &= (C - (1-\beta)C)A^{-1} = \beta\, CA^{-1}. \end{aligned}$$

6. Analysis of the Error for an Initial Value Problem

In this section we consider a general equation in the form

$$(\mathcal{X}(t))_t = A_\beta \mathcal{X}(t) + \frac{1}{\beta}Bf(t), \qquad \mathcal{X}(0) = -A^{-1}Bf(0) \tag{17}$$

where A_β is defined in (12). Here we assume the forcing term $f(t)$ is in $C_b^2[0,\infty)$ and $\mathcal{X}(t) \in \mathcal{Z}$. The solution to (17) is given in terms of the operator semigroup $\mathrm{e}^{A_\beta t}$ (the semigroup generated by A_β) by the variation of parameters formula. In particular, we can write

$$\mathcal{X}(t) = \mathrm{e}^{A_\beta t}\mathcal{X}(0) + \int_0^t \mathrm{e}^{A_\beta (t-\tau)} \frac{1}{\beta} Bf(\tau)\,\mathrm{d}\tau.$$

Since $f(t)$ and $\mathrm{e}^{A_\beta t}$ are continuously differentiable we can apply integration by parts to the above integral to obtain

$$\begin{aligned} \mathcal{X}(t) = \mathrm{e}^{A_\beta t}\mathcal{X}(0) &+ (-A_\beta^{-1})\frac{1}{\beta}Bf(t) + A_\beta^{-1}\mathrm{e}^{A_\beta t}\frac{1}{\beta}Bf(0) \\ &+ A_\beta^{-1}\int_0^t \mathrm{e}^{A_\beta(t-\tau)}\frac{1}{\beta}Bf'(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Since $f(t)$ is in C^2 the above integral exists. Then

$$\begin{aligned} C\mathfrak{X}(t) = &- C\mathrm{e}^{A_\beta t} A^{-1} Bf(0) + C\mathrm{e}^{A_\beta t} A_\beta^{-1} \frac{1}{\beta} Bf(0) \\ &+ (-CA_\beta^{-1}) \frac{1}{\beta} Bf(t) + CA_\beta^{-1} \int_0^t \mathrm{e}^{A_\beta (t-\tau)} \frac{1}{\beta} Bf'(\tau)\,\mathrm{d}\tau. \end{aligned}$$

By our special choice of initial condition in (17), application of the properties of A_β^{-1}, A^{-1} in Section 5 and our assumption in (2) we can simplify the above and apply C to obtain

$$\begin{aligned} C\mathfrak{X}(t) = &- C\mathrm{e}^{A_\beta t} A^{-1} Bf(0) + C\mathrm{e}^{A_\beta t} A^{-1} Bf(0) \\ &- CA_\beta^{-1} \frac{1}{\beta} Bf(t) + CA_\beta^{-1} \int_0^t \mathrm{e}^{A_\beta (t-\tau)} \frac{1}{\beta} Bf'(\tau)\,\mathrm{d}\tau \\ = &- CA^{-1} Bf(t) + CA^{-1} \int_0^t \mathrm{e}^{A_\beta (t-\tau)} Bf'(\tau)\,\mathrm{d}\tau \\ = &\; f(t) + CA^{-1} \int_0^t \mathrm{e}^{A_\beta (t-\tau)} Bf'(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Thus we can write

$$f(t) - C\mathfrak{X}(t) = -CA^{-1} \int_0^t \mathrm{e}^{A_\beta (t-\tau)} Bf'(\tau)\,\mathrm{d}\tau.$$

Let

$$K = -CA^{-1}\mathrm{e}^{A_\beta t} B$$

then we can write

$$f(t) - C\mathfrak{X}(t) = K * f'$$

where $*$ denotes the convolution, i.e.,

$$K * \phi = \int_0^t K(t-\tau)\phi(\tau)\mathrm{d}\tau.$$

Finally we can rewrite the error at the nth iteration as a convolution

$$e_n = e_{n-1} - C\overline{\Pi}_n(t)$$

so that

$$e_n = -CA^{-1} \int_0^t \mathrm{e}^{A_\beta (t-\tau)} Be'_{n-1}(\tau)\,\mathrm{d}\tau = K * e'_{n-1}. \tag{18}$$

7. Estimates of the Error for Harmonic Tracking

In this section, we will apply the recursive formula (18) to obtain an explicit estimate of the error in the nth step of the β-iteration in the special case of a sinusoidal signal, i.e., a solution of the harmonic oscillator equation $y_r''(t) + \alpha^2 y_r(t) = 0$ which has solution given by sines or cosines with frequency α. Thus the main objective of this section is to obtain a bound on $|e_n(t)|$ when t approaches infinity, relative to this sinusoidal (harmonic) reference signal.
To be more specific consider the following initial value problem

$$y_r''(t) + \alpha^2 y_r(t) = 0, \qquad y(0) = a, \qquad y'(0) = b$$

then

$$y(t) = a\cos(\alpha t) + \frac{b}{\alpha}\sin(\alpha t)$$

which can be written in terms of $\theta = \tan^{-1}(b/(\alpha a)$ and $\mathrm{Amp} = \sqrt{\alpha^2 a^2 + b^2}$ as

$$y_r = \mathrm{Amp}\sin(\alpha t + \theta). \tag{19}$$

The desired estimates will be obtained in a series of lemmas taken from the Master's Thesis [7]. For a couple of more technical proofs we refer the reader to this thesis which is available through the Texas Tech University Library.

Lemma 3. *Let A be a (unbounded) linear operator which generates an exponentially stable C_0 semigroup in the Hilbert space $\mathcal{Z}$. Let $\beta \in (0,1)$ as in Lemma 1 so that the operator A_β generates an exponentially stable C_0 semigroup in $\mathcal{Z}$. Then for any arbitrary bounded operators $L_1, L_2 \in \mathcal{L}(\mathcal{Z})$ and for $t \geq 0$*

$$\lim_{t\to\infty} \left\| \int_0^t \mathrm{e}^{A_\beta(t-\tau)}\, L_1 \mathrm{e}^{A_\beta \tau}\, L_2 \mathrm{d}\tau \right\| = 0.$$

Proof: This result follows very easily based on the semigroup estimate (1)

$$\|\mathrm{e}^{At}\| < M\mathrm{e}^{-\omega t}$$

and, from Lemma 1, the similar estimate given in (14) for the norm of the semigroup generated by A_β

$$\|\mathrm{e}^{A_\beta t}\| < M\mathrm{e}^{-\omega_\beta t}$$

where $\omega_\beta \geq (\omega - \gamma M\|BC\|)$ and $\gamma = \frac{(1-\beta)}{\beta}$. We then have

$$\left\| \int_0^t \mathrm{e}^{A_\beta(t-\tau)}\, L_1 \mathrm{e}^{A_\beta \tau}\, L_2 \mathrm{d}\tau \right\| \leq \int_0^t \|\mathrm{e}^{A_\beta(t-\tau)}\| \|\|L_1\|\|\mathrm{e}^{A_\beta \tau}\|\|\|\, L_2\| \mathrm{d}\tau$$

$$\leq \int_0^t \mathrm{e}^{-\omega_\beta(t-\tau)}\, \|L_1\|\mathrm{e}^{-\omega_\beta \tau}\|\, L_2\|\mathrm{d}\tau = M^2 \|\, L_1\|\|\, L_2\|(t\mathrm{e}^{-\omega_\beta t}) \xrightarrow{t\to\infty} 0.$$

■

Let

$$\mathcal{Y}_r(t) = \big(y_r(t), \qquad y_r'(t)\big), \qquad S_\alpha = \begin{pmatrix} 0 & -\alpha^2 \\ 1 & 0 \end{pmatrix}$$

and introduce the operators, R_1, $R_2 \in \mathcal{L}(\mathcal{Z})$ defined by

$$R_1 = \beta^{-1}\left(A_\beta^2 + \alpha^2 I\right)^{-1}, \qquad R_2 = \beta^{-1}A_\beta^{-1}R_1 = \beta^{-1}A_\beta^{-1}(A_\beta^2 + \alpha^2 I)^{-1}.$$

Lemma 4. *Under the assumption $y_r''(t) = -\alpha^2 y_r(t)$ we have*

$$-CA^{-1}\int_0^t \mathrm{e}^{A_\beta(t-\tau)} B y_r(\tau)\,\mathrm{d}\tau = \mathcal{Y}_r(t)F_1 + \mathcal{Y}_r(0)F_2$$

where

$$F_1 = \begin{pmatrix} CR_1B \\ CR_2B \end{pmatrix}, \qquad F_2 = \begin{pmatrix} -CR_1\mathrm{e}^{A_\beta t}B \\ -CR_2\mathrm{e}^{A_\beta t}B \end{pmatrix}$$

$$-CA^{-1}\int_0^t \mathrm{e}^{A_\beta(t-\tau)} B y_r'(\tau)\,\mathrm{d}\tau = \mathcal{Y}_r(t)F_3 + \mathcal{Y}_r(0)F_4$$

with

$$F_3 = \begin{pmatrix} -\alpha^2 CR_2B \\ CR_1B \end{pmatrix}, \qquad F_4 = \begin{pmatrix} \alpha^2 CR_2\mathrm{e}^{A_\beta t}B \\ -CR_1\mathrm{e}^{A_\beta t}B \end{pmatrix}.$$

The proof of this lemma is somewhat lengthy but quite elementary involving integration by parts twice, using the assumption that $y_r''(t)+\alpha^2 y_r(t) = 0$ and applying the identities in Lemma 2. For the first equation in Lemma 4, just as in calculus, the desired integral appears again on the right hand side. We move this term to the left hand side and combine the two terms to arrive at the expression $(I + \alpha^2 A_\beta^{-2})$ multiplied by the desired integral. Noting that the operator $(I + \alpha^2 A_\beta^{-2})$ is invertible we apply its inverse to obtain a formula for integral. The complete details can be found in the Master's Thesis [7].

Theorem 5. *Let $y_r''(t) + \alpha^2 y_r(t) = 0$, then*

$$e_1(t) = -K * y_r'(t), \qquad e_n(t) = -K * e_{n-1}'(t) \ \text{ for } n \geq 2$$

where $K = CA^{-1}\mathrm{e}^{A_\beta t}\,B$. Let $\beta \in (0,1)$ such that the operator A_β generates an exponentially stable C_0 semigroup in $\mathcal{Z}$. Then

$$e_n(t) = \mathcal{Y}_r(0)\mathcal{E}_n(t) + E_n(t)$$

where

$$\lim_{t\to\infty} \|\mathcal{E}_n\|_\infty \to 0$$

exponentially fast

$$\mathcal{Y}_r(t) = \big(y_r(t),\ y_r'(t)\big)$$

and

$$E_1(t) = \alpha^2 \mathcal{Y}_r(t) S_\alpha^{-1} \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix}, \qquad E_2(t) = \alpha^2 \mathcal{Y}_r(t) S_\alpha^{-1} G \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix}$$

$$E_n(t) = \alpha^2 \mathcal{Y}_r(t) S_\alpha^{-1} G^{n-1} \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} \text{ for all } n \geq 1$$

where

$$G = \begin{pmatrix} -\alpha^2 \mathcal{R}_2 & -\alpha^2 \mathcal{R}_1 \\ \mathcal{R}_1 & -\alpha^2 \mathcal{R}_2 \end{pmatrix}, \qquad S_\alpha = \begin{pmatrix} 0 & -\alpha^2 \\ 1 & 0 \end{pmatrix}, \qquad S_\alpha^{-1} = \frac{1}{\alpha^2} \begin{pmatrix} 0 & \alpha^2 \\ -1 & 0 \end{pmatrix}$$

$$\mathcal{R}_1 = C R_1 B, \qquad \mathcal{R}_2 = C R_2 B$$

$$R_1 = \beta^{-1}(A_\beta^2 + \alpha^2 I)^{-1}, \qquad R_2 = \beta^{-1} A_\beta^{-1} (A_\beta^2 + \alpha^2 I)^{-1}. \tag{20}$$

Proof: These results are essentially contained in the Master's Thesis [7]. ■

We now turn to a detailed analysis of the error terms $E_n(t)$. To this end we note that

$$G = X \Lambda X^{-1} \text{ where } X = \begin{pmatrix} \mathrm{i}\alpha & -\mathrm{i}\alpha \\ 1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}, \qquad \zeta_1 = \alpha(-\alpha \mathcal{R}_2 + \mathcal{R}_1 \mathrm{i}), \qquad \zeta_2 = \alpha(-\alpha \mathcal{R}_2 - \mathcal{R}_1 \mathrm{i})$$

which implies

$$G^{n-1} = X \Lambda^{n-1} X^{-1}$$

and therefore

$$E_n(t) = \alpha^2 \mathcal{Y}(t) S_\alpha^{-1} X \Lambda^{n-1} X^{-1} \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix}.$$

Now we note that

$$\Lambda^{n-1} = \begin{pmatrix} \zeta_1^{n-1} & 0 \\ 0 & \zeta_2^{n-1} \end{pmatrix}$$

$$S_\alpha^{-1} X = \frac{1}{\alpha^2} \begin{pmatrix} \alpha^2 & \alpha^2 \\ -\mathrm{i}\alpha & \mathrm{i}\alpha \end{pmatrix}, \quad X^{-1} \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} = \frac{1}{2\mathrm{i}\alpha} \begin{pmatrix} \mathcal{R}_1 + \mathrm{i}\alpha \mathcal{R}_2 \\ -\mathcal{R}_1 + \mathrm{i}\alpha \mathcal{R}_2 \end{pmatrix} = \frac{-1}{2\alpha^2} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

and therefore

$$\Lambda^{n-1} X^{-1} \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} = \frac{-1}{2\alpha^2} \Lambda^n.$$

Combining the above results we have

$$\alpha^2 S_\alpha^{-1} X \Lambda^{n-1} X^{-1} \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} = \begin{pmatrix} \alpha^2 & \alpha^2 \\ -\mathrm{i}\alpha & \mathrm{i}\alpha \end{pmatrix} \frac{-1}{2\alpha^2} \begin{pmatrix} \zeta_1^n \\ \zeta_2^n \end{pmatrix} = \frac{-1}{2\alpha^2} \begin{pmatrix} \alpha^2 \left[\zeta_1^n + \zeta_2^n\right] \\ -\mathrm{i}\alpha \left[\zeta_1^n - \zeta_2^n\right] \end{pmatrix}.$$

Then from (19)

$$\mathcal{Y}(t) = \big(\mathrm{Amp} \sin(\alpha t + \theta),\ \alpha\, \mathrm{Amp} \cos(\alpha t + \theta)\big)$$

and we finally obtain an exact representation of $E_n(t)$ in terms of $\mathcal{Y}(t)$, $\mathcal{R}_1$ and $\mathcal{R}_2$

$$E_n(t) = \frac{\text{Amp}}{2}\big(- \sin(\alpha t + \theta)\,[\zeta_1^n + \zeta_2^n] + \mathrm{i}\cos(\alpha t + \theta)\,[\zeta_1^n - \zeta_2^n]\,\big). \qquad (21)$$

A sharp estimate of formula (21) can be derived using the definition of the resolvent of A, $R(\lambda, A)$, for the evaluation of $\mathcal{R}_1$ and $\mathcal{R}_2$. This it is quite cumbersome. Instead we prefer to use rougher estimates which will give us the desired geometric convergence. However, in the example section we will show that using the resolvent gives a better result.

Returning to (21) we can make the following crude estimates for $E_n(t)$. In particular, we need to estimate the terms

$$\mathcal{R}_1 = CR_1B, \qquad \text{and} \qquad \mathcal{R}_2 = CR_2B$$

where from (20)

$$R_1 = \beta^{-1}(A_\beta^2 + \alpha^2 I)^{-1}, \qquad R_2 = \beta^{-1}A_\beta^{-1}(A_\beta^2 + \alpha^2 I)^{-1}.$$

Let us first consider an estimate for the norm of the resolvent $(A_\beta^2 + \alpha^2 I)^{-1}$ which can be written as

$$(A_\beta^2 + \alpha^2 I)^{-1} = (i\alpha I - A_\beta)^{-1}(-\mathrm{i}\alpha I - A_\beta)^{-1}.$$

Recalling that the spectrum of A_β satisfies $\sigma(A_\beta) \subset \mathbb{C}^-_{-\omega_\beta}$ and applying a very well known resolvent estimate from the Hille-Yosida Theorem (see [8, p 20, Theorem 5.3]) we have

$$\|(\lambda I - A_\beta)^{-1}\| \leq \frac{M_\beta}{|\Re(\lambda) - \omega_\beta|}, \qquad \text{for all} \qquad \Re(\lambda) > -\omega_\beta$$

Therefore, we have

$$\|A_\beta^{-1}\| \leq \frac{M_\beta}{\omega_\beta}\,, \qquad \|(\pm\mathrm{i}\alpha I - A_\beta)^{-1}\| \leq \frac{M_\beta}{\omega_\beta}$$

and

$$|\mathcal{R}_1| \leq \frac{M_\beta^2\|B\|\|C\|}{\beta\omega_\beta^2}, \qquad |\mathcal{R}_2| \leq \frac{M_\beta^3\|B\|\|C\|}{\beta\omega_\beta^3}.$$

Next, for $j = 1, 2$ we have

$$|\zeta_j| \leq \alpha(\alpha|\mathcal{R}_2| + |\mathcal{R}_1|)$$

which implies

$$|\zeta_1^n + \zeta_2^n| \leq 2(\alpha^2|\mathcal{R}_2| + \alpha|\mathcal{R}_1|)^n$$

and

$$|\zeta_1^n - \zeta_2^n| \leq 2(\alpha^2|\mathcal{R}_2| + \alpha|\mathcal{R}_1|)^n$$

Therefore

$$
\begin{aligned}
|E_n(t)| &\le \mathrm{Amp}(\alpha^2|\mathcal{R}_2| + \alpha|\mathcal{R}_1|)^n \\
&\le \mathrm{Amp}\left(\frac{\alpha^2 M_\beta^3\|B\|\|C\|}{\beta\eta_\beta^0(\omega_\beta)^2} + \frac{\alpha M_\beta^2\|B\|\|C\|}{\beta(\omega_\beta)^2}\right)^n \\
&\le \mathrm{Amp}\left(\frac{\alpha M_\beta^2\|B\|\|C\|}{\beta(\omega_\beta)^2}\right)^n\left(\frac{\alpha M_\beta}{\omega}+1\right)^n
\end{aligned}
$$

so that

$$
|E_n(t)| \le \mathrm{Amp}\left(\frac{\alpha M_\beta^2\|B\|\|C\|}{\beta(\omega_\beta)^2}\right)^n\left(\frac{\alpha M_\beta}{\omega}+1\right)^n. \tag{22}
$$

Thus we obtain geometric convergence provided that

$$
D = \left(\frac{\alpha M_\beta^2\|B\|\|C\|}{\beta\omega_\beta^2}\right)\left(\frac{\alpha M_\beta}{\omega_\beta}+1\right) < 1. \tag{23}
$$

Remark 6. *We note that the above estimate is stated for a very general class of operators A requiring only that they be the generator of a asymptotically stable C_0 semigroup. Very often in practice the operator A is self-adjoint with compact resolvent or at least a scalar spectral operator whose eigenvalues form a Riesz basis. This is the case in our examples given in Section 8 and 9 below. In such a case the estimate in* (22) *can be significantly enhanced. For example, if A is self-adjoint with eigenvalues $\{-\lambda_j\}_{j=1}^\infty$ consisting of an infinite set of real negative simple eigenvalues satisfying*

$$
0 < \omega = \lambda_1 < \lambda_2 < \dots \qquad \textit{and} \qquad \lim_{n\to\infty} \lambda_n = \infty
$$

and there is an orthonormal basis of eigenfunctions $\{\varphi_j\}_{j=1}^\infty$ for which the spectral theory of self-adjoint operators gives

$$
\mathrm{e}^{At}\phi = \sum_{j=1}^\infty \mathrm{e}^{-\lambda_j t}\langle\phi,\varphi_j\rangle\varphi_j \qquad \textit{and} \qquad (\lambda I - A)^{-1}\phi = \sum_{j=1}^\infty \frac{\langle\phi,\varphi_j\rangle\varphi_j}{\lambda+\lambda_j}.
$$

These formulas show that the growth bound in (1) *is $\omega = \lambda_1$ and $M = 1$, i.e.,*

$$
\|\mathrm{e}^{At}\| \le e^{-\omega t} \quad \textit{and} \quad \|(\lambda I - A)^{-1}\| \le \frac{1}{\eta^\lambda} \quad \textit{for} \quad \Re(\lambda) > -\lambda_1
$$

where

$$
\eta^\lambda = \min_j |\lambda + \lambda_j|.
$$

Under the additional assumption given in Schwartz [9]

$$
\sum_{j=1}^\infty \frac{1}{d_j^2} < \infty \quad \textit{where} \quad d_j = \inf_{k\neq j}|\lambda_j - \lambda_k|
$$

on the growth rate of the eigenvalues $\{-\lambda_j\}_{j=1}^{\infty}$ *we can conclude that the bounded perturbation* A_β *is a scalar spectral operator whose eigenvalues* $\{-\lambda_j^\beta\}_{j=1}^{\infty}$, *satisfy*

$$0 < \omega_\beta < \Re(\lambda_1^\beta) < \Re(\lambda_2) < \dots \quad \textit{and} \quad \lim_{n\to\infty} \Re(\lambda_n) = \infty$$

and whose eigenvectors form a Riesz basis $\{\phi_j\}_{j=1}^{\infty}$ *with biorthogonal family* $\{\psi_j\}_{j=1}^{\infty}$, *i.e.,*

$$\langle \phi_j, \psi_k \rangle = \delta_{j,k}$$

and there are constants M_1 *and* M_2 *so that for every* $\phi \in \mathcal{Z}$

$$M_1 \sum_{j=1}^{\infty} |\langle \phi, \psi_j \rangle|^2 \le \|\phi\|^2 \le M_2 \sum_{j=1}^{\infty} |\langle \phi, \psi_j \rangle|^2$$

and

$$\frac{1}{M_2} \sum_{j=1}^{\infty} |\langle \phi, \phi_j \rangle|^2 \le \|\phi\|^2 \le \frac{1}{M_1} \sum_{j=1}^{\infty} |\langle \phi, \phi_j \rangle|^2.$$

The semigroup and resolvent for A_β *are given by*

$$\mathrm{e}^{A_\beta t}\phi = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j^\beta t} \langle \phi, \psi_j \rangle \phi_j \quad \textit{and} \quad (\lambda I - A_\beta)^{-1}\phi = \sum_{j=1}^{\infty} \frac{\langle \phi, \psi_j \rangle \phi_j}{\lambda + \lambda_j^\beta}$$

$$\|\mathrm{e}^{A_\beta t}\| \le M_\beta \mathrm{e}^{-\omega_\beta t} \quad \textit{where} \quad \omega_\beta = \Re(\lambda_1^\beta), \quad M_\beta = \sqrt{\frac{M_2}{M_1}}$$

and

$$\|(\lambda I - A_\beta)^{-1}\| \le \frac{M_\beta}{\eta_\beta^\lambda} \quad \textit{where} \quad \eta_\beta^\lambda = \min_j |\lambda + \lambda_j^\beta|. \tag{24}$$

For a discussion of the above results see, for example, [5].

8. Numerical Example: Tracking Sinusoidal Signals

In our numerical simulations we consider a simple control problem governed by a one dimensional heat equation on a unit, $0 \le x \le 1$, interval with homogenous Dirichlet boundary condition at $x = 0$ and homogenous Neumann boundary condition at $x = 1$. In this case the control system (3) is

$$z_t(x,t) = z_{xx}(x,t) + Bu(t), \qquad z(x,0) = \phi(x), \qquad y(t) = Cz(t).$$

The control input $u(t) \in \mathbb{R}$ enters through the interior of the interval

$$Bu = b(x)u \quad \text{where} \quad b(x) = \mathbf{1}_{[3/4,1]}(x) = \begin{cases} 0, & \text{for } 0 \le x < 3/4 \\ 1, & \text{for } 3/4 \le x < 1. \end{cases}$$

The measured output operator C is defined as the average temperature over the interval $[1/2, 3/4]$. Thus we have

$$C\phi = \langle \phi, c\rangle, \quad \text{where} \quad c(x) = 4 \times \mathbf{1}_{[1/2,3/4]}(x) = \begin{cases} 0, & \text{for } 0 \le x < 1/2 \\ 4, & \text{for } 1/2 \le x < 3/4 \\ 0, & \text{for } 3/4 \le x < 1. \end{cases}$$

The Hilbert state space $\mathcal{Z} = L^2(0,1)$ with inner product given by

$$\langle \phi_1, \phi_2\rangle = \int_0^1 \phi_1(x)\phi_2(x)\,\mathrm{d}x \quad \text{for all} \quad \phi_1, \phi_2 \in \mathcal{Z}$$

and norm $\|\phi\| = \langle \phi, \phi\rangle^{1/2}$. The heat diffusion operator A is defined by the unbounded, self-adjoint operator

$$A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \text{with} \quad \mathcal{D}(A) = \{\phi \in H^2(0,1)\,;\, \phi(0) = 0,\ \phi'(1) = 0\}.$$

The spectrum of the operator A consists of an infinite set of real negative eigenvalues $\lambda_j = -\mu_j^2$ where $\mu_j (j - 1/2)\pi$, for $j = 1, 2, \ldots$. A generates an exponentially stable C_0-semigroup given explicitly in terms of the corresponding orthonormal eigenfunctions $\phi_j(x) = \sqrt{2}\sin(\mu_j x)$, by

$$\mathrm{e}^{At}\phi = \sum_{j=1}^{\infty} \mathrm{e}^{\lambda_t t}\langle \phi, \phi_j\rangle.$$

It is easy to show that this is a contraction semigroup satisfying

$$\|\mathrm{e}^{At}\| \le \mathrm{e}^{-(\pi^2/4)t}.$$

Thus we have $M = 1$ and $\omega = \pi^2/4$.

Recall that, in our estimates we have assumed in (2) that $CA^{-1}B = \langle A^{-1}b, c\rangle = -1$. To achieve this we need to normalize our definition of b as it follows. First we solve the ordinary differential equation

$$X'' = -b, \qquad X(0) = 0, \qquad X'(1) = 1$$

which gives

$$X = \begin{cases} x, & \text{for } 0 \le x < 3/4 \\ -(x^2 - 4x + 9/8), & \text{for } 3/4 \le x \le 1. \end{cases}$$

Applying C to X, we set

$$G = CX = 4\int_{1/2}^{3/4} x\,\mathrm{d}x = \frac{5}{8}\cdot$$

Finally we replace our original $b(x)$ by

$$b(x) = \frac{1}{G}\mathbf{1}_{[3/4,1]}(x) = \begin{cases} 0, & \text{for } 0 \le x < 3/4 \\ 1/G, & \text{for } 3/4 \le x < 1. \end{cases}$$

The main purpose of this section is to exhibit the geometric convergence of the iterative scheme and show how the frequency α of a sinusoidal signal and choice of β influence the numerical results. In particular we show that for a sinusoidal signal $y_r = \sin(\alpha t)$ for smaller frequency α a wide range of β values produce very acceptable tracking errors. However for rapidly oscillating signals, i.e., for large α, there is no value of β for which the iterative scheme will produce successively smaller tracking errors. Indeed, the tracking errors can actually increase with each β iteration, just as the error estimates predict.

In the sequel we will consider a range of α consisting of $2, 6, 10, 14, 18$, and a range of β consisting of $\beta = .1, .05, 0.025$, for which we provide the first three eigenvalues of A_β. All the other remaining eigenvalues lie even farther in the left half of the complex plane.

For $\beta = 0.1$ we have

$$\lambda_1^\beta = -25.78 + 12.30\mathrm{i}, \qquad \lambda_2^\beta = -25.78 - 12.30\mathrm{i}, \qquad \lambda_3^\beta = -42.57$$

for $\beta = 0.05$ we have

$$\lambda_1^\beta = -37.72 + 34.67\mathrm{i}, \qquad \lambda_2^\beta = -37.72 - 34.67\mathrm{i}, \qquad \lambda_3^\beta = -27.27$$

and for $\beta = 0.025$ we have

$$\lambda_1^\beta = -46.80 + 60.45\mathrm{i}, \qquad \lambda_2^\beta = -46.80 - 60.45\mathrm{i}, \qquad \lambda_3^\beta = -25.97.$$

These values are used to find the lower bounds η_β^λ in (24) for $\lambda = 0, \pm\alpha\mathrm{i}$.

In the following tables we display the results to demonstrate the geometric convergence of the β-iteration scheme. Our theoretical results predict the existence of a constant D so that for t sufficiently large

$$E_n(t) \le D^n$$

since the amplitude, Amp, of the signal is equal to one.

In the Table 1 we first compute

$$\widetilde{e}_1 = \max_{4 \le t \le 10} |e_1(t)|, \qquad \widetilde{e}_2 = \max_{4 \le t \le 10} |e_2(t)|, \qquad \widetilde{e}_3 = \max_{4 \le t \le 10} |e_3(t)|$$

then we evaluate the following ratios

$$\frac{\widetilde{e}_2}{\widetilde{e}_1} \quad \text{and} \quad \frac{\widetilde{e}_3}{\widetilde{e}_2}.$$

As expected we observe that

$$\frac{\widetilde{e}_2}{\widetilde{e}_1} \cong \frac{\widetilde{e}_3}{\widetilde{e}_2}$$

to several decimal places. Then we set

$$D_1 = \frac{\widetilde{e}_3}{\widetilde{e}_2}.$$

In the Table 2 we compute the values of D_2 using formula (23). Note that for any pair of α and β we have

$$D_1 < D_2$$

demonstrating the validity of estimate (23). However while for $\beta = 0.1$ the estimate seems quite accurate for large values of α and small values of β these estimates become overly conservative. This is easily explained by the fact that in finding equation (23) some rough assumptions were made in estimating the coefficients $\mathcal{R}_1$ and $\mathcal{R}_2$. In this special case we can apply the results expressed in Remark 6 using η_β^λ with $\lambda = \pm i\alpha$. Even sharper estimates can be obtained starting from equality (21) and using the resolvent of A, $R(\lambda, A)$, in this case the values of the geometric constant D_3 are summarized in Table 3. Once more notice that as predicted for any pair of α and β we have

$$D_1 < D_3.$$

However in this case notice that the values of the constant D_3 is more accurate with respect to the values D_2 and especially for small β and large α.

Table 1: D_1, numerical evaluation.

α \ β	.1	.05	.025
2	0.0844	0.0423	0.0211
6	0.2533	0.1283	0.0642
10	0.4205	0.2184	0.1094
14	0.5796	0.3136	0.1577
18	0.7222	0.4136	0.2098

Table 2: D_2 using Equation (23).

α \ β	.1	.05	.025
2	0.178	0.3674	0.8124
6	0.6684	1.1968	2.648
10	1.3244	2.0398	4.4922
14	2.0698	2.7784	6.0698
18	2.7794	3.347	7.2426

Table 3: D_3 using formula with $R(\mathrm{i}\alpha, A)$.

α \ β	.1	.05	.025
2	0.1097	0.0568	0.0306
6	0.6073	0.3673	0.2033
10	0.9579	0.6832	0.3954
14	1.1225	0.9381	0.6004
18	1.1770	1.1028	0.803

In Figs. 1–8 below we show the behavior of $y(t)$, $y_r(t)$, $e(t)$, and $e_1(t), e_2(t), e_3(t)$ for $\alpha = 2, 10$, and for $\beta = 0.1, 0.05$.

Case 1: $\alpha = 2$

We begin with $\beta = 0.1$.

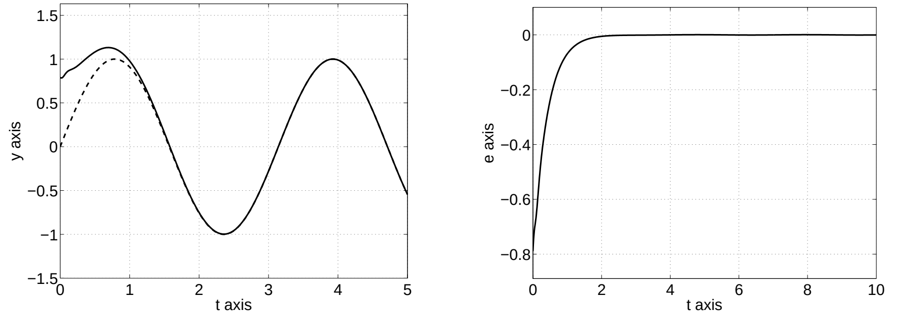

Figure 1. $\alpha = 2, \beta = 0.1$. Left: y, y_r (dashed), Right: e.

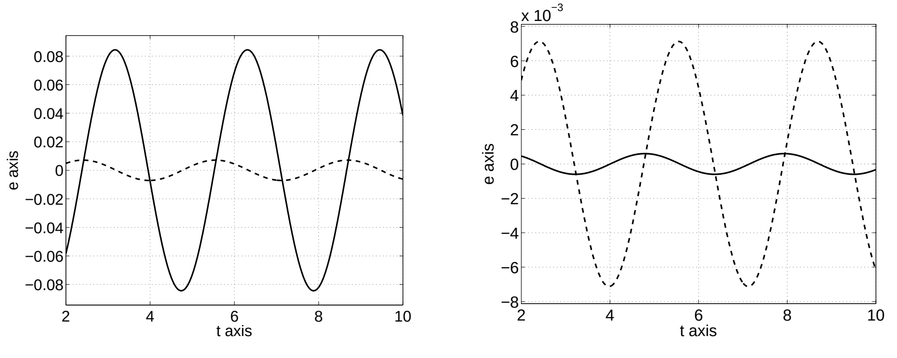

Figure 2. $\alpha = 2, \beta = 0.1$. Left: e_1, e_2, Right: e_2, e_3.

Next we take $\beta = 0.05$.

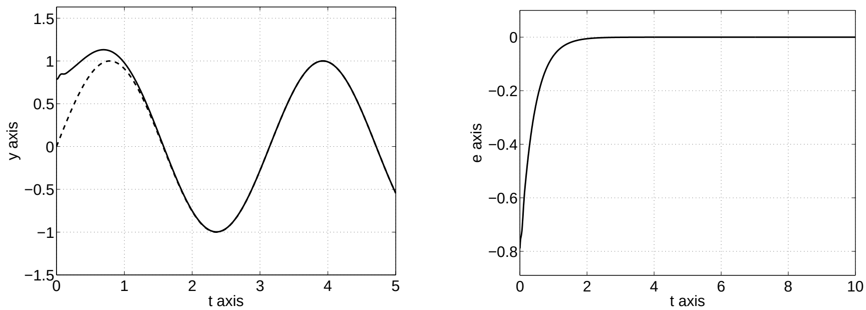

Figure 3. $\alpha = 2, \beta = 0.05$. Left: y, y_r (dashed), Right: e.

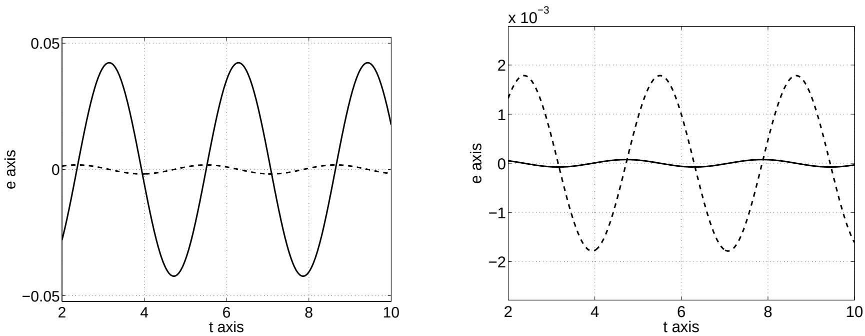

Figure 4. $\alpha = 2, \beta = 0.05$. Left: e_1, e_2, Right e_2, e_3.

Case 2: $\alpha = 10$

We begin with $\beta = 0.1$.

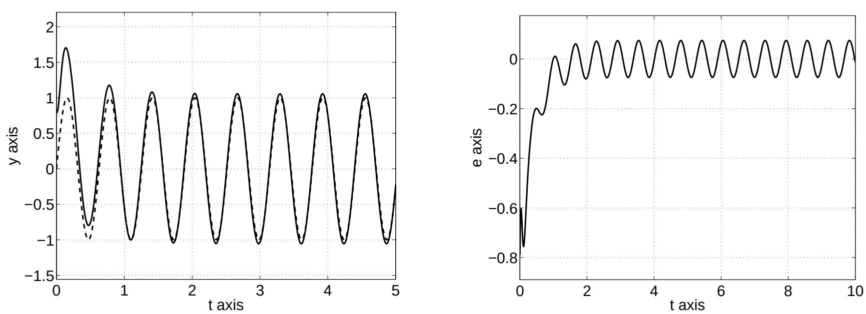

Figure 5. $\alpha = 10, \beta = 0.1$. Left: y, y_r (dashed), Right: e.

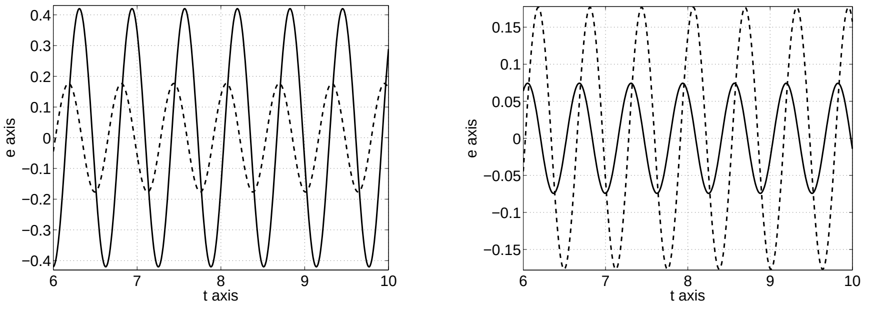

Figure 6. $\alpha = 10, \beta = 0.1$: (Left) e_1, e_2 , (Right) e_2 , e_3.

Next we set $\beta = 0.05$.

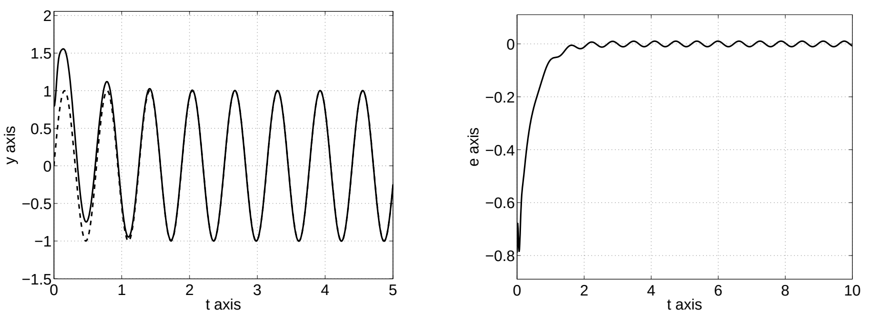

Figure 7. $\alpha = 10, \beta = 0.05$. Left: y, y_r (dashed), Right: e.

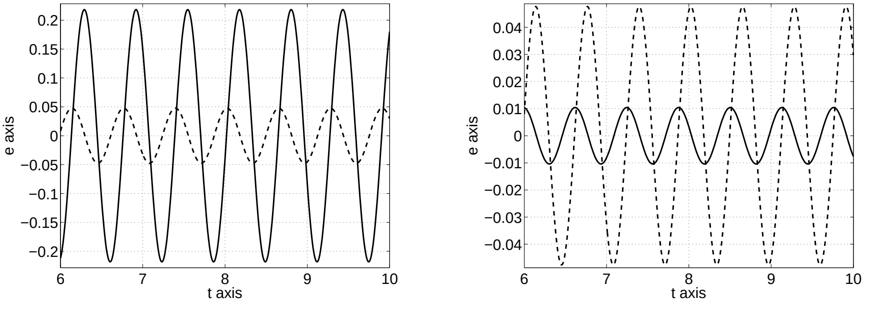

Figure 8. $\alpha = 10, \beta = 0.05$. Left e_1, e_2, Right: e_2, e_3.

9. Numerical Example: Tracking General Periodic Signals

In this section we consider tracking a continuous function that can be approximated by a Fourier series. In order to apply the geometric theory with a finite-dimensional exo-system we could truncate the Fourier series expansion to obtain an approximate reference signal. The problem with that is that the resulting regulator equations given in (5) would become rather complicated to solve both analytically and numerically. On the other hand the β-iteration method can be easily used for tracking a finite sum of simple harmonic functions.

Let us assume that y_r is a periodic function with period $2P$ with Fourier series representation:

$$y_r(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left\{ a_j \cos\left(\frac{j\pi t}{P}\right) + b_j \sin\left(\frac{j\pi t}{P}\right) \right\}$$

and we assume that y_r is in the class $C^{(p)}[0, 2P]$, for some $p \geq 1$.

Truncating the infinite series (if it is indeed infinite) at the value K we obtain

$$y_r(t) \simeq y_r^K(t) = \frac{a_0}{2} + \sum_{j=1}^{K} \left\{ a_j \cos\left(\frac{j\pi t}{P}\right) + b_j \sin\left(\frac{j\pi t}{P}\right) \right\}.$$

Thus we consider

$$y_r^K(t) = y_r(0) + \sum_{j=1}^{K} y_r(j)(t)$$

where

$$y_r(j)(t) = \{ a_j \cos(\alpha_j t) + b_j \sin(\alpha_j t) \}$$

is a harmonic function with frequency

$$\alpha_j = \frac{j\pi}{P}, \qquad j = 1, 2, \ldots, K.$$

From (19) we see that

$$\max_{[0,2P]} |y_r(j)(t)| = \sqrt{\alpha_j^2 a_j^2 + b_j^2}$$

and

$$\max_{[0,2P]} |y_r'(j)(t)| = \alpha_j \sqrt{\alpha_j^2 a_j^2 + b_j^2}.$$

We will set

$$M_K = \max_{j=1,\ldots,K} \begin{cases} \sqrt{\alpha_j^2 a_j^2 + b_j^2}, & \alpha_j \leq 1 \\ \alpha_j \sqrt{\alpha_j^2 a_j^2 + b_j^2}, & \alpha_j > 1. \end{cases}$$

The first term $\frac{a_0}{2}$ can be considered as a harmonic expression with $\alpha_0 = 0$. But it is easier to solve a set-point problem to find the part of the control corresponding to a harmonic expression with 0 frequency. We also note that $a_0 = 0$ if the average value of the signal is zero. For simplicity of the exposition we will make this assumption.

In this case, following the development in Section 4 we can express the DRE in terms of a sum of K unknowns $\overline{\Pi}(j)(t)$ and $\overline{\Gamma}(j)(t)$, $j = 1, \ldots, K$, where for each j we employ the iterative algorithm to find a sequence of $\overline{\Pi}^n(j)(t)$and $\overline{\Gamma}^n(j)(t)$ in the form

$$\overline{\Pi}^n(j)(t) = \sum_{\ell=1}^{n} \overline{\Pi}(j)_\ell(t), \qquad \overline{\Gamma}^n(j)(t) = \sum_{\ell=1}^{n} \overline{\Gamma}(j)_\ell(t), \qquad j = 1, \ldots, K$$

and we have for each j

$$\overline{\Pi}^n(j)(t) \to \overline{\Pi}(j)(t) \quad \text{and} \quad \overline{\Gamma}^n(j)(t) \to \overline{\Gamma}(j)(t).$$

Linearity allows us to express the DRE in terms of K systems of equations

$$\overline{\Pi}(j)(t)_t = A_\beta \overline{\Pi}(j)(t)(t) + \frac{1}{\beta} B f(t), \qquad \overline{\Pi}(j)(0) = -A^{-1} B y_r(j)(0)$$

for which we may then apply Lemmas 3, 4, and Theorem 5 to obtain for each $j = 1, \ldots, K$

$$e_n(j)(t) = \mathcal{Y}_r(j)(0)\mathcal{E}_n(j)(t) + \mathcal{Y}_r(j)(t)E_n(j)$$

with

$$\mathcal{Y}_r(j)(t) = \big(y_r(j)(t),\ y_r'(j)(t)\big)$$

satisfying

$$|\mathcal{Y}_r(j)(t)| \leq \max_{\substack{t\in[0,2P] \\ j=1,\ldots,K}} \{|y_r(j)(t)|, |y_r'(j)(t)|\} \leq M_K.$$

Then just as before, for each $j = 1, \ldots, K$

$$\lim_{t\to\infty} \|\mathcal{E}_n(j)\|_\infty \to 0$$

exponentially fast.

The remaining t dependence in the other terms can be bounded by M_K above and we have

$$|\mathcal{Y}_r(j)(t)E_n(j)| \leq M_K \left(\frac{\alpha_j M^2 \|B\| \|C\|}{\beta \omega_\beta^2}\right)^n \left(\frac{\alpha_j M}{\omega_\beta} + 1\right)^n.$$

Example 7 (Tracking a Triangle Function)**.** In our specific numerical example we consider the periodic triangle function defined to be $f(t)$ on the interval $0 \leq t < 2P$ and then extended periodically to the whole real line. The Fourier series expansion for $f(t)$ is given by

$$f(t) = \frac{8}{\pi^2} \sum_{j=1}^{\infty} \frac{\sin\left(\frac{(2j-1)\pi t}{P}\right)}{(2j-1)^2}.$$

We take $y_r(t)$ to be a finite truncation of this infinite sum

$$y_r(t) = f_K(t) = \frac{8}{\pi^2} \sum_{j=1}^{K} \frac{\sin\left(\frac{(2j-1)\pi t}{P}\right)}{(2j-1)^2}.$$

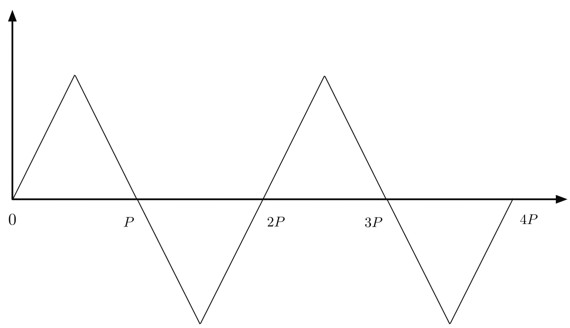

Figure 9. Plot of $f(t)$ and $f_6(t)$.

In Fig. 9 we have plotted both $f(t)$ and its approximation $f_6(t)$. Note that, already for $K = 6$, the two functions are visually indistinguishable.
Applying the results found in the previous sections by simply defining

$$y_r(j)(t) = \left(\frac{8}{\pi^2}\right) \frac{\sin\left(\dfrac{(2j-1)\pi t}{P}\right)}{(2j-1)^2}, \qquad j = 1\ldots, K.$$

In this case we can consider

$$\alpha_j = \frac{(2j-1)\pi}{P}, \qquad j = 1\ldots, 6$$

and set $\overline{\Pi}(j)^n(t)$ in (17) to obtain

$$(\overline{\Pi}^n(j)(t))_t = A_\beta \overline{\Pi}^n(j)(t) + \frac{1}{\beta} B y_r(j)(t), \qquad (\overline{\Pi}^n(j)(0)) = -A^{-1} B y_r(j)(0).$$

In our numerical simulation we have set $P = 5$ so the period is $2P = 10$ and we have truncated the infinite Fourier series at $K = 6$. In addition to our figures depicting the results of the numerical study we also include maximum steady state error values

$$\widetilde{e}_j = \max_{5 \le t \le 40} |e_j(t)| \qquad \text{for} \qquad j = 1, 2, 3, 4.$$

Then we computed the values

$$\frac{\widetilde{e}_2}{\widetilde{e}_1} = 0.0719, \qquad \frac{\widetilde{e}_3}{\widetilde{e}_2} = 0.0587, \quad \text{and} \quad \frac{\widetilde{e}_4}{\widetilde{e}_3} = 0.1086.$$

For each α_j for $j = 1, \ldots, 6$ we have also estimated the computed values of E_j using $\beta = 0.05$ and the corresponding values for ω_β used in the previous example. Thus for each term of the Fourier series we have computed the values

$$D_j = \left(\frac{\alpha_j M^2 \|B\| \|C\|}{\beta \omega_\beta^2} \right) \left(\frac{\alpha_j M}{\omega_\beta} + 1 \right)$$

and obtained

$$D_1 = 0.0553, \quad D_2 = 0.1734, \quad D_3 = 0.3015, \quad D_4 = 0.4396, \quad D_5 = 0.5876.$$

Notice that these are all less than 1 and we see that the geometric convergence occurs as the theory predicts. In the actual numerical error estimates we are also aided by the rapid decrease of the amplitudes of the individual signals $y_r(j)(t)$ given by $8/(\pi^2(2j-1)^2)$. In Fig. 10 we have depicted, on the left, the numerically computed measured output $y(t)$ and the reference signal $y_r(t)$ with $K = 6$ and, on the right the error $e(t) = y_r(t) - y(t)$ which is on the order of 10^{-5}. In Fig. 11 we have plotted, on the left, both e_1 and e_2 and, on the right, e_2 and e_3. Finally in Fig. 12 we have plotted the errors e_3 and e_4 and we note the decrease of the errors predicted by our theoretical results.

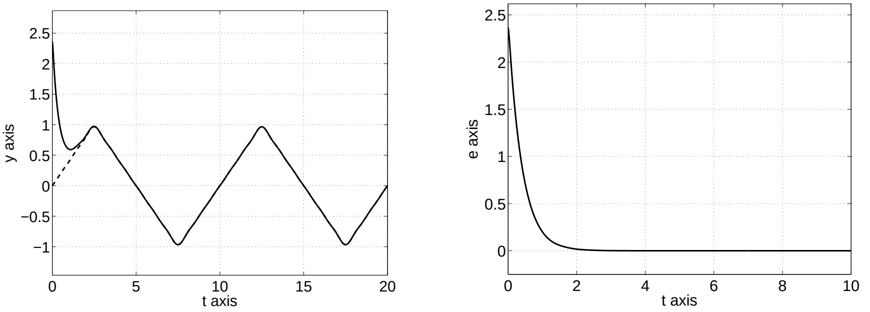

Figure 10. Left: y, y_r, Right: e.

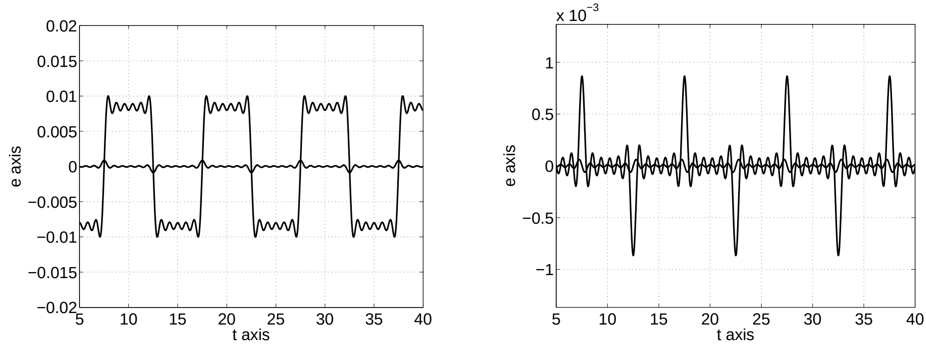

Figure 11. Left: e_1, e_2, Right: e_2, e_3.

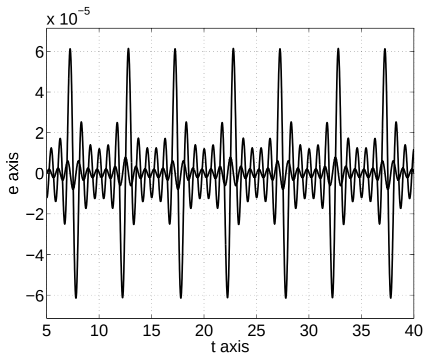

Figure 12. e_3, e_4.

10. Acknowledgments

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